Pavel Krejčí

Periodic solutions of partial differential equations with hysteresis

In: Jaromír Vosmanský and Miloš Zlámal (eds.): Equadiff 6, Proceedings of the International Conference on Differential Equations and Their Applications held in Brno, Czechoslovakia, Aug. 26 - 30, 1985. J. E. Purkyně University, Department of Mathematics, Brno, 1986. pp. [221]--226.

Persistent URL: http://dml.cz/dmlcz/700173

PERIODIC SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS WITH HYSTERESIS

P. KREJČÍ
Mathematical Institute, Czechoslovak Academy of Sciences
115 67 Prague 1, Czechoslovakia

Introduction.

In mechanics of plastic-elastic bodies or in the theory of electromagnetic field in ferromagnetic media we are led to the consideration of hysteresis phenomena. There are various approaches to the mathematical description of hysteresis (cf. [1]). Existence results for PDE's with hysteresis nonlinearities are due to Visintin (see e.g. [5]). We give here a survey of results of [2], [3], [4], where we prove the existence of periodic solutions to the problems

$$u_{tt} - u_{xx} \pm F(u) = H(t,x) \;, \quad u(t,0) = u(t,\pi) = 0 \quad ([2]), \tag{P1}$$

$$u_t - (F(u_x))_x = H(t,x) \;, \quad u(t,0) = u(t,1) = 0 \quad ([3]), \tag{P2}$$

$$u_{tt} - (F(u_x))_x = H(t,x) \;, \quad u_x(t,0) = u_x(t,\pi) = 0 \quad ([4]), \tag{P3}$$

where F is the Ishlinskiĭ hysteresis operator and H is a given time-periodic function with an arbitrary period $\omega > 0$.

1. Ishlinskiĭ operator (cf. [1], [2]).

We first define simple hysteresis operators $v \to \ell_h(v)$, $f_h(v)$ for $h > 0$ and for piecewise monotone continuous inputs $v : [0,T] \to R^1$ as follows:

$$\ell_h(v)(t) = \begin{cases} \max\,\{\ell_h(v)(t_0),\, v(t) - h\}, & t \in [t_0,t_1] \,, \\ \quad \text{if } v \text{ is nondecreasing in } [t_0,t_1] & \\ \min\,\{\ell_h(v)(t_0),\, v(t) + h\}, & t \in [t_0,t_1] \,, \\ \quad \text{if } v \text{ is nonincreasing in } [t_0,t_1] \,, & \end{cases} \tag{1.1}$$

$$\ell_h(v)(0) = \begin{cases} 0 , & \text{if } |v(0)| \leq h \\ v(0) - h , & v(0) > h \\ v(0) + h , & v(0) < - h . \end{cases}$$

$$f_h(v)(t) = v(t) - \ell_h(v)(t) . \tag{1.2}$$

For v , w continuous and piecewise monotone we have (see [1], p. 16)

$$|\ell_h(v)(t) - \ell_h(w)(t)| \leq \max \{ |v(s) - w(s)| ; s \in [0,t] \} . \tag{1.3}$$

This property enables us to define $\ell_h(v)$, $f_h(v)$ for arbitrary continuous inputs. Moreover, it follows from (1.3) that ℓ_h , f_h map continuously the space $C([0,T])$ of all continuous functions into itself.

Let us introduce the space C_ω , $\omega > 0$, of all continuous ω-periodic functions $v : R^1 \to R^1$ with sup-norm $||\cdot||$. For $v \in C_\omega$ the functions $\ell_h(v)$, $f_h(v)$ are ω-periodic for $t \geq \omega$, hence ℓ_h , f_n can be considered as continuous operators $C_\omega \to C_\omega$.

Let further $g : [0,\infty) \to [0,\infty)$ be a function of class $C^2(0,\infty)$ satisfying

(1.4)

(i) g *is increasing,* $g(0) = 0$, $0 < g'(0+) < \infty$.

(ii) $g(h) \leq a \cdot h^\alpha$ *for some* $a > 0$, $\alpha \in (0,1)$ *and for every* $h \geq 0$.

(iii) *Put* $\gamma(r) = \inf \{-g''(h); 0 < h \leq r\}$. *We require the existence of constants* $b > 0$, $\beta \in (0,\alpha]$ *such that* $\liminf_{r\to\infty} \gamma(r)\, r^{2-\beta} = b$.

(iv) $\beta > 1/3$, $3\alpha < 4\beta$.

For $v \in C_\omega$ we define

(1.5)

(i) $$F(v)(t) = - \int_0^\infty f_h(v)(t)\, g''(h)\, dh$$

(ii) $$L(v)(t) = \int_0^\infty \ell_h(v)(t)\, (g^{-1})''(h)\, dh + (g^{-1})'(0+)\, v(t) ,$$

where $g(g^{-1}(h)) = g^{-1}(g(h)) = h$.

Roughly speaking ,the dependence of F on v (L on v) can be represented by a system of hysteresis loops constituted by parts of the graph of g (g^{-1}, respectively) for v nondecreasing and $-g$ ($-g^{-1}$, respectively) for v nonincreasing. The operators F, L have the following properties:

(i) $F, L : C_\omega \to C_\omega$ *are continuous*

(ii) $L = F^{-1}$

(iii) $||F(v)|| = g(||v||)$

(iv) $$g'(\max\{||v||, ||w||\})\,||v-w|| \leq ||F(v)-F(w)|| \leq g'(0+)\,||v-w||$$

(v) *For* v *absolutely continuous* $F(v)$, $L(v)$ *are absolutely continuous and the inequalities*

$$|(F(v))'(t)| \leq g'(0+)\,|v'(t)|,$$
$$|(L(v))'(t)| \leq (g^{-1})'(||v||)\,|v'(t)| \tag{1.6}$$

are satisfied almost everywhere.

(vi) *For* v' *absolutely continuous we have*

$$\int_\omega^{2\omega} (F(v))'v'' \geq \frac{1}{4}\,\gamma(||v||)\int_0^\omega |v'|^3$$

$$\int_\omega^{2\omega} (L(v))'v'' \leq -\frac{1}{4}\,\tilde{\gamma}(||v||)\int_0^\omega |v'|^3,$$

where $\tilde{\gamma}(r) = \inf\{(g^{-1})''(h);\ 0 < h \leq r\}$.

For functions of two variables $u(t,x)$, $t \in R^1$, $x \in I$, such that $u(.,x) \in C_\omega$ for each $x \in I$ we define

$$F(u)(t,x) = F(u(.,x))(t), \quad L(u)(t,x) = L(u(.,x))(t).$$

2. Existence results.

We introduce the Banach spaces or time-periodic functions $u(t,x)$, $u : R^1 \times [0,\ell] \to R^1$, $\ell > 0$, $u(t+\omega, x) = u(t,x)$, $\omega > 0$:

$C_\omega([0,\ell])$: the space of all continuous functions with norm

$$|||u||| = \max\{|u(t,x)|;\ t \quad R^1,\ x \in [0,\ell]\};$$

$L^p_\omega(0,\ell)$, $1 \leq p \leq \infty$: the space of all measurable functions such that

$$|u|_p = \left(\int_0^\omega\int_0^\ell |u(t,x)|^p\,dx\,dt\right)^{1/p} < \infty \quad \text{for } p < \infty,$$

$$|u|_\infty = \sup\text{ess}\{|u(t,x)|;\ t \quad R^1,\ x \in [0,\ell]\} < \infty,$$

with norm i• i p ;

$L^2(0,\ell; iJ) = \{u \in 1^\wedge(0,\ell); |u|_{2>1} = (j(j|u(t,x)| dt) dx)$

with norm I * I o i •

Theorem. (2.1)

(i) *Let* H G L^2_{OJ}(0,TT) i?e *given such that* H_{tt} 6 $L^{3/2}_0$(0,TT) *and* Z-et (1.4) (i) - (iii) *hold. Then there exists at least one solution* u € C (To, TT]) , u_{tt} - u_{xx} € L (0,TT) , u_t £ L^3_{OJ} (0,TT) to t/ze *problém* (P1) suc/z t/zat (P1) £s satts-*fied almost everywhere.*

(ii) Let H 6 C (F0,11) £e *given such that* H_t e L (0,1) aná £et (1.4)(i) - (iii) *hold. Then there exists at least one classical solution* **u 6 C (£0, í]) , u , u , (F(u)) £** C_u([°.1]) > u_{tt} e L"(0,1), u_{tx} 6 L3(0.1), u_{xx} e L^(0(1) to t/ze *problém* (P2) .

(iii) Let H 6 L (0, TT) fce *given such that* **H 6 L ' (O.TOH** L^2(0,TT; L^1) , i ÍH(t,x) dx dt = 0 , ***and*** *let* (1.4)(i) - (iv) *hold.. Then there exists at least one solution* u *E* C ([p,Tr]) , u_x « V^0'1^ ' u_{tt}> ($F(u_x)_x$ e L>>*> • u_{tx} e ^J0'^ *° t/ze *problém* (P3) *such that* (P3) is *satisfied almost every-where* .

Sketch of the proofs.

(i) We use the classical Galerkin method. We denote

$$w_{jk}(t,x) = e_j(t) \sin kx \;, \; j \text{ integer}, \; k \text{ natural}, \tag{2.2}$$

where $e_j(t) = \sin \frac{j}{2} t$ for $j > 0$ and $\cos \frac{j}{2} t$ for $j < 0$.

For m > 1 put

$$u_m(t,x) = \sum_{j=-m}^{m} \sum_{k=1}^{m} u_{jk} w_{jk}(t,x) \;, \tag{2.3}$$

where the coefficients u_{jk} are solutions of the algebraic systém

$$\int_0^{2\pi}\int_0^{\pi} (u_{tt} - u_{xx} \pm F(u))w_{jk}\, dx\, dt = \int_0^{2\pi}\int_0^{\pi} H\, w_{jk}\, dx\, dt \;, \tag{2.4}$$

$j = -m,\dots,m\ ,\quad k = 1,\dots,m\ .$

Multiplying (2.4) by $\left(\frac{2\pi j}{\omega}\right)^3 u_{-jk}$, summing over j and k and using (1.6)(vi) we get

$$\gamma(|||u_m|||)\,|u_{m\,t}|_3^2 \leqq \text{const.}$$

Similarly we have $|u_{m\,tt} - u_{m\,xx}|_2 \leqq \text{const.}\left(1 + |||F(u_m)|||\right)$. Classical embedding theorems and (1.4), (1.6)(iii) yield $|||u_m||| \leqq$ $\text{const.}\left(|u_{m\,tt} - u_{m\,xx}|_2 + |u_{m\,t}|_3\right) \leqq \text{const.}\left(|||u_m|||^{1-\beta/2} + |||u_m|||^{\alpha} + 1\right)$, hence $|||u_m|||\ ,\ |u_{m\,t}|_3\ ,\ |u_{m\,tt} - u_{m\,xx}|_2 \leqq \text{const.}$

These estimates imply the solvability of (2.4). On the other hand there exists a subsequence $\{u_n\}$ of $\{u_m\}$ such that $u_{n\,tt} - u_{n\,xx} \to$ $u_{tt} - u_{xx}$ in $L^2_\omega(0,\pi)$ weak, $u_{n\,t} \to u_t$ in $L^3_\omega(0,\pi)$ weak and $u_n \to u$ in $C_\omega([0,\pi])$ strong. Thus, we can pass to the limit in (2.4) and the proof is complete.

(ii) We replace the problem (P2) by the following system of ordinary differential equations (space discretization):

$$\begin{aligned} &v_j' - n\left(F(n(v_{j+1} - v_j)) - F(n(v_j - v_{j-1}))\right) = h_j\ ,\\ &v_0 = v_n = 0\ , \end{aligned} \tag{2.5}$$

where $h_j(t) = n \displaystyle\int_{j/n}^{(j+1)/n} H(t,x)\,dx\ ,\quad j = 1,\dots,n-1\ .$

Using (1.6)(iv), (vi) we derive apriori estimates for v_j independent of n which ensure the existence of periodic solutions of (2.5). We further put

$$\begin{aligned} u_n(t,x) = {}& v_j(t) + (nx - j)(v_{j+1}(t) - v_j(t)) + \tfrac{1}{2}\left[(v_{j+1}(t) - v_j(t))\right.\\ &\left. + (nx - j)^2 (v_{j+2}(t) - 2v_{j+1}(t) + v_j(t))\right]\ , \end{aligned}$$

$$t \in R^1\ ,\quad x \in [j/n,\ (j+1)/n]\ ,\quad j = 0,1,\dots,n-1\ ,\quad v_{n+1} = -\,v_{n-1}\ .$$

A straigtforward computation and the compactness argument show that there exists a subsequence of u_n which converges in suitable topologies to a solution u of (P2).

(iii) We first solve the auxiliary problem

$$L(v_t - \psi)_t - v_{xx} = \tilde{H}_x\ ,\quad v(t,0) = v(t,\pi) = 0\ ,$$

where $\psi(x) = \frac{1}{\omega}\int_0^x\int_0^\omega H(t,\xi)\,dt\,d\xi$, $x \in [0,\pi]$,

$$\tilde{H}(t,x) = \int_0^t H(\tau,x)\,d\tau - \frac{t}{\omega}\int_0^\omega H(\tau,x)\,d\tau\ , \quad t \in R^1\ , \quad x \in [0,\pi]\ .$$

We use again the Galerkin approximation scheme of the type (2.2), (2.3), (2.4). In an analogous way we derive the following estimates:

$$\tilde{\gamma}(|||\underset{m}{v}_t - \psi|||)\,|\underset{m}{v}_{tt}|_3^2 \leqq \text{const.}$$

$$|\underset{m}{v}_{xt}|_2^2 \leqq (g^{-1})'(|||\underset{m}{v}_t - \psi|||)\,|\underset{m}{v}_{tt}|_2^2 + \text{const.}\,|\underset{m}{v}_t|_3\ .$$

Following (1.4) there exists $\delta > 0$ such that $1/\tilde{\gamma}(r) \leqq$ const.$(r^{3-1/\beta-\delta} + 1)$ for every $r > 0$. The space $\{u \in L^1_\omega(0,\pi);\ u_t \in L^3_\omega(0,\pi)\ ,\ u_x \in L^2_\omega(0,\pi)\}$ is compactly embedded into $C_\omega([0,\pi])$, hence $|||\underset{m}{v}_t - \psi||| \leqq \text{const.}(|\underset{m}{v}_{tt}|_3 + |\underset{m}{v}_{tx}|_2 + 1) \leqq \text{const.}(|||\underset{m}{v}_t - \psi|||^{1-\delta/2} + 1)$.
Similarly as above we get

$$|||\underset{m}{v}_t|||\ ,\ |\underset{m}{v}_{tt}|_3\ ,\ |\underset{m}{v}_{tx}|_2\ ,\ |\underset{m}{v}_{xx}|_2 \leqq \text{const.},$$

so that we can repeat the argument of (i).

The solution u of (P3) is then given by the formula

$$u(t,x) = \int_\omega^t (v_x + \tilde{H})(\tau,x)\,d\tau - \frac{t}{\omega}\int_\omega^{2\omega}(v_x + \tilde{H})(\tau,x)\,d\tau + \int_0^x L(v_t - \psi)(\omega,\xi)\,d\xi + \text{const.},\quad t \in R^1\ ,\quad x \in [0,\pi]\ .$$

References

[1] KRASNOSELSKIĬ, M. A. and POKROVSKIĬ, A. V.: *Systems with hysteresis*. Nauka, Moscow, 1983 (Russian).

[2] KREJČÍ, P.: *Hysteresis and periodic solutions of nonlinear wave equations*. To appear.

[3] KREJČÍ, P.: *Periodic solutions to a parabolic equation with hysteresis*. To appear.

[4] KREJČÍ, P.: *Periodic solutions to a quasilinear wave equation with hysteresis*. To appear.

[5] VISINTIN, A.: *On the Preisach model for hysteresis*. Nonlin. Anal., Theory, Meth. & Appl. 8 (1984), pp. 977-996.