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BOUNDARY INTEGRAL EQUATIONS OF ELASTICITY IN DOMAINS WITH PIECEWISE SMOOTH BOUNDARIES

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0. Introduction

In the author's papers [1-3] a method for investigation of boundary integral equations arising in problems of mechanics of continuum in domains with piecewise smooth boundaries was proposed. Traditionally, equations of the potential theory are studied directly by methods of the Fredholm and singular integral operator theories. In the case of a non-smooth boundary this way leads to difficulties that have not been overcome until now. In a sense our approach is opposite to the traditional one. It is based on the well-known fact that solutions of integral equations can be expressed in terms of solutions of some exterior and interior boundary value problems. These are studied with the help of the theory developed in [4 - 8] and, as a result, theorems on solvability of equations of the potential theory are obtained. For these equations we can get differentiability properties and asymptotics of solutions near singularities of the boundary using the same approach [3, 9 - 11]. In [3] our method of construction of the potential theory was illustrated by the example of three boundary value problems of linear isotropic elasticity, namely, the first, the second and mixed, as well as of the same problems for the Laplace operator under the hypothesis that there exist a finite number of non-intersecting smooth edges on the boundary.

In the present lecture we study the first two boundary value problems for the Lamè system in domains with boundary singularities of the type of edges, conic points and polyhedral angles. New results on the harmonic potential theory are also reported.

1. Domains and function spaces

Let $G^{(i)}$ be a domain in R^3 with compact closure and with the boundary S, $Q^{(e)} = \mathbb{R}^3 \setminus \overline{G^{(i)}}$. We suppose that S is the union of a finite number of "faces" $\{F\}$, "edges" $\{E\}$ and "vertices" $\{Q\}$ and that openings of all angles are non zero. Confining ourselves only to the above visual description we refer the reader to [8] for the exact definition of the class of domains to be considered. In any case it contains domains with polyhedral boundaries. We place the origin into $G^{(i)}$.

Let $\{\mathcal{U}\}$ be a sufficiently small finite covering of $\overline{G^{(i)}}$ by open sets satisfying: a) $\mathcal{U}$ contains not more than a single vertex Q, b) if $\overline{\mathcal{U}}$ does not contain vertices then $\overline{\mathcal{U}}$ intersects not more than a single edge E. With any vertex Q and any edge E we associate real numbers β_Q and γ_E, respectively.

By means of the partition of unity subordinate to the covering $\{\mathcal{U}\}$ we define *the space* $C^{1,\alpha}_{\beta,\gamma}(G^{(i)})$ with $0 < \alpha < 1$, $\beta = \{\beta_Q\}$, $\gamma = \{\gamma_E\}$. If $\overline{\mathcal{U}}$ intersects no singularities of S then the $C^{1,\alpha}_{\beta,\gamma}(G^{(i)})$-norm of a function with support in $\mathcal{U}$ is equivalent to the norm in the usual Hölder space $C^{1,\alpha}$. In the case $\mathcal{U} \cap E \neq \emptyset$, $\overline{\mathcal{U}} \cap \{Q\} = \emptyset$ and $\operatorname{supp} u \subset \mathcal{U}$ we have

$$||u||_{C^{1,\alpha}_{\beta,\gamma}(G^{(i)})} \sim \sup_{x \in G^{(i)}} |u(x)| + \sup_{x,y \in G^{(i)}} \frac{|r_E(x)^{\gamma_E} \nabla u(x) - r_E(y)^{\gamma_E} \nabla u(y)|}{|x-y|^{\alpha}} ,$$

where $r_E(x)$ is the distance from the point x to E. If $\mathcal{U}$ contains the vertex Q and $\operatorname{supp} u \subset \mathcal{U}$ then

$$||u||_{C^{1,\alpha}_{\beta,\gamma}(G^{(i)})} \sim \sup_{x \in G^{(i)}} |u(x)| + \sup_{x,y \in G^{(i)}} \frac{|\rho_Q(x)^{\beta_Q} \prod_{\{E: Q \in \overline{E}\}} r_E(x)^{\gamma_E} \nabla u(x) - \rho_Q(y)^{\beta_Q} \prod_{\{E: Q \in \overline{E}\}} r_E(x)^{\gamma_E} \nabla u(y)|}{|x-y|^{\alpha}} ,$$

where $\rho_Q(x) = |x - Q|$.

Replacing here $G^{(i)}$ by $G^{(e)}$ and $\sup_{x \in G^{(i)}} |u(x)|$ by $\sup_{x \in G^{(e)}} (1 + |x|)|u(x)|$ we obtain the definition of *the space* $C^{1,\alpha}_{\beta,\gamma}(G^{(e)})$. By $C^{1,\alpha}_{\beta,\gamma}(S)$ we denote the space of traces on $\cup F$ of functions in $C^{1,\alpha}_{\beta,\gamma}(G^{(i)})$ or $C^{1,\alpha}_{\beta,\gamma}(G^{(e)})$.

Let us introduce another *space* $C^{0,\alpha}_{\beta,\gamma}(S)$ of functions on $\cup F$. If $\mathcal{U} \cap E \neq \emptyset$, $\overline{\mathcal{U}} \cap \{Q\} = \emptyset$ and $\operatorname{supp} u \subset \mathcal{U}$ then

$$||u||_{C^{0,\alpha}_{\beta,\gamma}(S)} \sim \sup_{x \in \cup F} r_E(x)^{\gamma_E - \alpha} |u(x)| +$$

$$+ \sup_{x,y \in \cup F} \frac{|r_E(x)^{\gamma_E} u(x) - r_E(y)^{\gamma_E} u(y)|}{|x-y|^\alpha} .$$

If $\mathcal{U}$ contains the vertex Q and $\operatorname{supp} u \subset \mathcal{U}$ then

$$||u||_{C^{0,\alpha}_{\beta,\gamma}(S)} \sim \sup_{x \in \cup F} \rho_Q(x)^{\beta_Q} \prod_{\{E:Q \in \bar{E}\}} r_E(x)^{\gamma_E} \sum_{\{E:Q \in \bar{E}\}} \frac{1}{r_E^\alpha} |u(x)| +$$

$$\sup_{x,y \in \cup F} \frac{|\rho_Q(x)^{\beta_Q} \prod_{\{E:Q \in \bar{E}\}} r_E(x)^{\gamma_E} u(x) - \rho_Q(y)^{\beta_Q} \prod_{\{E:Q \in \bar{E}\}} r_E(y)^{\gamma_E} u(y)|}{|x-y|^\alpha} .$$

If $\bar{\mathcal{U}}$ intersects no singularities of the boundary and $\operatorname{supp} u \subset \mathcal{U}$ then the norm in $C^{0,\alpha}_{\beta,\gamma}(S)$ is equivalent to the norm in $C^{0,\alpha}(S)$.

2. Boundary value problems of elasticity

We consider *interior* and *exterior Dirichlet problems:*

$$\mu\Delta u + (\lambda + \mu) \nabla \operatorname{div} u = 0 \quad \text{in } G^{(i)} , \quad u = g \quad \text{on } \cup F ; \qquad (D^{(i)})$$

$$\left.\begin{array}{l} \mu\Delta u + (\lambda + \mu) \nabla \operatorname{div} u = 0 \quad \text{in } G^{(e)} , \quad u = g \quad \text{on } \cup F , \\ u(x) = O\left((1 + |x|)^{-1}\right) \end{array}\right\} (D^{(e)})$$

and *interior* and *exterior Neumann problems:*

$$\mu\Delta u + (\lambda + \mu) \nabla \operatorname{div} u = 0 \quad \text{in } G^{(i)} , \quad Tu = h \quad \text{on } \cup F , \qquad (N^{(i)})$$

$$\left.\begin{array}{l} \mu\Delta u + (\lambda + \mu) \nabla \operatorname{div} u = 0 \quad \text{in } G^{(e)} , \quad Tu = h \quad \text{on } \cup F , \\ u(x) = O\left((1 + |x|)^{-1}\right) , \end{array}\right\} (N^{(e)})$$

where $T = T(\partial_x, n)$ is the matrix with elements $T_{kj}(\partial_x, n) = \mu\delta_k^j \, \partial/\partial n + \lambda n_k \, \partial/\partial x_j + \mu n_j \, \partial/\partial x_k$ and $n = (n_1, n_2, n_3)$ is an outer normal to $G^{(i)}$.

We introduce a collection of reals $\{\delta_E\}$ which appears in the statement of the next lemma and will be used in the sequel. Let $\phi(z) \in (0, 2\pi)$ be the angle between the tangent half-planes to S at the point $z \in E$ from the side of $G^{(i)}$. We put $\omega_E = \inf_{z \in E} (\pi + |\pi - \phi(z)|)$. Let δ_E be the root of the equation $\sin(\omega_E \delta) + \delta \sin \omega_E = 0$ with the least positive real part; δ_E is real and decreases as ω_E increases, $1/2 < \delta_E < 1$.

We formulate a theorem on solvability of all above mentioned boundary value problems, which is sufficient to justify the boundary integral equations method.

THEOREM 1. *Let* $\{\gamma_E\}$ *and* $\{\beta_Q\}$ *satisfy*

$$0 < 1 - \gamma_E + \alpha < \delta_E \quad \textit{for all} \quad E \,, \tag{1}$$

$$|\beta_Q + \sum_{\{E:Q \in \bar{E}\}} \gamma_E - \alpha - 3/2| < \varepsilon_Q \quad \textit{for all} \quad Q \,, \tag{2}$$

where $\{\varepsilon_Q\}$ *is a collection of positive numbers in* $(0,1)$ *which depend on the tangent cone to* S *with the vertex* Q *)*.

Then (i) $(D^{(i)})$ *and* $(D^{(e)})$ *are uniquely solvable in* $C^{1,\alpha}_{\beta,\gamma}(G^{(i)})$ *and* $C^{1,\alpha}_{\beta,\gamma}(G^{(e)})$ *for all* $g \in C^{1,\alpha}_{\beta,\gamma}(S)$; (ii) $(N^{(e)})$ *is uniquely solvable in* $C^{1,\alpha}_{\beta,\gamma}(G^{(e)})$ *for all* $h \in C^{0,\alpha}_{\beta,\gamma}(S)$; (iii) $(N^{(i)})$ *is solvable in* $C^{1,\alpha}_{\beta,\gamma}(G^{(i)})$ *for all* $h \in C^{0,\alpha}_{\beta,\gamma}(S)$ *with zero principal vector and principal moment. Its solution is unique up to a rigid displacement.*

For $(D^{(i)})$ and $(D^{(e)})$ this result is contained in Theorem 11.5 [8]. The proof for $(N^{(i)})$ and $(N^{(e)})$ requires minor technical modifications.

3. Solution of $(D^{(i)})$ and $(D^{(e)})$ by a simple layer potential

Let $V\sigma$ be *the elastic simple layer potential* with the density σ .

THEOREM 2. *Let* $\{\gamma_E\}$ *and* $\{\beta_Q\}$ *satisfy* (1) *and* (2). *Then the operators* $C^{0,\alpha}_{\beta,\gamma}(S) \ni \sigma \to (V\sigma)|_{G^{(e)}} \in C^{1,\alpha}_{\beta,\gamma}(G^{(e)})$ *and* $C^{0,\alpha}_{\beta,\gamma}(S) \ni \sigma \to (V\sigma)|_{G^{(i)}} \in C^{1,\alpha}_{\beta,\gamma}(G^{(i)})$ *are bounded. There exists a bounded inverse* V^{-1} : $C^{1,\alpha}_{\beta,\gamma}(S) \to C^{0,\alpha}_{\beta,\gamma}(S)$.

The first part of the theorem can be checked directly. The second part follows from Theorem 2 and the identity $2\sigma = Tu^{(i)} - Tu^{(e)}$ where $u^{(i)}$ and $u^{(e)}$ are restrictions of $V\sigma$ to $G^{(i)}$ and $G^{(e)}$, respectively.

4. Solution of $(D^{(i)})$, $(D^{(e)})$, $(N^{(i)})$, $(N^{(e)})$ by a double layer potential

Let $W\tau$ be *the double layer potential* with the density τ , i.e.

$$(W\tau)(x) = \frac{1}{2\pi} \int_S \left(T(\partial_y, n_y)\Gamma(x,y)\right)^* \tau(y)\, ds_y \,,$$

*) The numbers ε_Q can be defined by some spectral boundary value problems in spherical domains (cf. [8]) but we shall not use it in what follows. For the problems $(D^{(i)})$ and $(D^{(e)})$ it was shown in [12] that $\varepsilon_Q > 1/2$. The validity of the last inequality for $(N^{(i)})$, $(N^{(e)})$ remains an open question.

where $*$ denotes the adjoint operator and Γ is the Kelvin-Somigliana tensor.

If $u^{(i)} = W\tau$ then τ satisfies the system of singular integral equations

$$(-1 + W)\tau = g \quad \text{on} \quad \cup F . \tag{3}$$

The solution of $(D^{(e)})$ can be expressed as the sum $(W\tau)(x) + \Gamma(x,0)a + \operatorname{rot}\Gamma(x,0)b$ where a , b are unknown constant vectors. Then the triplet (τ,a,b) satisfies the system

$$(1 + W)\tau + \Gamma(\cdot,0)a + \operatorname{rot}\Gamma(\cdot,0)b = g \tag{4}$$

Representing the solutions of $(N^{(i)})$ and $(N^{(e)})$ as $W\tau$ we arrive at the systems

$$(1 + W^*)\tau = h , \tag{5}$$

$$(-1 + W^*)\tau = h . \tag{6}$$

<u>THEOREM 3</u>. *Let* $\{\gamma_E\}$ *and* $\{\beta_Q\}$ *satisfy* (1) *and* (2). *Then* (i) *the operators* W *and* W^* *are bounded in* $C^{1,\alpha}_{\beta,\gamma}(S)$ *and* $C^{0,\alpha}_{\beta}(S)$, (ii) *if* $g \in C^{1,\alpha}_{\beta,\gamma}(S)$ *then systems* (3) *and* (4) *are uniquely solvable in* $C^{1,\alpha}_{\beta,\gamma}(S)$ *and* $C^{1,\alpha}_{\beta}(S) \times R^3 \times R^3$, *respectively.*

Let us describe a scheme of the proof of the solvability confining ourselves to system (3). Let $u^{(i)}$ and $u^{(e)}$ be the solutions of $(D^{(i)})$ and $(D^{(e)})$. Then $2g = V(Tu^{(i)} - Tu^{(e)})$. We introduce the solution v of the problem $(N^{(e)})$ where $h = 1/2\ (Tu^{(i)} - Tu^{(e)})$. Since $v = 1/2(Wv - VTv)$ on $G^{(e)}$, then $(-1 + W)v = VTv = g$ on $\cup F$. Consequently, the vector-function $\tau = v|_{\cup F}$ is a solution of (3). The inclusion $\tau \in C^{1,\alpha}_{\beta,\gamma}(S)$ follows directly from Theorem 1.

To prove the unicity of the solution of (3) it suffices to establish the solvability in $C^{0,\alpha}_{\beta,\gamma}(S)$ of the formally adjoint system (6), where $h \in C^{0,\alpha}_{\beta,\gamma}(S)$. Let $v^{(e)}$ be a solution of $(N^{(e)})$. We consider the simple layer potential $V\sigma$ which coincides with $v^{(e)}$ on $G^{(e)}$ (see Theorem 2). It remains to note that the density σ satisfies (6).

The above argument contains all essential points for the proof of the following theorem on solvability of systems (5) and (6).

<u>THEOREM 4</u>. *There exists a bounded inverse:* $(-1 + W^*)^{-1}$ *in* $C^{0,\alpha}_{\beta,\gamma}(S)$. *System* (4) *is solvable in* $C^{0,\alpha}_{\beta,\gamma}(S)$ *for all* h *with zero principal vector and principal moment.*

For solution of systems of integral equations under consideration theorems on increasing smoothness and changing collections $\{\gamma_E\}$ and

$\{\beta_Q\}$ hold which are similar to the theorems of the same kind for solutions of boundary value problems (cf. [8]). Before giving an example let us introduce some function spaces.

Let ℓ be an integer, $\ell \geq 1$, $0 < \alpha < 1$. If in the definition of $C^{1,\alpha}_{\beta,\gamma}(G^{(i)})$ we replace ∇ by $\nabla_\ell = \{\partial^\ell/\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}\}$ then we obtain the definition of *the space* $C^{\ell,\alpha}_{\beta,\gamma}(G^{(i)})$. By $C^{\ell,\alpha}_{\beta,\gamma}(S)$ we mean the space of traces on S of functions in $C^{\ell,\alpha}_{\beta,\gamma}(G^{(i)})$. Here we can replace (i) by (e).

The following assertion which completes Theorem 3 holds.

THEOREM 5. *Let* $\{\gamma_E\}$, $\{\beta_Q\}$, $\{\gamma^*_E - \ell + 1\}$ *and* $\{\beta^*_Q - \ell + 1\}$ *satisfy* (1) *and* (2). *If* $g \in C^{1,\alpha}_{\beta,\gamma}(S) \cap C^{\ell,\alpha}_{\beta^*,\gamma^*}(S)$ *and* $\tau \in C^{1,\alpha}_{\beta,\gamma}(S)$ *is a solution of any system* (3) *and* (4) *then* $\tau \in C^{\ell,\alpha}_{\beta^*,\gamma^*}(S)$.

Similar complements can be made to Theorems 2 and 4.

The potential theory for plane boundary value problems can be developed with the help of the same arguments and even more easily. If by S we mean a piecewise smooth contour without cusps and by $\{Q\}$ we denote the set of its angular points then the statements of Theorems 1 - 5 remain valid up to obvious changes.

For harmonic and hydrodinamic potentials results similar to Theorems 1 - 5 can be obtained by the same method.

5. The Fredholm radius of harmonic potentials

Here we present a formula for the Fredholm radii of the double layer harmonic potential W and its formal adjoint W^* in certain Hölder spaces. We shall suppose that S contains no vertices. So the collection $\{\beta_Q\}$ in the notation of the function spaces is omitted.

Let L be a linear operator in a Banach space. The Fredholm radius $r(L)$ of L is the largest radius of a circle $C_r = \{\lambda \in \mathbb{C} : |\lambda| < r\}$ such that for all $\lambda \in C_r$ the operator $1 - \lambda L$ is Fredholm and $\mathrm{Ind}\,(1 - \lambda L) = 0$.

We introduce the operators

$$(W\tau)(x) = \frac{1}{2\pi}\int_S \frac{\cos(x - y,\ n(y))}{|x - y|^2}\,\tau(y)\,ds(y)\ , \tag{7}$$

$$(W^*\tau)(x) = -\frac{1}{2\pi}\int_S \frac{\cos(x - y,\ n(x))}{|x - y|^2}\,\tau(y)\,ds(y)\ , \tag{8}$$

where $x \in \cup F$.

The following result is proved in [9]:

THEOREM 6. *Let* $\alpha \in (0,1)$, $0 < 1 + \alpha - \beta < 1$, $\ell \geqq 0$ *and*

$$R = \min_{z \in \cup E} \left| \frac{\sin \pi(1 + \alpha - \beta)}{\sin\left((\pi - \phi(z))(1 + \alpha - \beta)\right)} \right| .$$

Let W *and* W^* *be operators in* $C^{\ell,\alpha}_{\beta+\ell}(S)$ *and* $C^{\ell+1,\alpha}_{\beta+\ell}(S)$, *respectively, defined by* (7) *and* (8). *Then* $r(W) = r(W^*) = R$.

We note that $R > 1$ if and only if

$$1 + \alpha - \beta < \frac{\pi}{\pi + |\pi - \phi(z)|} \quad \text{for all} \quad z \in \cup E .$$

The proof of Theorem 6 relies heavily on Theorem 7 where the following notation is used.

Let $G = G^{(i)} \cup G^{(e)}$ and let $u^{(i)}$ and $u^{(e)}$ be restrictions of the function u to $G^{(i)}$ and $G^{(e)}$. By $C^{\ell,\alpha}_{\beta}(G)$ we denote the space of functions u with $u^{(i)} \in C^{\ell,\alpha}_{\beta}(G^{(i)})$, $u^{(e)} \in C^{\ell,\alpha}_{\beta}(G^{(e)})$. The norm in $C^{\ell,\alpha}_{\beta}(G)$ is defined as the sum of the norms of $u^{(i)}$ and $u^{(e)}$ in $C^{\ell,\alpha}_{\beta}(G^{(i)})$ and $C^{\ell,\alpha}_{\beta}(G^{(e)})$. For $\ell \geqq 1$, $0 < \beta < \ell + \alpha$ we put

$$\hat{C}^{\ell,\alpha}_{\beta}(G) = \{u \in C^{\ell,\alpha}_{\beta}(G) : \Delta u \ \text{ in } \ G , \ u^{(i)} = u^{(e)} \ \text{ on } \ S\}$$

$$\check{C}^{\ell,\alpha}_{\beta}(G) = \{u \in C^{\ell,\alpha}_{\beta}(G) : \Delta u = 0 \ \text{ in } \ G , \ \frac{\partial u^{(i)}}{\partial n} = \frac{\partial u^{(e)}}{\partial n} \ \text{ on } \ \cup F\} .$$

Clearly, for $\ell \geqq 0$ the operators

$$L_1 : \hat{C}^{\ell+1,\alpha}_{\beta}(G) \ni u \to (1 - \lambda)\frac{\partial u^{(i)}}{\partial n} - (1 + \lambda)\frac{\partial u^{(e)}}{\partial n} \in C^{\ell,\alpha}_{\beta}(S) ,$$

$$L_2 : \check{C}^{\ell+1,\alpha}_{\beta}(G) \ni u \to (1 - \lambda)u^{(i)} - (1 + \lambda)u^{(e)} \in C^{\ell+1,\alpha}_{\beta}(S)$$

are bounded.

THEOREM 7. *Let* $\alpha \in (0,1)$, $0 < 1 + \alpha - \beta < 1$, $\ell \geqq 0$.

1) *If* $|\lambda| < R$ *then* L_k , $k = 1,2$, *is Fredholm and* ind $(L_k) = 0$.

2) *If* $|\lambda| = R$ *then the range of* L_k *is not closed.*

References

[1] MAZ'YA, V. G.: *Integral equations of the potential theory in domains with piecewise smooth boundaries.* Uspehi Matem.Nauk, v.36, n 4, 1981 p. 229-230.

[2] MAZ'YA, V. G.: *On solvability of integral equations in the classical*

theory of elasticity in domains with piecewise smooth boundaries. Republican School-Conf. General Mechanics and Elasticity Theory (Telavi, 1981), Abstracts of Reports, Tbilisi, 1981, pp. 55-56.

[3] MAZ'YA, V. G.: *The potential theory for the Lamè system in domains with piecewise smooth boundaries.* Proc. of the Conference on the Partial Diff. Equations in Memoriam of I. N. Vekua, Tbilisi 1982 (to appear).

[4] MAZ'YA, V. G. and PLAMENEVSKIĬ, B. A.: L_p*-estimates of solutions of elliptic boundary value problems in domains with edges.* Trudy Moskov. Mat. Obshch. 37 (1978), 49-93; English transl. in Trans. Moscow Math. Soc. 1980, no. 1 (37).

[5] ———— : *Estimates of the Green functions and Schauder estimates of the solutions of elliptic boundary value problems in a dihedral angle.* Sibirsk. Mat. Zh. 19 (1978), 49-93; English transl. in Siberian Math. J. 19 (1978).

[6] ———— : *Schauder estimates of solutions of elliptic boundary value problems in domains with edges on the boundary.* Partial Differential Equations (Proc. Sem. S. L. Sobolev, 1978, no. 2), Inst. Mat. Sibirsk. Otdel. Akad. Nauk SSSR, Novosibirsk, 1978, pp. 69-102; English transl. in Amer. Math. Soc. Transl. (2) 123 (1984).

[7] ———— : *Elliptic boundary value problems on manifolds with singularities.* Problemy Mat. Anal., vyp. 6, Izdat. Leningrad. Univ., Leningrad, 1977, pp. 85-142; English transl., to appear in J. Soviet Math.

[8] ———— : *The first boundary value problem for the classical equations of mathematical physics in domains with piecewise smooth boundaries* (I) Zeitschrift für Analysis und ihre Anwendungen, Bd. 2 (4) 1983, S. 335-359; (II) ibid. Bd. 2 (6) 1983, S.523-551.

[9] GRAČEV, N. V. and MAZ'YA, V. G.: *On the Fredholm radius for operators of the double layer potential type on piecewise smooth boundaries.* Vestn. Leningrad Univ. Math. (to appear).

[10] ZARGARYAN, S. S. and MAZ'YA, V. G.: *On singularities of solutions of a system of integral equations in potential theory for the Zaremba problem.* Vestn. Leningrad. Univ. Math. 1983, n 1. English transl.vol. 16 (1984), p. 49-55.

[11] ZARGARYAN, S. S. and MAZ'YA, V. G.: *On the asymptotic behavior of solutions of integral equations in potential theory in a neighborhood of corner points of the contour.* Prikl. Mat. Mekh. vol. 48, n 1, 1984; English transl. in J. Appl. Math. Mech. 48 (1984).

[12] MAZ'YA, V. G. and PLAMENEVSKIĬ, B. A.: *On properties of solutions of three-dimensional problems of elasticity theory and hydrodynamics in domains with isolated singular points.* Dinamika Sploshnoj Sredy ,Novosibirsk, n 50 (1981), p. 99-120 English transl. in Amer. Math. Soc. Transl. (2), vol. 123, 1984, p. 109-123.