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HIGHER REGULARITY OF WEAK SOLUTIONS OF STRONGLY NONLINEAR ELLIPTIC EQUATIONS

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In a bounded open set $G \subset R^N$ we consider the Dirichlet problem

$$Lu: = -\Delta u - \sum_{i=1}^{N} \partial_i g_i(\partial_i u) + g_0(u) = f \ , \qquad u\Big|_{\partial G} = 0 \ . \tag{1}$$

If f,g are measurable functions on G, we write $(f,g): = \int_G f\cdot g$ if $f\cdot g \in L^1(G)$. As usual, a weak solution of (1) is a function

(H.1)
$$\begin{cases} u \in H_0^{1,2}(G) \text{ such that } g_i(\partial_i u) \in L^1(G),\ i = 1,\ldots,N \\ (\partial_0 u := u) \text{ and} \\ \sum_i (\partial_i u, \partial_i \phi) + \sum_{i=1}^{N} (g_i(\partial_i u), \partial_i \phi) + (g_0(u), \phi) = (f, \phi) \qquad (2) \\ \text{holds for all } \phi \in C_0^\infty(G). \end{cases}$$

For the strong nonlinearities g_i we assume

(H.2)
$$\begin{cases} g_i \in C^0(R) \text{ are non-decreasing and} \\ g_i(t)\cdot t \geq 0 \text{ for } t \in R,\ i = 0,1,\ldots N\ . \end{cases}$$

Existence of those weak solutions was studied by a considerable number of authors. Observe that the nonlinearities depend on derivatives up to half the order of the equation. Existence for those problems was first proved in [3]. Let us emphasize that all our considerations hold true if $-\Delta$ is replaced by a strongly elliptic operator of order 2m and the nonlinearities may depend analogously to (1) up to the derivatives of order m. All existence-proves lead to weak solutions such that in addition to (H.1) we get

(H.3) $\quad g_i(\partial_i u)\cdot\partial_i u \in L^1(G) \quad (i = 0,1,\ldots N)$

Like as in the case of linear equations two questions arise: Under what conditions are the weak solutions unique? Do the weak solutions have better regularity properties? For star-shaped domains uniqueness and stability of weak solutions of (1) was proved in [4]. This result was considerable generalized to arbitrary domains with smooth boundary and to very general operators by M. Landes [2].

Concerning higher regularity properties, surprisingly it turns out that the meanwhile classical difference quotient method perfectly works in the underlying case to gain one more order of differentiability. As far as the author knows, this method goes back to S.Agmon (see e.g. [1]). For the nonlinearities we assume

(H.4)

Assume (H.2) and $g_i \in C^1(R)$, $i = 0,1,\dots,N$.

For $i = 1,\dots,N$ we assume

i) g_i' is non-decreasing in $[0,\infty)$

ii) There exists a constant $C \geq 1$ such that

$g_i'(-t) \leq C g_i'(t)$ for $t \in R$ ("nearly odd")

iii) Let $G_i(t) := \int_0^t g_i(s)ds$. Assume that G_i satisfies a Δ_2-condition: There exists a $K > 0$ such that

$G_i(2t) \leq K G_i(t)$ for $t \in R$.

iv) There exists $\gamma > 0$ such that

$|g_i(t)| \leq \gamma |g_k(t)|$ for $t \in R$ and $i,k = 1,\dots,N$

("isotropic")

An example is given by

$$g_i(t) := \alpha_i t|t|^{p-1}, \quad \alpha_i \in R, \; P > 1 \tag{3}$$

Then we can prove the following

<u>Theorem.</u> Assume (H.1)-(H.4). Then, for $G' \subset\subset G$ we have $u\big|_{G'} \in W^{2,2}(G')$, $g_i'(\partial_i u)\cdot(\partial_i \partial_k u)^2\big|_{G'} \in L^1(G')$,

$$g_0'(u)\cdot(\partial_k u)^2\big|_{G'} \in L^1(G') \quad (i,k = 1,\dots,N)$$

and there is a constant $K = K(G',G,g_i)$ such that

$$\sum_{i,k=1}^{N} \int_{G'} (\partial_i\partial_k u)^2 + \sum_{i,k=1}^{N} \int_{G'} g_i'(\partial_i u)\cdot(\partial_i\partial_k u)^2 +$$

$$+ \sum_{k=1}^{N} \int_{G'} g_0'(u)(\partial_k u)^2 \leq K\cdot\Big(\|f\|^2_{L^2(G)} + \|u\|^2_{H_0^{1,2}(G)}$$

$$+ \sum_{i=0}^{N} \int_G g_i(\partial_i u)\cdot\partial_i u\Big)$$

As mentioned above, the proof is done by means of the difference quotient method. We can not give details here. In any case, the proof is completely elementary although it demands are careful analysis. To see what is going on, we assume now that the weak solution is arbitrarily smooth and sketch how to get the a-priori-estimate of the theorem. Roughly spoken, one can prove with this method all those regularity properties which can be read of an a-priori-estimate like as in the theorem.

For this purpose, let $\phi \in C_0^\infty(G)$ and for $k = 1,\dots,N$ put $\partial_k\phi$ as a test function in (2) and integrate at the left hand side by parts, which leads to

$$\sum_{i=1}^{N} (\partial_i\partial_k u, \partial_i\phi) + \sum_{i=1}^{N} (g_i'(\partial_i u)\partial_i\partial_k u, \partial_i\phi) + (g_0'(u)\partial_k u, \phi) = \tag{4}$$

$$= -(f, \partial_k\phi) \;.$$

Let now $\zeta \in C_0^\infty(G)$ such that $\zeta \equiv 1$ in G'. Put $\phi := \partial_k u\cdot\zeta^2$ in (4) which gives

$$\sum_{i=1}^{N} \int (\partial_i\partial_k u)^2\cdot\zeta^2 + \sum_{i=1}^{N} \int \partial_i\partial_k u\,\partial_k u\,2\zeta\cdot\partial_i\zeta +$$

$$+ \sum_{i=1}^{N} \int g_i'(\partial_i u)\cdot(\partial_i\partial_k u)^2\cdot\zeta^2 + \sum_{i=1}^{N} \int g_i'(\partial_i u)\,\partial_i\partial_k u\,\partial_k u\,2\zeta\,\partial_i\zeta + \tag{5}$$

$$+ \int g_0'(u)(\partial_k u)^2\cdot\zeta^2 = -(f, \partial_k\partial_k u\zeta^2) - (f, \partial_k u\,2\zeta\,\partial_k\zeta)$$

The first, third and fifth expression at the left of (5) are that we have to estimate. E.g. the second admits trivially the estimate for $\varepsilon > 0$

$$\Big|2\sum_{i=1}^{N} \int (\partial_i\partial_k u\zeta)\cdot(\partial_k u)\cdot\partial_i\zeta\Big| \leq$$

$$\leq \varepsilon\cdot\sum_{i=1}^{N} \int (\partial_i\partial_k u)^2\zeta^2 + \varepsilon^{-1}\cdot\sum_{i=1}^{N} \int (\partial_k u)^2(\partial_i\zeta)^2$$

analogous for the right hand side of (5). Cumbersome seems the fourth term. To estimate it, observe $g_i'(t) \geq 0$. For $\delta > 0$ we get:

$$\begin{aligned} &\left| 2 \sum_{i=1}^{N} \int (\sqrt{g_i'(\partial_i u)} \cdot \partial_i \partial_k u \zeta)(\sqrt{g_i'(\partial_i u)}\, \partial_k u \partial_i \zeta) \right| \\ &\leq \delta \cdot \sum_{i=1}^{N} \int g_i'(\partial_i u) \cdot (\partial_i \partial_k u)^2 \cdot \zeta^2 + \delta^{-1} \cdot \int g_i'(\partial_i u) \cdot (\partial_k u)^2 \cdot (\partial_i \zeta)^2 \quad . \end{aligned} \tag{6}$$

To estimate the second expression on the right hand side of (6), we make use of the following

Lemma: Assume (H.4). Then

i) There is a constant $C > 0$ such that
$G_i(t) \leq C \cdot G_k(t)$ for $t \in \mathbb{R}$ and $i,k = 1,\dots,N$

ii) There is a constant $C' > 0$ such that
$g_i'(t) s^2 \leq C' \cdot (g_i(t) \cdot t + G_i(s))$

Remark: Property ii) is no surprise if we consider e.g. $g(t) := t|t|^{p-1}$, $p > 1$. Then, $g'(t) = p \cdot |t|^{p-1}$, $g(t) \cdot t = |t|^{p+1}$ and $G(t) = (p+1)^{-1} \cdot |t|^{p+1}$. By the inequality $a \cdot b \leq \lambda^{-1} \cdot a^{\lambda} + \lambda'^{-1} \cdot b^{\lambda'}$ for $1 < \lambda, \lambda' < \infty$, $\lambda^{-1} + \lambda'^{-1} = 1$, we get with $\lambda := \frac{p+1}{p-1}$ and therefore $\lambda' = \frac{p+1}{2}$

$$g'(t) \cdot s^2 \leq p \cdot \frac{p-1}{p+1} \cdot g(t) \cdot t + \frac{2p}{p+1} \cdot G(s)$$

from which ii) follows. The assumptions in (H.4) (especially iii)) guarantee Lemma 2, property ii) in general.

By means of the Lemma we are now able to estimate the second expression at the right hand side of (6), first pointwise:

$$\begin{aligned} g_i'(\partial_i u) \cdot (\partial_k u)^2 &\leq C'(g_i(\partial_i u) \cdot \partial_i u + G_i(\partial_k u)) \\ &\leq C'(g_i(\partial_i u) \cdot \partial_i u + C \cdot G_k(\partial_k u)) . \end{aligned}$$

If we observe (H.2) and the definition of G_k, we conclude $G_k(\partial_k u) \leq g_k(\partial_k u) \partial_k u$. Multiplying by $(\partial_i \zeta)^2$, integrating and combining all inequalities, we have proved the desired estimate.

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