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CHARACTERISTIC FUNCTIONS AND PRODUCTS OF DERIVATIVES

0. Introduction. The main result of this note is Theorem 18 that describes in simple terms the system of sets $S \subset R = (-\infty, \infty)$ whose characteristic function χ_s can be expressed as the product of two (or more) derivatives. Suppose that S is such a set. Then χ_s is a function of Baire class 1 and, clearly, $S = \{x \in R; \chi_s(x) > 0\} = \{x \in R; \chi_s \geqq 1\}$. We see that S is at the same time an F_σ -set and a G_δ-set; such sets are called ambiguous. It has been proved in [1] that χ_s can be expressed as the product of two nonnegative derivatives if and only if S is ambiguous and each point of the set $T = R \setminus S$ is a point of density of T. Theorem 18 shows that we obtain a larger system of sets S, if we drop the requirement of nonnegativity of the derivatives with product χ_s.

1. Notation. The outer Lebesgue measure of a set $A \subset R$ will be denoted by $|A|$. The word interval means a connected set $A \subset R$ with $|A| > 0$.

Let $c \in R$ and let $J_1, J_2, \ldots$ be intervals. We say that the sequence $\langle J_n \rangle$ has property P_c if $diam(J_n \cup \{c\}) \to 0$ and $\sup\{diam(J_n \cup \{c\})/|J_n|;\ n = 1, 2, \ldots\} < \infty$.

Let $T \subset R$ and $c \in R$. We say that T is porous at c if there is a sequence $\langle J_n \rangle$ with property P_c such that $J_n \cap T = \emptyset$ for each n. If such a sequence does not exist, we say that T is nonporous at c. A set $T \subset R$ is called nonporous if it is nonporous at each of its points.

The word function means a mapping to R. For each interval J let $\Delta(J)$ be the system of all functions (finitely) differentiable on J and let $\mathcal{D}(J) = \{F'; F \in \Delta(J)\}$. (At a boundary point of J belonging to J we mean the corresponding unilateral derivative of F.) We write $\mathcal{D} = \mathcal{D}(R)$. If $a, b \in R$ and if F is a function defined at a and at b, then $[F]_a^b$ means, as usual, $F(b) - F(a)$. If $a, b \in R$, $J = [a, b]$ and if $f \in \mathcal{D}(J)$, then $\int_a^b f$ or $\int_J f$ means $[F]_a^b$, where $F' = f$ on J; $\int_a^a f = 0$. A symbol like $\int_A f$ sometimes means the corresponding Lebesgue integral. It is well-known that these two definitions of integral do not contradict each other.

If F is a function on a set $A \subset R$, then osc (F, A) means $\sup\{|F(y) - F(x)|;\ x, y \in A\}$.

For each $A \subset R$ let int A and $c\ell\ A$ be the interior and the closure of A, respectively.

In what follows S is a subset of R and T is its complement.

2. Lemma. Let S be ambiguous and let A be a nonempty G_δ-set in R. Then there is an open interval I such that $I \cap A \neq \emptyset$ and that either $I \cap A \subset S$ or $I \cap A \subset T$.

Proof. Since $\{x \in R;\ \chi_s(x) > \alpha\}$ and $\{x \in R;\ \chi_s(x) < \alpha\}$ are F_σ-sets for each $\alpha \in R$, χ_s is a function of Baire class 1. Thus there is a $c \in A$ and an open interval I such that $c \in I$ and that χ_s is constant on $I \cap A$. This completes the proof.

3. Lemma. Let $c, a_n, b_n \in R$, $a_n < b_n$ $(n = 1, 2, \ldots)$ and let the sequence $\langle (a_n, b_n) \rangle$ have property P_c. Let F be a function differentiable at c. Then $(F(b_n) - F(a_n))/(b_n - a_n) \to F'(c)$.

Proof. If $x, y \in R$, $x \neq c \neq y \neq x$ and if F is defined at x and at y, then

$$\frac{F(y) - F(x)}{y - x} = \frac{F(y) - F(c)}{y - c} + \frac{x - c}{y - x}\left(\frac{F(y) - F(c)}{y - c} - \frac{F(x) - F(c)}{x - c}\right).$$

This easily implies our assertion.

4. Proposition. Let $c \in R$. Let T be a Lebesgue measurable set that is porous at c whose right and left densities at c have common value δ. Then $\delta = 0$.

Proof. Let F be an indefinite integral of χ_s. Then $F'(c) = 1 - \delta$. Let $\langle J_n \rangle$ be a sequence with property P_c such that $J_n \subset S$ for each n. Let $\{a_n, b_n\}$ be the boundary of J_n. Then $(F(b_n) - F(a_n))/(b_n - a_n) = 1$ for each n. By 3 we have $1 = F'(c) = 1 - \delta$ whence $\delta = 0$.

Remark. A nonporous ambiguous set can have density 0 at some of its points. Example: For $n = \pm 1, \pm 2, \ldots$ let J_n be the open interval with boundary $\{\frac{1}{n}, \frac{1}{n} + \frac{1}{n^3}\}$; let $T_0 = \cup J_n$, $T = \{0\} \cup T_0$. It is easy to prove that T is a nonporous ambiguous set whose density at 0 is 0.

5. Proposition. Let $c \in R$ and let T be porous at c. Let m be a natural number, $f_1, \ldots, f_m \in \mathcal{D}$ and $\chi_s = \prod f_j$. Then $c \in S$.

Proof. Let L be an interval, $L \subset S$. Since sgn f_j is constant on L for each j, we have by Hölder's inequality $1 = \frac{1}{|L|} \int_L (\prod f_j)^{1/m} \leqq (\prod \frac{1}{|L|} \int_L f_j)^{1/m}$. Now it follows easily from 3 that $1 \leqq \prod f_j(c)$. Hence $c \in S$.

Remark. If χ_s can be expressed as the product of two (or more) derivatives, then (as stated in the introduction) S is ambiguous and, by 5, T is nonporous. Theorem 18 says that the converse of this assertion is also correct. To prove it we shall first establish some further properties of sets S fulfilling the mentioned conditions.

6. Proposition. Let S be ambiguous and let T be nonporous. Let F be the boundary of S and let F_0 be the set of all accumulation points of F. Then F and F_0 are closed and $F \setminus F_0 \subset S$.

Proof. It is easy to see that F and F_0 are closed. Now let $c \in F \setminus F_0$. There are $u, v \in R$ such that $u < c < v$ and that $(u, v) \cap F = \{c\}$.

Suppose that $c \in T$. Since T is nonporous, we cannot have $(u, c) \subset S$; hence $(u, c) \subset T$. Similarly $(c, v) \subset T$ so that $(u, v) \subset T$, $(u, v) \cap F = \emptyset$ - a contradiction.

7. Proposition. Let S and T be as in 6. Let L be an open interval such that $L \cap S \neq \emptyset \neq L \cap T$. Then there are the following two possibilities:

(A) There is a $c \in L$ such that $L \cap S = L \cap [c, \infty)$.

(B) There are $p, q \in L$ such that $p < q$, $p \in S$ and $(p, q) \subset T$.

Proof. Let F and F_0 be as in 6. Clearly $L \cap F \neq \emptyset$. We distinguish several cases.

Case 1. $L \cap F$ is a singleton, say $\{c\}$. By 6 we have $c \in S$ so that either $L \cap S = L \cap (-\infty, c]$ or $L \cap S = \{c\}$ or $L \cap S = L \cap [c, \infty)$. Thus our assertion holds.

Case 2. There are $x_1, x_2, x_3 \in L$ such that $x_1 < x_2 < x_3, (x_1, x_3) \cap F = \{x_2\}$ and $x_1 \in S$ or $x_3 \in T$. By 6 we have $x_2 \in S$. Let $x_1 \in S$. If $(x_1, x_2) \subset T$, we choose $p = x_1, q = x_2$. If $(x_1, x_2) \subset S$, then $(x_2, x_3) \subset T$ and we choose $p = x_2, q = x_3$. Now let $x_3 \in T$. Since T is nonporous, we cannot have $(x_2, x_3) \subset S$. Thus we may choose $p = x_2, q = x_3$ again.

Case 3. There is an open interval $I \subset L$ such that each point of $I \cap F$ is isolated and $I \cap F$ has at least two points. Then there are $x_1, x_2, x_3 \in I$ such that $x_1 < x_2 < x_3$, $x_1, x_2 \in F$ and $(x_1, x_3) \cap F = \{x_2\}$. By 6 we have $x_1 \in S$ and so Case 2.

Case 4. $L \cap F_0 \neq \emptyset$ and each component of $L \setminus F_0$ contains at most one point of F. According to 2 with $A = L \cap F_0$ there is a $c \in F_0$ and an interval $M = (u, v) \subset L$ such that $c \in M$ and that either $M \cap F_0 \subset S$ or $M \cap F_0 \subset T$.

a) Let $M \cap F_0 \subset S$. Let $\mathcal{J}_1$ be the system of all components J of $M \setminus F_0$ for which $u \notin c\ell\, J$. If there is a $J \in \mathcal{J}_1$ with $J \cap F \neq \emptyset$, we have Case 2. Thus suppose that $J \cap F = \emptyset$ for each $J \in \mathcal{J}_1$. If $J = (y_1, y_2) \in \mathcal{J}_1$ and $J \subset T$, we choose $p = y_1$, $q = y_2$. Now let $J \subset S$ for each $J \in \mathcal{J}_1$. Then $[r, v) \subset S$ for each $r \in M \cap F_0$. In particular, $[c, v) \subset S$ so that $(c, v) \cap F = \emptyset$. If $(u, c) \cap F_0 = \emptyset$, then, by assumption, $(u, c) \cap F$ has at most one point so that $c \notin F_0$ - a contradiction. Hence there is an $r \in (u, c) \cap F_0$; thus $[r, v) \subset S$, $c \notin F$ - a contradiction again.

b) Let $M \cap F_0 \subset T$. Let $\mathcal{J}_2$ be the system of all components J of $M \setminus F_0$ for which $v \notin c\ell\, J$. If there is a $J \in \mathcal{J}_2$ with $J \cap F \neq \emptyset$, we have Case 2. Thus suppose that $J \cap F = \emptyset$ for each $J \in \mathcal{J}_2$. Since T is nonporous, we must have $J \subset T$ for each $J \in \mathcal{J}_2$. Therefore $(u, r] \subset T$ for each $r \in M \cap F_0$ which leads to a contradiction as in a).

Remark. Let S and T be as before and let $\emptyset \neq S \neq R$. Proposition 7 says, in particular, that, unless $S = [c, \infty)$ for some $c \in R$, there are $p, q \in R$ such that $p < q$, $p \in S$ and $(p, q) \subset T$. Similarly, unless $S = (-\infty, c]$ for some $c \in R$, there are $p, q \in R$ such that $p < q$, $q \in S$ and $(p, q) \subset T$. This might lead to the conjecture that T always has an open component. This conjecture, however, is false, as the following example shows:

Let $G_1, G_2, \ldots$ be the components of $[0, 1] \setminus C$, where C is the Cantor set. Let p_n be the midpoint of G_n and let $S = \{p_1, p_2, \ldots\}$. It is easy to see that S is ambiguous and that T is nonporous, but no component of T is open.

8. Proposition. Let S and T be as in 6. Let J be an interval such that $J \cap T \neq \emptyset$. Then $J \cap T$ contains an interval.

Proof. Let $I = \operatorname{int} J$. We may suppose that $I \cap S \neq \emptyset$. If $I \cap T = \emptyset$, then we choose a $c \in (J \setminus I) \cap T$ and we see that T is porous at c - a contradiction. Hence $I \cap T \neq \emptyset$ and we apply 7.

9. Lemma. Let $a, b, \alpha, \beta, \gamma, \delta, \varepsilon, P, Q \in R$, $a < b$, $\alpha\beta = \gamma\delta = 0$, $\varepsilon > 0$. Then there are functions f, g piecewise linear on $J = [a, b]$ such that $f(a) = \alpha$, $g(a) = \beta$, $f(b) = \gamma$, $g(b) = \delta$, $fg = 0$ on J, $\int_J f = P$, $\int_J g = Q$, $\int_J |f| < |P| + \varepsilon$ and $\int_J |g| < |Q| + \varepsilon$.

Proof. Choose numbers $x_1, \ldots, x_5$ such that $a < x_1 < \ldots < x_5 < b$, $(|\alpha| + |\beta|)(x_1 - a) + (|\gamma| + |\delta|)(b - x_5) < \varepsilon$ and set

$$P_1 = (2P - \alpha(x_1 - a) - \gamma(b - x_5))/(x_3 - x_1),$$

$$Q_1 = (2Q - \beta(x_1 - a) - \delta(b - x_5))/(x_5 - x_3).$$

Define $f(a) = \alpha$, $f(x_1) = 0$, $f(x_2) = P_1$, $f(x_3) = f(x_4) = f(x_5) = 0$, $f(b) = \gamma$, $g(a) = \beta$, $g(x_1) = g(x_2) = g(x_3) = 0$, $g(x_4) = Q_1$, $g(x_5) = 0$, $g(b) = \delta$ and let f and g be linear on each of the intervals $[x_{j-1}, x_j]$ $(j = 1, \ldots, 6)$, where $x_0 = a$, $x_6 = b$. It is easy to see that $fg = 0$ on J. We have $\int_J f = \frac{1}{2}(\alpha(x_1 - a) + P_1(x_3 - x_1) + \gamma(b - x_5)) = P$, $\int_J |f| \leqq |\alpha|(x_1 - a) + |P| + |\gamma|(b - x_5) < |P| + \varepsilon$. Similarly for g.

10. Notation. In 11-17 S is a fixed subset of R. For each interval J let $\mathfrak{f}(J)$ be the system of all pairs (f, g) such that $f, g \in \mathcal{D}(J)$, $f = g = 1$ on $J \cap S$ and $fg = 0$ on $J \cap T$. Let $\mathcal{J}$ be the system of all intervals J such that $\mathfrak{f}(J) \neq \emptyset$.

11. Lemma. Let $a_j, b_j \in R$, $a_1 < a_2 < b_1 < b_2$. Let $(f_j, g_j) \in \mathfrak{f}([a_j, b_j])$ $(j = 1, 2)$. Then there is a pair $(f, g) \in \mathfrak{f}([a_1, b_2])$ such that $f = f_1$, $g = g_1$ on $[a_1, a_2]$ and $f = f_2$, $g = g_2$ on $[b_1, b_2]$.

Proof. If there is a $c \in [a_2, b_1] \cap S$, we have $f_1(c) = \cdots = g_2(c) = 1$. Then we set $f = f_1$, $g = g_1$ on $[a_1, c]$ and $f = f_2$, $g = g_2$ on $[c, b_2]$. Otherwise we have $f_1 g_1 = f_2 g_2 = 0$ on $[a_2, b_1]$. Then we choose a number $\alpha \in (a_2, b_1)$ and construct functions f, g such that $f = f_1$, $g = g_1$ on $[a_1, a_2]$, $f(\alpha) = g(\alpha) = 0$, $f = f_2$, $g = g_2$ on $[b_1, b_2]$ and that f and g are linear on the intervals $[a_2, \alpha]$ and $[\alpha, b_1]$. It is easy to see that $(f, g) \in \mathfrak{f}([a_1, b_2])$.

12. Lemma. Let L be an open interval. Suppose that for each $x \in L$ there is an open interval I such that $x \in I \in \mathcal{J}$. Then $L \in \mathcal{J}$.

Proof. Choose numbers $x_n \in L$ $(n = 0, \pm 1, \pm 2, \ldots)$ such that

$$x_{n-1} < x_n, \ \inf\{x_n\} = \inf L, \ \sup\{x_n\} = \sup L. \tag{1}$$

It follows easily from 11 that $[x_{n-1}, x_{n+1}] \in \mathcal{J}$ for each n. Applying 11 once more we get $L \in \mathcal{J}$.

13. Lemma. Let S and T be as in 6. Let L be an open interval. Let $L \in \mathcal{J}$ and let w be a positive continuous function on L. Then there are $F, G \in \Delta(L)$ such that $(F', G') \in \mathfrak{f}(L)$ and

$$|F(x) - x| < w(x), \ |G(x) - x| < w(x) \text{ for each } x \in L.$$

Proof. Let F_0, G_0 be functions such that $(F_0', G_0') \in \mathfrak{f}(L)$. There are numbers $x_n \in L$ $(n = 0, \pm 1, \pm 2, \ldots)$ fulfilling (1) such that, if we define $J_n = [x_{n-1}, x_n]$, we have

$$3|J_n| + 4 \operatorname{osc}(F_0, J_n) + 4 \operatorname{osc}(G_0, J_n) < \min w(J_n) \text{ for each } n. \tag{2}$$

Choose an n. If $J_n \subset S$, set $f = g = 1$ on J_n. Otherwise, by 8, there are $a, b \in R$ such that $x_{n-1} < a < b < x_n$ and that $F_0' G_0' = 0$ on $[a, b]$. Set $P = |J_n| - [F_0]_{x_{n-1}}^{a} - [F_0]_{b}^{x_n}$, $Q = |J_n| - [G_0]_{x_{n-1}}^{a} - [G_0]_{b}^{x_n}$, $\varepsilon = |J_n|$. According to 9 there are functions f, g such that $f = F_0'$, $g = G_0'$ on $[x_{n-1}, a] \cup [b, x_n]$, f and g are continuous on $[a, b]$, $fg = 0$ on $[a, b]$, $\int_{J_n} f = \int_{J_n} g = |J_n|$ and $\int_a^b |f| < |P| + \varepsilon$, $\int_a^b |g| < |Q| + \varepsilon$. Thus $\int_a^b |f| < 2|J_n| + 2 \operatorname{osc}(F_0, J_n)$. If $x \in J_n$, then

$$\Big|\int_{x_{n-1}}^{x} f\Big| \leqq 2 \operatorname{osc}(F_0, J_n) + \int_a^b |f| < 4 \operatorname{osc}(F_0, J_n) + 2|J_n|; \tag{3}$$

similarly for g. In this way we define functions f and g on L. It is easy to see that there are functions F and G such that $F' = f$, $G' = g$ on L and that $F(x_n) = G(x_n) = x_n$ for each n. If $x \in J_n$, then, by (2) and (3), $|F(x) - x| \leqq |F(x) - F(x_{n-1})| + |x_{n-1} - x| \leqq 4 \operatorname{osc}(F_0, J_n) + 3|J_n| < w(x)$; similarly $|G(x) - x| < w(x)$. Clearly $(F', G') \in \mathfrak{f}(L)$.

14. Notation. For each $x \in R$ let $\mathfrak{A}(x)$ be the system of all intervals J such that $x \in J \subset S$. If $\mathfrak{A}(x) = \emptyset$, we set $\varphi(x) = 0$; otherwise we define $\varphi(x) = \sup\{|J|;\ J \in \mathfrak{A}(x)\}$.

15. Lemma. Let $c \in T$ and let T be nonporous at c. Then $\varphi'(c) = 0$.

Proof. Clearly $\varphi(c) = 0$. Suppose that our assertion fails. Then there is an $\varepsilon \in (0, 1)$ and points $c_n \neq c$ such that $c_n \to c$ and $\varphi(c_n) > \varepsilon |c_n - c|$ $(n = 1, 2, \ldots)$. There are intervals $J_n \subset S$ such that $c_n \in J_n$ and $|c_n - c| \geqq |J_n| > \varepsilon |c_n - c|$. Then $diam(J_n \cup \{c\}) \leqq |c - c_n| + |J_n| \leqq |J_n|(1 + \varepsilon^{-1})$ so that $\langle J_n \rangle$ has property P_c. This is a contradiction.

16. Lemma. Let S, T, L and w be as in 13. Then there are $F, G \in \Delta(L)$ such that $(F', G') \in \mathfrak{f}(L)$ and that for each $x \in L$ there is a $y \in L$ with

$$|y - x| < dist(x, R \setminus L) \tag{4}$$

and

$$\max(|F(x)|,\ |G(x)|) < \varphi(y) + w(x). \tag{5}$$

Proof. Let F_0, G_0 be functions such that $(F_0', G_0') \in \mathfrak{f}(L)$. There are points $x_n \in L$ $(n = 0, \pm 1, \pm 2, \ldots)$ fulfilling (1) such that, if we define $J_n = [x_{n-1}, x_n]$ and $\varepsilon_n = \frac{1}{5} \min w(J_n)$, we have

$$\text{(6)} \qquad \operatorname{osc}(F_0, J_n) + \operatorname{osc}(G_0, J_n) < \varepsilon_n \text{ and } |J_n| < dist(J_n, R \setminus L)$$

for each n. Set $F(x_0) = G(x_0) = 0$. Suppose that n is a natural number and that $F(x_{n-1})$, $G(x_{n-1})$ have been defined. If $J_n \subset S$, we set $F(x) = F(x_{n-1}) + x - x_{n-1}$, $G(x) = G(x_{n-1}) + x - x_{n-1}$ for each $x \in J_n$. Otherwise there are, by 8, numbers a, b such that $x_{n-1} < a < b < x_n$ and that $F_0' G_0' = 0$ on $[a, b]$. Set

$$P_n = -F(x_{n-1}) - [F_0]_{x_{n-1}}^{a} - [F_0]_{b}^{x_n}, \; Q_n = -G(x_{n-1}) - [G_0]_{x_{n-1}}^{a} - [G_0]_{b}^{x_n}.$$

By 9 there are functions f, g on J_n such that $f = F_0'$, $g = G_0'$ on $[x_{n-1}, a] \cup [b, x_n]$, f and g are continuous on $[a, b]$, $\int_a^b f = P_n$, $\int_a^b g = Q_n$, $\int_a^b |f| < |P_n| + \varepsilon_n$, $\int_a^b |g| < |Q_n| + \varepsilon_n$ and $fg = 0$ on $[a, b]$. For each $x \in J_n$ set $F(x) = -\int_x^{x_n} f$, $G(x) = -\int_x^{x_n} g$. Thus we have defined functions F and G on $[x_0, \sup L)$. It is easy to see that $F(x_n) \geqq 0$ for each $n \geqq 0$; we have $F(x_n) > 0$ if and only if $J_n \subset S$. Similarly for G.

Let $x \in J_n$, $n > 0$. If $F(x_n) > 0$, we find an integer $m < n$ such that $F(x_m) = 0$ and that $F(x_k) > 0$ for $k = m + 1, \ldots, n$. Then $[x_m, x_n] \subset S$ and $F(t) = G(t) = t - x_m$ for $t \in [x_m, x_n]$. In particular, $0 \leqq F(x) = G(x) = x - x_m \leqq x_n - x_m \leqq \varphi(x_n)$. Thus the relation (5) is fulfilled with $y = x_n$. If $F(x_n) = 0$, then $|F(x)| < 2 \operatorname{osc}(F_0, J_n) + |P_n| + \varepsilon_n \leqq F(x_{n-1}) + 4 \operatorname{osc}(F_0, J_n) + \varepsilon_n$. By the preceding argument we have $F(x_{n-1}) \leqq \varphi(x_{n-1})$ so that, by the first inequality in (6), $|F(x)| < \varphi(x_{n-1}) + w(x)$; similarly for G. Hence (5) holds with $y = x_{n-1}$. The relation (4) follows from the second inequality in (6). On $(\inf L, x_0)$ we define F and G analogously. It is obvious that $(F', G') \in \mathfrak{f}(L)$.

17. Proposition. Let S be ambiguous and T nonporous. Then $R \in \mathcal{J}$.

Proof. Let $B = \{x \in R$; there is an open interval $J \in \mathcal{J}$ such that $x \in J\}$ and let $A = R \setminus B$. Then A is closed. Suppose that $A \neq \emptyset$. Let w be a function continuous on R such that $w = w' = 0$ on A and $w > 0$ on B. By 2 there is an open interval I such that $I \cap A \neq \emptyset$ and either $I \cap A \subset S$ or $I \cap A \subset T$. It follows from 12 that each component of $I \cap B$ belongs to $\mathcal{J}$.

(a) $I \cap A \subset S$. On each component L of $I \cap B$ we construct functions F and G according to 13. This defines functions F and G on $I \cap B$. Set $F(x) = G(x) = x$ for each $x \in I \cap A$. Then $|F(x) - x| \leqq w(x)$, $|G(x) - x| \leqq w(x)$ for each $x \in I$.

(b) $I \cap A \subset T$. On each component of $I \cap B$ we construct functions F and G according to 16. This defines functions F and G on $I \cap B$. Set $F = G = 0$ on

$I \cap A$. Let $a \in A \cap I$, $x \in I$, $x \neq a$. According to the construction of F there is a $y \in I$ such that $|y - x| < |x - a|$ and that $|F(x)| \leqq w(x) + \varphi(y)$. (If $x \in A$, we may choose $y = x$.) Obviously $|y - a| < 2|x - a|$ so that

$$\left|\frac{F(x)}{x-a}\right| \leqq \left|\frac{w(x)}{x-a}\right| + 2\left|\frac{\varphi(y)}{y-a}\right|.$$

By 15 we have $\varphi'(a) = 0$. Hence $F'(a) = 0$. Similarly $G'(a) = 0$.

Now it is easy to see that in either case $(F', G') \in \mathfrak{f}(I)$. It follows that $I \subset B$ which is a contradiction. Hence $A = \emptyset$. By 12 we have $R = B \in J$.

18. Theorem. Let $S \subset R$, $T = R \setminus S$. Then the following three conditions are equivalent:

1) There is a natural number m and functions $f_1, \ldots, f_m \in \mathcal{D}$ such that $f_1 \cdots f_m = \chi_s$.

2) S is ambiguous and T is nonporous.

3) There are $f, g \in \mathcal{D}$ such that $f = g = 1$ on S and $fg = 0$ on T.

Proof. Let 1) hold. Since χ_s is of Baire class 1, S is ambiguous and, by 5, T is nonporous. The implication 2) $\Rightarrow$ 3) has been proved in 17 and the implication 3) $\Rightarrow$ 1) is obvious.

Remark. Suppose that χ_s can be expressed as the product of two derivatives. It is natural to ask whether we can require something more of one or both of these factors. For example, by 18, we may require both to be identically 1 on S; in particular, we may require both to be continuous on int S. It is not difficult to prove that we may also require both to be continuous on int T. On the other hand it is clear that at each boundary point of S at least one of the factors must be discontinuous. This, however, does not yet exclude the possibility that one of the factors could be continuous everywhere; but, according to 21, in nontrivial cases this is actually impossible.

We may ask similar questions replacing continuity by, e.g., local summability or boundedness (above and/or below) or nonnegativity of one or both factors. It seems that it is not easy to answer some of these questions. For example, I do not know whether the following assertion is true: Let S be ambiguous, T nonporous. Then there are $f, g \in \mathcal{D}$ such that $f \geqq 0$ and $fg = \chi_s$.

What follows are some partial results that may serve as illustrations to the mentioned problems.

19. Proposition. Let S be ambiguous, T nonporous. Let $c \in (c\ell\, S) \cap (c\ell\, T)$. Let $f, g \in \mathcal{D}$, $fg = \chi_s$ and $f(c) \neq 0$. Then f is discontinuous at c.

Proof. Suppose that f is continuous at c. There is an open interval L such that $c \in L$ and that $f \neq 0$ on L. By 7 there are $p, q \in L$ such that $p < q$, $\{p, q\} \cap S \neq \emptyset$ and $(p, q) \subset T$. Then $g = 0$ on (p, q) so that $g = 0$ on $\{p, q\}$, $\{p, q\} \subset T$ which is a contradiction.

Remark. If $c \in S \cap c\ell\, T$, then both f and g are discontinuous at c. If, however, $c \in T \cap c\ell\, S$, then f may be continuous at c, as the following example shows.

20. Example. For $n = 1, 2, \ldots$ let $z_n = 2^{-n}$, $x_n = z_n - z_{n+1}/n^2$, $y_n = (x_n + z_n)/2$. It is easy to see that $x_n \geqq z_{n+1}$. Let f_n, g_n be nonnegative derivatives such that $f_n = g_n = 0$ on $R \setminus (x_n, z_n)$, $f_n \vee g_n \leqq 2$ on R, $f_n(y_n) = g_n(y_n) = 1$, $f_n g_n = 0$ on $R \setminus \{y_n\}$. Set $f = \sum f_n/n$, $g = \sum n g_n$. Let $z_{n+1} < x < z_n$. Then $\int_0^x g \leqq \sum_{k=n}^{\infty} k \int_{x_k}^{z_k} g_k \leqq \sum_{k=n}^{\infty} 2k(z_k - x_k) = \sum_{k=n}^{\infty} 2k\, z_{k+1}/k^2 \leqq \frac{1}{n} \sum_{k=n}^{\infty} z_k = 2z_n/n < 4x/n$. This easily implies that $g \in \mathcal{D}$. It is clear that $f \in \mathcal{D}$ and that f is continuous at 0. We have $fg = \chi_s$, where $S = \{y_1, y_2, \ldots\}$ so that $0 \in T \cap c\ell\, S$.

21. Proposition. Let $\emptyset \neq S \neq R$, $f, g \in \mathcal{D}$ and let $fg = \chi_s$. Then f is not continuous.

Proof. Let B be the boundary of S. Suppose first that $B \cap S = \emptyset$. Then S is open. Let (a, b) be a component of S and let, e.g., $a \in R$. Then T is porous at a which contradicts 5. Thus let $c \in B \cap S$. Then $f(c) \neq 0$ and, by 19, f is discontinuous at c.

22. Proposition. Let $f, g \in \mathcal{D}$, $\emptyset \neq S \neq R$, $fg = \chi_s$. Let $Q \in R$ and let $|f| \vee |g| \leqq Q$. Then $Q \geqq 2$.

Proof. According to 7 there are $p, q \in R$ such that $p < q$, $(p, q) \subset T$ and $\{p, q\} \cap S \neq \emptyset$. Let, e.g., $p \in S$. Then $f(p)g(p) = 1$. We may suppose that $f(p) > 0$. Clearly $|f| + |g| \leqq Q$ on (p, q). Hence for each $x \in (p, q)$ we have $\int_p^x (f + g) \leqq Q(x - p)$ so that $f(p) + g(p) \leqq Q$. Since $t + t^{-1} \geqq 2$ for each $t \in (0, \infty)$, we have $Q \geqq 2$.

23. Proposition. Let $f, g \in \mathcal{D}$, $fg = \chi_s$, $c \in T$. Let $f \geqq 0$ on S and let f and g be bounded below. Then the lower density of T at c is positive.

Proof. Let $M \in R$ and let $f \wedge g \geqq -M$. Let $x \in (c, \infty)$, $S_x = S \cap (c, x)$, $T_x = T \cap (c, x)$. We have $|S_x|^2 = (\int_{S_x} \sqrt{fg})^2 \leqq \int_{S_x} f \cdot \int_{S_x} g = (\int_c^x f - \int_{T_x} f) \cdot (\int_c^x g - \int_{T_x} g) \leqq (\int_c^x f + M|T_x|) \cdot (\int_c^x g + M|T_x|)$. Let δ be the right lower density of T at c. Choose $x_1, x_2, \ldots \in (c, \infty)$ such that $x_n \to c$ and $|T_{x_n}|/(x_n - c) \to \delta$. Then $(1 - \delta)^2 \leqq (f(c) + M\delta)(g(c) + M\delta)$ and $f(c)g(c) = 0$ so that $\delta > 0$. It can be proved similarly that the left lower density of T at c is positive.

Remark 1. It follows from 23 and from the example in 4 that the following two assertions (where S is ambiguous and T nonporous) are false:

A1. There are $f, g \in \mathcal{D}$ such that $f \geqq 0$, $\inf g > -\infty$ and $fg = \chi_s$.

A2. There are $f, g \in \mathcal{D}$ such that $f \wedge g > -1$ and $fg = \chi_s$.

Remark 2. Theorem 18 was stated without proof in [2].

References

[1] A.M. Bruckner, J. Mařík and C.E. Weil, Baire one, null functions, Contemporary Mathematics, Vol. 42, 1985, pp. 29-41.

[2] J. Mařík, Characteristic functions that are products of derivatives, Real Analysis Exchange, Vol. 12, No. 1, 1986-87, pp. 67-68.

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