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PERIODIC SOLUTIONS TO ABSTRACT DIFFERENTIAL EQUATIONS

by OTTO VEJVODA, IVAN STRAŠKRABA

The search for periodic solutions to different types of partial differential equations has shown the usefulness of the study of periodic solutions to abstract differential equations of the form

$$P\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)u(t) + \mathscr{A}u(t) = g(t) + \mathscr{F}(t, u(t)), \qquad t \geqq 0, \tag{1}$$

where $P(Z)$ is a polynomial and $\mathscr{A}$ is an operator in Banach or Hilbert space. In the sequel we suppose that $\mathscr{A}$ is a generally unbounded, closed and densely defined linear operator and g and $\mathscr{F}$ are ω-periodic in t and sufficiently smooth.

Periodic solutions to the equation (1) of the first order, i.e.

$$u_t(t) + \mathscr{A}u(t) = g(t) + \mathscr{F}(t, u(t)) \tag{2}$$

have been studied by several authors.

C. T. Taam in [1], [2] assuming $-\mathscr{A}$ to generate a strongly continuous exponentially decreasing semigroup $\exp(t\mathscr{A})$, $(t \geqq 0)$ with g and $\mathscr{F}$ sufficiently small, proves that the integral operator

$$\int_{-\infty}^{0} \exp(s\mathscr{A})\,(g(t+s) + \mathscr{F}(t+s, u(t+s))\,\mathrm{d}s$$

is contractive and its fixed point represents an ω-periodic solution to (1).

F. E. Browder in [3] shows the existence of a mild ω-periodic solution to (2) (with $\mathscr{A} = \mathscr{A}(t)$) provided that $\mathscr{A}(t)$ is a family of positive operators in a Hilbert space $\mathbf{H}$, that the equation

$$u(t) = \mathscr{U}(t, 0)\,\varphi + \int_0^t \mathscr{U}(t, s)\,(g(s) + \mathscr{F}(s, u(s))\,\mathrm{d}s$$

(where $\mathscr{U}(t, s)$ is the evolution operator) has a solution and that $\mathscr{F}$ is a monotone operator in u.

In the paper [4] of L. Zend the existence of an ω-periodic solution to (2) is proved under the assumption that $\mathscr{A}$ is a strongly positive operator having a completely continuous inverse and that $\mathscr{F}$ fulfils the coercieveness condition

$$\frac{\|\mathscr{F}(t, u)\|}{\|\mathscr{A}^{\varepsilon}u\|} \to 0 \quad \text{uniformly in} \quad t \geqq 0 \quad \text{as} \quad \|\mathscr{A}^{\varepsilon}u\| \to \infty \quad \text{with some} \quad 0 \leqq \varepsilon < 1.$$

(The assumptions for the existence of a solution to (2) with the boundary-value condition

$$\mu u(0) - u(\omega) = h$$

as given in the paper [5] of A. A. DEZIN, are too complicated to be described here.)

A very recent result concerning (2) where $\mathscr{A}$ is a positive operator generating an analytic semigroup and having a compact inverse and with $\mathscr{F}$ such that the limits

$$\lim_{n\to\infty} \frac{1}{n} \int_0^n \mathscr{F}(t, u)\,\mathrm{d}t, \qquad \lim_{n\to\infty} \frac{1}{n} \int_0^n \mathscr{F}'_u(t, u)\,\mathrm{d}t$$

exist in an appropriate topology on corresponding sets of u, is due to I. B. SIMONENKO [6] who applies a modification of the averaging principle to this problem.

We have investigated the equation (2) in a Hilbert space $\mathbf{H}$ provided that $\mathscr{A}$ is a selfadjoint and bounded from below (i.e. $(\mathscr{A}u, u) \geqq m(u, u)$, $u \in \mathbf{D}(\mathscr{A})$), zero is an isolated point of the spectrum $\mathfrak{S}(\mathscr{A})$ of $\mathscr{A}$ and $\mathscr{F}$ is sufficiently small. Writing the solution of the equation (2) with the initial condition $u(0) = \varphi$ in the form

$$u(\varphi)(t) = \exp(-t\mathscr{A})\,\varphi + \int_0^t \exp[-(t-\tau)\,\mathscr{A}]\,(g(\tau) + \mathscr{F}(\tau, u(\varphi)(\tau)))\,\mathrm{d}\tau$$

and making use of the Poincaré method, i.e. investigating under which conditions there exists an element $\varphi \in \mathbf{D}(\mathscr{A})$ such that

$$u(\varphi)(\omega) - u(\varphi)(0) = 0,$$

we get the following result:

A necessary and sufficient condition for the existence of an ω-periodic solution to (2) *is that the equation*

$$\int_0^\omega (g(\tau) + \mathscr{F}(\tau, u(\varphi)(\tau)), \xi)\,\mathrm{d}\tau = 0, \quad \xi \in \mathbf{N}(\mathscr{A}), \tag{*}$$

where $(.,.)$ *is the scalar product in* $\mathbf{H}$ *and* $\mathbf{N}(\mathscr{A}) = \{\mathscr{A}u = 0\}$, *has a solution* $\varphi \in \mathbf{D}(\mathscr{A})$. (In practice, we must usually replace (*) by some sufficient conditions, cf. [7]). Let us note that for the Schrödinger equation

$$u_t(t) + i\mathscr{A}u(t) = g(t) + \mathscr{F}(t, u(t))$$

an assertion similar to that obtained for the equation (7) may be stated.

To a study of an abstract equation of the second order is devoted the paper [8] by K. MASUDA. It deals with the equation

$$u_{tt}(t) + 2\alpha u_t(t) + \mathscr{A}u(t) = g(t) + \mathscr{F}(t, u(t)), \tag{3}$$

where $\alpha > 0$, $\mathscr{A}$ is a strongly positive and selfadjoint operator in a Hilbert space $\mathbf{H}$, g is sufficiently small and $\mathscr{F}$ behaves roughly speaking like a polynomial of the order $r > 1$ in u. Replacing the equation (3) with the initial conditions

$$u(0) = \varphi \qquad u_t(0) = \psi \tag{4}$$

by the equivalent integral equation

$$u(t) = w(\varphi, \psi; u)(t) \equiv \mathrm{e}^{-\alpha t} \cos t \sqrt{\mathscr{A} - \alpha^2}\varphi + \\ + \mathrm{e}^{-\alpha t} \frac{\sin t \sqrt{\mathscr{A} - \alpha^2}}{\sqrt{\mathscr{A} - \alpha^2}} (\psi - \alpha\varphi) + \int_0^t \mathrm{e}^{-(t-\tau)\alpha} \frac{\sin (t - \tau) \sqrt{\mathscr{A} - \alpha^2}}{\sqrt{\mathscr{A} - \alpha^2}} \times \\ \times \left(g(\tau) + \mathscr{F}(\tau, u(\tau))\right) \mathrm{d}\tau, \tag{5}$$

and investigating the operator $w(\varphi, \psi; u)(t + n\omega)$ for $n \to \infty$ (analogously as B. A. Fleishman and F. A. Ficken did in [9]) Masuda proves the existence of an asymptotically stable ω-periodic solution of (3).

We have studied the equation (3) under somewhat more general assumptions, namely that $\mathscr{A}$ is only bounded from below instead of beeing strongly positive and $\mathscr{F}$ may be any sufficiently small nonlinear operator. Provided that 0 is an isolated point of the spectrum of $\mathscr{A}$ we have found a necessary and sufficient condition for the solution of (3), (4) to be ω-periodic (i.e. for the existence of an ω-periodic solution of (2)) in the form of the following bifurcation equations

$$\int_0^\omega (g(\tau) + \mathscr{F}(\tau, w(\varphi, \psi; u)(\tau)), \xi)\, \mathrm{d}\tau = 0, \quad \xi \in \mathbf{N}(\mathscr{A}) \tag{6}$$

for the initial conditions φ, ψ. In addition we have obtained similar results (where in (6) $\xi \in \mathbf{N}(\mathscr{A} + \gamma)$) for the equation

$$u_{tt}(t) + 2(\alpha + \beta\mathscr{A})\, u_t(t) + (\mathscr{A} + \gamma)\, u(t) = g(t) + \mathscr{F}(t, u(t))$$

with $\alpha \geqq 0$, $\beta \geqq 0$, $\alpha + \beta > 0$, γ real, $\mathscr{A}$ strongly positive.

Finally examining the equation

$$u_{tt}(t) + \mathscr{A}u(t) = g(t) + \mathscr{F}(t, u(t)) \tag{7}$$

under the assumption that there exist numbers $c > 0$ and $\varrho \geqq 0$ such that

$$\inf_{k \in M} \left| \frac{\omega}{2} \sqrt{\lambda} - k\pi \right| \geqq \frac{c}{\lambda^\varrho}, \qquad \lambda \in \mathfrak{S}(\mathscr{A}) - \left\{ \frac{4l^2\pi^2}{\omega^2} \right\}_{l \in M},$$

(where $M = \{0, \pm 1, \pm 2, \ldots\}$)
and that the operator $\mathscr{F}(t, u)$ maps $\mathbf{D}(\mathscr{A}^\varepsilon)$ into $\mathbf{D}(\mathscr{A}^\delta)$ with prescribed $\varepsilon > 0$, $\delta > 0$ depending on ϱ and on the required smoothness of the solution, we derived the

bifu-rcation equations representing necessary and sufficient conditions for the existence of an ω-periodic solution in the form

$$\int_0^\omega (\varphi, \cos \tau \sqrt{\mathscr{A}}(g(\tau) + \mathscr{F}(\tau, u(\tau)))\,d\tau = 0, \qquad \varphi \in \mathbf{N}\left(\sin \frac{\omega}{2} \sqrt{\mathscr{A}}\right),$$

$$\int_0^\omega \left(\psi, \frac{\sin \tau \sqrt{\mathscr{A}}}{\sqrt{\mathscr{A}}}(g(\tau) + \mathscr{F}(\tau, u(\tau))\right) d\tau = 0, \qquad \psi \in \mathbf{N}\left(\frac{\sin \frac{\omega}{2} \sqrt{\mathscr{A}}}{\sqrt{\mathscr{A}}}\right).$$

As for the equations of an arbitrary order inspired by the method applied in the paper [10] by Ju. A. Dubinskij dealing with periodic solutions to the elliptic-parabolic problem

$$P\left(\frac{d}{dt}\right) u(x, t) + \sum_{|\alpha| \leqq 2m} A_\alpha(x)\, D^\alpha u(x, t) = f(x, t), \qquad (x, t) \in G \times S^1 = Q,$$

$$B_j\left(x, D, \frac{d}{dt}\right) u \Big|_{\partial Q} = g_j, \qquad j = 1, \ldots, m,$$

where $G \subset R^n$, S^1 is a circle, $P(z) = \sum_{g=0}^{s} a_q z^q$.

M. Sova has derived similar results for an abstract equation

$$\Theta u \equiv P\left(\frac{d}{dt}\right) u(t) + \mathscr{A} u(t) = f(t)$$

in a Hilbert space $\mathbf{H}$. He proves Θ to have a closed range in $\mathscr{L}_2(\langle 0, 2\pi \rangle; \mathbf{H})$ and to be smooth of order $r \geqq 0$ iff there exists a constant $\gamma > 0$ such that for any integer k and for any $v \in \mathbf{D}(\mathscr{A}) \cap \mathbf{N}(P(ik) + \mathscr{A})^\perp$

$$\| P(ik)\, v + \mathscr{A} v \| \geqq \gamma(1 + |k|^r) \| v \|;$$

from this one easily gets the conditions on $f(t)$ that there exists a periodic solution in the linear case and bifurcation equations for $\mathscr{F}(t, u(t))$ in the nonlinear case. (These results were presented at a Soviet-Czechoslovak conference held in Novosibirsk in 1971 and have not yet been published.) A further generalization may be found in the paper of M. Sova published in the present volume.

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