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AN APPLICATION OF GREEN'S FUNCTION IN THE DIFFERENTIAL EQUATIONS

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In solving of various types of problems in the theory of ordinary and partial differential equations, difference equations, there occurs a notion of Green's function. With help of it many problems of various character from the theory of ordinary and partial differential equations, especially from nonlinear equations, can be reduced to an integral equation of Hammerstein's type and thus can be studied from a uniform standpoint. This enables us to carry over the methods and the results from a one group of the problems to another group and, of course, to use the results of the theory of integral equations and in the main, of functional analysis.

The aim of this lecture is to show some methods for obtaining the sufficient conditions for the existence and partly for the uniqueness of the solution of a nonlinear boundary value problem, using the fixed point theorems. The methods may be used in solving of related problems too.

Notations and Assumptions.

Let R^n mean the n-dimensional real Euclidean space and if $x, y \in R^n$, let $|x, y|$ be their distance. $|x, S|$ will mean the distance between the point x and the set $S \subset R^n$. If $x \in R^n$, $\delta > 0$, then $B(x, \delta) = \{y : y \in R^n, |x, y| < \delta\}$. $j \in R^m$ denotes the vector with all its components equal to 1.

Let $D \subset R^n$ be a region, $\bar{D}$ the closure of D, $\emptyset \neq S \subset D$ a set. These sets will satisfy

Assumption 1. *Let $D \cup S$ be compact.*
From this assumption it follows that $D \cup S = \bar{D}$ and hence D is bounded and S contains the boundary of D.

Denote $E = (D \cup S) \times R^m$, $E^0 = D \times R^m$ and if $b \geq 0$, let

$$E_b = (D \cup S) \times \underbrace{\langle -b, b\rangle \times \ldots \times \langle -b, b\rangle}_{m\text{-times}}.$$

Further, let U be partially ordered Banach space of all real $m \times 1$ vector functions $u(x) = (u_1(x), \ldots, u_m(x))$, $u(x) \in C_0(D \cup S)$, $x = (x_1, \ldots, x_n)$, with the norm $||u|| = \max_{k=1,\ldots,m} \max_{x \in D \cup S} |u_k(x)|$. If $u, v \in U$, then $u \leqq v$ if and only if for every $k = 1, \ldots, m$, $x \in D \cup S$, $u_k(x) \leqq v_k(x)$ holds. Similarly the sharp inequality is valid and also the inequality in R^m. Denote $|u(x)| = (|u_1(x)|, \ldots, |u_m(x)|)$ and analogically, if $u \in R^m$, then $|u| = (|u_1|, \ldots, |u_m|)$. As usual, if $v_1 \leqq v_2$, then $\langle v_1, v_2 \rangle = \{u : u \in U, v_1 \leqq u \leqq v_2\}$. $\langle v_1, v_2 \rangle$ is a closed, convex and bounded set in U. $E_{v_1,v_2} = \{(x, u) : (x, u) \in E, v_{1k}(x) \leqq u_k \leqq v_{2k}(x), k = 1, \ldots, m\}$, where $v_1(x) = (v_{11}(x), \ldots, v_{1m}(x)) \leqq v_2(x) = (v_{21}(x), \ldots, v_{2m}(x)) \in U$, $u = (u_1, \ldots, u_m)$. $U_b = \{u : u \in U, ||u|| \leqq b\}$.

Similarly as for the vector functions, the matrix function $|G(x, t)|$ is defined by $|G(x, t)| = (|G_{kl}(x, t)|)$ if $G(x, t) = (G_{kl}(x, t))$, $k, l = 1, \ldots, m$. For $H(x) = (H_{kl}(x)) \in C_0(D \cup S)$, $k, l = 1, \ldots, m$, it is $||H(x)|| = \max_{k,l=1,\ldots,m} \max_{x \in D \cup S} |H_{kl}(x)|$. $G(x, t) < H(x, t)$ if and only if $G_{kl}(x, t) < H_{kl}(x, t)$ for every (x, t) of their common domain and all $k, l = 1, \ldots, m$. J (εJ) is the $m \times m$ matrix whose all elements are equal to 1 (are equal to ε). J_0 is the unit $m \times m$ matrix.

In what follows, the matrices and the vectors will be supposed to be of the type $m \times m$ and $m \times 1$, respectively.

Consider the (boundary-value) problem

$$L(u) = f(x, u), \qquad x \in D, \tag{1}$$

$$M(u) = g(x), \qquad x \in S, \tag{2}$$

where $f(x, u) = (f_1(x, u), \ldots, f_m(x, u))$ is a real vector function of the variables $x = (x_1, \ldots, x_n)$, $u = (u_1, \ldots, u_m)$ defined in E, $g(x)$ is a real vector function defined in S, L is a linear differential operator, and M is a linear operator. These functions and operators will be supposed to satisfy an assumption. By a solution of the problem (1), (2) will be meant every $u \in U$ satisfying the equations (1), (2) and possessing as many continuous derivatives as one usually requires from the solution of the problem (1), (2).

Assumption 2. *Let the problem*

$$L(v) = 0j, \qquad x \in D \tag{3}$$

$$M(v) = 0j, \qquad x \in S \tag{4}$$

have only the trivial solution, let there exist a solution $v(x)$ of the problem

$$L(v) = 0j, \qquad x \in D$$

$$M(v) = g(x), \qquad x \in S$$

and the matrix function $G(x, t)$, so called Green's function of the problem (3), (4), *with the following properties:*

1. $\int_D |G(x, t)| \, \mathrm{d}t$ *exists for each* $x \in D \cup S$.

2. *Given any* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that* $\int_D |G(x, t) - G(y, t)| \, dt < \varepsilon J$ *whenever* $|x, y| < \delta$, $x, y \in D \cup S$.

3. *The alternative holds: Either for every* $r(x) \in U$ *the function*

$$w(x) = v(x) + \int_D G(x, t) \, r(t) \, dt, \qquad x \in D \cup S \tag{5}$$

is a solution of the problem

$$L(w) = r(x), \qquad x \in D \tag{6}$$

$$M(w) = g(x), \qquad x \in S \tag{7}$$

or for every $r(x) \in U$ *the function* (5) *satisfies a Hölder's condition and for every* $r(x) \in U$ *satisfying a Hölder's condition the function* (5) *is a solution of the problem* (6), (7).

Remark 1. By the assumption on the problem (3), (4), the solutions $v(x)$, $w(x)$, as well as $G(x, t)$, are uniquely determined ($G(x, t)$ except on a set of Lebesguemeasure zero).

Lemma 1. *Let Assumption* 1 *be fulfilled and let the matrix function* $G(x, t)$ *possess the following properties*:

1. $G(x, t)$ *is defined and continuous for every* $x \in D \cup S$, $t \in D$, $t \neq x$.

2. *For* $x \neq t$ *the function* $G(x, t)$ *is almost uniformly bounded in the sense that, for any* $\delta > 0$, *there exists an* $N = N(\delta) > 0$ *such that* $|G(x, t)| < NJ$ *for all* $x \in D \cup S$, $t \in D$, $|x, t| \geqq \delta$.

3. $\int_D |G(x, t)| \, dt$ *is uniformly convergent for every* $x \in D \cup S$, *that is, given any* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that* $\int_{D \cap B(x,\delta)} |G(x, t)| \, dt < \varepsilon J$ *for all* $x \in D \cup S$.

Then the function $G(x, t)$ *possesses the properties* 1 *and* 2 *from Assumption* 2.

Proof. Obviously $G(x, t)$ has the property 1 from Assumption 2. The property 2 can be shown in this way. By the property 3, there exists $\delta > 0$ such that $\int_{D \cap B(x,\delta)} |G(x, t)| \, dt < \frac{\varepsilon}{3} J$. Suppose $y \in B\left(x, \frac{\delta}{4}\right) \cap (D \cup S)$. Then $\int_{D \cap B(x,\delta)} |G(x, t) - G(y, t)| \, dt \leqq \int_{D \cap B\left(x, \frac{\delta}{2}\right)} |G(x, t)| \, dt + \int_{D \cap B\left(x, \frac{\delta}{2}\right)} |G(y, t)| \, dt < \frac{2\varepsilon}{3} J$. With respect to the property 1 of $G(x, t)$ $\lim_{y \to x} |G(x, t) - G(y, t)| = 0J$ for all $t \in D - B\left(x, \frac{\delta}{2}\right)$ $\left(\text{that is, for all } t \in D \text{ such that } |t, x| \geqq \frac{\delta}{2}\right)$. Further the function $|G(x, t)| + N\left(\frac{\delta}{4}\right) J$ is an integrable majorant for $|G(x, t) - G(y, t)|$. By the Lebesgue theorem there exists $0 < \delta_1 = \delta_1(x, \varepsilon) <$

$< \frac{\delta}{4}$ such that $\int\limits_{D-B\left(x,\frac{\delta}{2}\right)} |G(x, t) - G(y, t)| \, dt < \frac{\varepsilon}{3} J$ for $|x, y| < \delta_1$ and hence, $\int\limits_D |G(x, t) - G(y, t)| \, dt < \varepsilon J$. Finally Assumption 1 implies that δ_1 does not depend on x.

Remark 2. If Assumptions 1 and 2 hold, then the function $H(x) = \int\limits_D |G(x, t)| \, dt$ is continuous on $D \cup S$ and $C = ||H(x)|| < \infty$.

Assumption 3. *Let $f(x, u) \in C_0(E)$ and if in Assumption 2 the second part of the alternative is true, let $f(x, u)$ satisfy on every bounded subset $Z \subset E$ a Hölder's condition with constants which may depend on Z.*

Study of the problem (1), (2).

First, the equivalence of this problem to an integral equation will be shown.

Lemma 2. *Let Assumptions 1, 2 and 3 be satisfied. Then, if the first part of alternative in Assumption 2 holds, the boundary-value problem (1), (2) is equivalent to the integral equation*

$$u(x) = v(x) + \int\limits_D G(x, t) f(t, u(t)) \, dt, \qquad x \in D \cup S. \tag{8}$$

If the second part of alternative is valid, every solution of (8) is a solution of the problem (1), (2) satisfying a Hölder's condition and conversely, every solution of the problem (1), (2) which satisfies a Hölder's condition is a solution of the equation (8), too. Here the only request on the solution of (8) is to be of U.

The equation (8) is a functional equation of the type

$$u = Tu. \tag{9}$$

The properties of the operator T defined for every $u \in U$ by

$$Tu = v + \int\limits_D G(x, t) f(t, u(t)) \, dt \tag{10}$$

will now be considered.

Lemma 3. *If Assumptions 1, 2 and 3 hoid, the operator T given by (10) is continuous, compact and $TU \subset U$.*

Proof. Let $\varepsilon > 0$ and $b > 0$ be arbitrary numbers. From the inequality $|Tu_1 - Tu_2| \leqq \int\limits_D |G(x, t)| \, |f(t, u_1(t)) - f(t, u_2(t))| \, dt$ and from the uniform continuity of $f(x, u)$ on E_b follows the existence of such a $\delta = \delta(b, \varepsilon)$ that $||Tu_1 - Tu_2|| < \varepsilon m C$ for $u_1, u_2 \in U_b$, $||u_1 - u_2|| < \delta$. Thus T is continuous on U_b. Denote $K_b = \max\limits_{k=1,\ldots,m} \max\limits_{(x,u) \in E_b} |f_k(x, u)|$. If $u \in U_b$, then $|Tu(x) -$

$- Tu(y)| \leqq |v(x) - v(y)| + K_b \int_D |G(x, t) - G(y, t)|\, j\, dt$. Hence for a sufficiently small $\delta > 0$, on the basis of Assumption 2, $|Tu(x) - Tu(y)| < \varepsilon(1 + mK_b)\, j$ follows from $|x, y| < \delta$, $x, y \in D \cup S$. Finally, $|Tu(x)| \leqq (||v|| + K_b C m)\, j$. By the Ascoli theorem one gets that TU_b is relatively compact. At the same time $TU \subset U$ was proved.

Consider the interval $\langle v - bj, v + bj \rangle$, $b > 0$. Let $K_{v,b} = \max |f_k(x, u)|$ for $k = 1, \ldots, m$, $(x, u) \in E_{v-bj,v+bj}$. For $u \in \langle v - bj, v + bj \rangle$ the inequality $|Tu - v| \leqq mC.\ K_{v,b} j$ is valid. From it, using Lemmas 2, 3 and Schauder's fixed point theorem ([1], p. 355) one obtains

Theorem 1. *Let Assumptions* 1, 2 *and* 3 *be satisfied. Let* $b > 0$ *exist, for which*

$$mCK_{v,b} \leqq b.$$

Then there exists at least one solution of the problem (1), (2) *contained in the interval* $\langle v - bj, v + bj \rangle$ *(which satisfies a Hölder's condition if in Assumption* 2 *the second part of the alternative holds).*

With help of the Schauder theorem a generalization of the first Fredholm theorem was proved by another Polish mathematician A. LASOTA. This affirms that a nonlinear equation has at least one solution if a certain system of homogeneous linear equations possesses only the trivial solution.

Let R be a Banach space. Let $L_s(R, R)$ be the space of all linear (additive and homogeneous) operators on R into R. In the space $L_s(R, R)$ the simple convergence is defined as follows: The sequence $\{A_n\} \subset L_s(R, R)$ converges simply to $A \in L_s(R, R)$ ($A_n \underset{s}{\to} A$) if for each $z \in R$ $A_n z \to Az$.

Lasota's Theorem ([2], p. 89—91). *Let* $Q \subset L_s(R, R)$ *be a set satisfying the following conditions:*

1. *Each sequence* $\{A_n\} \subset Q$ *contains a subsequence* $A_{n_k} \underset{s}{\longrightarrow} A \in Q$.

2. *The set* $\bigcup_{A \in Q, ||z|| = 1} A\, z$ *is relatively compact in* R.

Suppose that for each $A \in Q$ *the equation*

$$z = A\, z$$

has only the trivial solution.

Further let $A = A(z)$ *be the operator on* R *into* Q *such that*

3. $z_n \to z$ *implies* $A(z_n) \underset{s}{\longrightarrow} A(z)$.

Finally, let $b(z)$ *be the operator which maps* R *into* R *and satisfies the conditions:*

4. $b(z)$ *is compact.*

5. $\lim_{||z|| \to \infty} (||z||^{-1} ||b(z)||) = 0.$

Under these assumptions there exists at least one solution of the equation

$$z = A(z)\, z + b(z).$$

From this theorem, by method used in the paper [3], one gets

Theorem 2. *Let Assumptions* 1, 2 *and* 3 *be fulfilled. Let* $f(x, u)$ *satisfy the inequality*

$$(11) \qquad |f(x, u)| \leqq (N + \sum_{l=1}^{m} L_l |u_l|)\, j$$

on E, *where* $N \geqq 0$, $L_l \geqq 0$, $l = 1, \ldots, m$, *are arbitrary constants. Let the equation*

$$u(x) = \int_D G(x, t)\, F(t)\, u(t)\, \mathrm{d}t$$

have only the trivial solution for every matrix function $F(x) = (F_{kl}(x))$, *where* $F_{kl}(x) = a_l(x)$, k, $l = 1, \ldots, m$, $a_l(x)$ *are measurable on* D *and satisfy the inequality*

$$|a_l(x) \leqq L_l, \qquad l = 1, \ldots, m.$$

Then the problem (1), (2) *has at least one solution.*

Proof. Defining the vector functions

$$p_k(x, u) = f(x, u)\, (N + \sum_{l=1}^{m} L_l |u_l|)^{-1} L_k \eta(u_k), \qquad k = 1, \ldots, m,$$

$$q(x, u) = f(x, u) - \sum_{l=1}^{m} p_l(x, u)\, u_l,$$

where the scalar function $\eta(u) = u$ for $|u| \leqq 1$, $\eta(u) = \operatorname{sgn} u$, $|u| > 1$, (here u is scalar variable) the equation (8) can be rewritten in the form

$$(12) \qquad u(x) = \int_D G(x, t)[\sum_{l=1}^{m} p_l(t, u(t))\, u_l(t)]\, \mathrm{d}t + \\ + \int_D G(x, t)\, q(t, u(t))\, \mathrm{d}t + v(x).$$

The functions $p_k(x, u)$, $q(x, u) \in C_0(E)$ and, by (11), they satisfy

$$(13) \qquad |p_k(x, u)| \leqq L_k j, \qquad |q(x, u)| \leqq (N + \sum_{l=1}^{m} L_l) j.$$

Denote the set of all matrix functions F satisfying the assumption of Theorem 2, by M_F. Let Q be the set of all operators A from U into U defined by the relation

$$(14) \qquad w = Au = \int_D G(x, t)\, F(t)\, u(t)\, \mathrm{d}t, \qquad F(x) \in M_F.$$

By the assumption the equation $u = Au$ has for each $A \in Q$ only the trivial solution. Further for $||u|| = 1$, $u \in U$ and $L_0 = L_1 + \ldots + L_m$ $|Au(x) - Au(y)| \leqq m \varepsilon L_0 j$ for $|x, y| < \delta$ by Assumption 2. Moreover $|Au(x)| \leqq mCL_0 j$, $x \in D \cup S$, and thus by Ascoli's Theorem the set $\bigcup_{A \in Q,\, ||u||=1} Au$ is relatively compact.

Denote $A_n u = \int_D G(x, t)\, F_n(t)\, u(t)\, dt = \int_D G(x, t) \sum_{l=1}^{m} a_{l,n}(t)\, u_l(t)\, j\, dt$. In view of $|a_{l,n}(x)| \leqq L_l$, for each $l = 1, \ldots, m$ the set $\{a_{l,n}(x)\}$ is weakly compact in $L_1(D)$ and therefore there exists $a_l(x) \in L_1(D)$ and a subsequence $\{a_{l,n_k}(x)\}$ such that for every $g(x) \in L_1(D)$

$$\lim_{k \to \infty} \int_D g(t)\, a_{l,n_k}(t)\, dt = \int_D g(t)\, a_l(t)\, dt$$

holds. Obviously $|a_l(x)| \leqq L_l$ and besides, we can reach that $\{n_k\}$ is the same for all $l = 1, \ldots, m$. Thus for each $x \in D \cup S$ and $u \in U$ there exists

$$\lim_{k \to \infty} \int_D G(x, t) \sum_{l=1}^{m} a_{l,n_k}(t)\, u_l(t)\, j\, dt = \int_D G(x, t) \sum_{l=1}^{m} a_l(t)\, u_l(t)\, j\, dt. \tag{15}$$

The functions (14) being equicontinuous on $D \cup S$, the convergence (15) is uniform.

For each $u \in U$ define the operator $A(u) \in Q$ by the relation

$$w = A(u)\, y = \int_D G(x, t) \sum_{l=1}^{m} p_l(t, u(t))\, y_l(t)\, dt.$$

If $u_n \to u$, then $A(u_n) \underset{s}{\to} A(u)$. In fact, denoting $w_n = A(u_n)\, y$, $w = A(u)\, y$, the inequality

$$|w_n - w| \leqq ||y|| \int_D |G(x, t)| \sum_{l=1}^{m} ||p_l(t, u_n(t)) - p_l(t, u(t))||\, j\, dt \leqq$$
$$\leqq ||y||\, m^2 \max_{l=1,\ldots,m} ||p_l(t, u_n(t)) - p_l(t, u(t))||\, Cj$$

holds, which implies $||w_n - w|| \to 0$ from $||u_n - u|| \to 0$.

Consider now the operator

$$w = bu = \int_D G(x, t)\, q(t, u(t))\, dt + v(x).$$

From (13) follows $||bu|| \leqq (N + L_0)\, mC + ||v||$, so that the operator b is bounded. Obviously it is also continuous. Finally, from the inequality $|bu(x) - bu(y)| \leqq (N + L_0 + 1)\, m\, \varepsilon\, j$ for $|x, y| < \delta$, δ is sufficiently small, follows the relative compactness of bU in U.

Thus, all assumptions of Lasota's Theorem being satisfied, the equation (12) has at least one solution in U.

In the following, some theorems will be proved, where the properties of the partially ordered space U will be used. The first ones will be the theorems of a comparison character. Examples of such theorems can be found in the paper [4]. Here the following definition will be of use.

The function $h(x, u)$ defined on $E_{v_1 v_1}$ will be said to be nondecreasing (nonincreasing) in u on $E_{v_1 v_2}$ if for each $x \in D \cup S$ $h(x, u_1) \leqq h(x, u_2)$ $(h(x, u_1 \geqq \geqq h(x, u_2)$ whenever $v_1(x) \leqq u_1 \leqq u_2 \leqq v_2(x)$.

Theorem 3. *Let Assumptions* 1, 2 *and* 3 *hold. Let the Green's function* $G(x, t) \geqq 0J$ $(\leqq 0J)$ *for all points of its domain. Let there exist the vector functions* $h_j(x, t)$, $j = 1, 2$, *with the following properties*:

a. $h_j(x, t)$ *satisfy Assumption* 3.

b. The problem

$$L(u) = h_j(x, u), \qquad x \in D$$
$$M(u) = g(x), \qquad x \in S$$

has a solution $v_j(x)$ *and* $v_1 \leqq v_2$. *If the second part of the alternative in Assumption* 2 *is valid, then* $v_j(x)$ *satisfy a Hölder's condition.*

c. If $G(x, t) \geqq 0J$ $(\leqq 0J)$. *the functions* $h_j(x, u)$ *are nondecreasing (nonincreasing) in* u *on* E_{v_1, v_2} *and satisfy the inequalities*

$$h_1(x, u) \leqq f(x, u) \leqq h_2(x, u)$$
$$(h_1(x, u) \geqq f(x, u) \geqq h_2(x, u))$$

there. Then the problem (1), (2) *has at least one solution in* $\langle v_1, v_2 \rangle$.

Proof. With respect to Lemma 3 it suffices to prove that $T\langle v_1, v_2 \rangle \subset \subset \langle v_1, v_2 \rangle$. Assume $G(x, t) \geqq 0J$. If $u \in \langle v_1, v_2 \rangle$, then $G(x, t)\, h_1(t, v_1(t)) \leqq \leqq G(x, t)\, h_1(t, u(t)) \leqq G(x, t) f(t, u(t)) \leqq G(x, t)\, h_2(t, u(t)) \leqq G(x, t)\, h_2(t, v_2(t))$. From these inequalities the assertion of the theorem follows. The case $G(x, t) \leqq 0J$ is proved analogically.

Theorem 4. *Let Assumptions* 1, 2 *and* 3 *hold. Let in Assumption* 2 *mentioned Green's function* $G(x, t) \geqq 0J$ $(\leqq 0J)$ *and the solution* $v(x) \geqq 0j$ $(\leqq 0j)$ *for all-points of their domain. Let there exist a vector function* $h(x, u)$ *with the properties*

a. $h(x, t) \geqq 0j$.

b. $h(x, t)$ *satisfies Assumption* 3.

c. The problem

$$L(u) = h(x, u), \qquad x \in D$$
$$M(u) = g(x), \qquad x \in S$$

has a solution $v_0(x)$ *(satisfying a Hölder's condition if the second part of the alternative in Assumption* 2 *holds).*

d. If $G(x, t) \geqq 0J$ $(\leqq 0J)$, *then* $h(x, u)$ *is nondecreasing (nonincreasing) in* u *on* E_{-v_0, v_0} $(E_{v_0, -v_0})$ *and the inequality*

$$|f(x, u)| \leqq h(x, u)$$

holds there.

Then the problem (1), (2) *has at least one solution contained in the interval* $\langle -v_0, v_0 \rangle$ ($\langle v_0, -v_0 \rangle$).

Proof. Let $G(x, t) \geqq 0J$, $v(x) \geqq 0j$. Then for $-v_0 \leqq u \leqq v_0$ the inequalities $-G(x, t)\, h(t, v_0(t)) \leqq -G(x, t)\, h(t, u(t)) \leqq G(x, t) f(t, u(t)) \leqq G(x, t)\, h(t, u(t)) \leqq$ $\leqq G(x, t)\, h(t, v_0(t))$ hold, whence it follows that $-v_0 + 2v \leqq Tu \leqq v_0$.

The case $G(x, t) \leqq 0J$, $v(x) \leqq 0j$ is proved analogically.

A further result can be obtained by using the method developed in [1], p. 277—280. This method is based on the assumption that the operator T given by (10) is decomposable into a sum of an isotone operator T_1 and an antitone operator T_2, $T_1 U \subset U$, $T_2 U \subset U$.

If two elements $v_0, w_0 \in U$ are chosen, by the relations

$$\begin{aligned} v_{n+1} &= T_1 v_n + T_2 w_n \\ w_{n+1} &= T_1 w_n + T_2 v_n, \qquad n = 0, 1, \ldots, \end{aligned}$$

the sequences $\{v_n\}$, $\{w_n\}$ are defined. If

$$v_0 \leqq w_0, \qquad v_0 \leqq v_1, \qquad w_1 \leqq w_0$$

hold, then for all $n = 0, 1, 2, \ldots,$

$$v_n \leqq w_n, \qquad v_n \leqq v_{n+1}, \qquad w_{n+1} \leqq w_n \tag{16}$$

and $T\langle v_n, w_n \rangle \subset \langle v_{n+1}, w_{n+1} \rangle$. Assuming T is continuous and compact there exists $\lim\limits_{n \to \infty} v_n = \bar{v}$, $\lim\limits_{n \to \infty} w_n = \bar{w}$, $\bar{v} \leqq \bar{w}$. The operator T has at least one fixed point in the interval $\langle \bar{v}, \bar{w} \rangle$. Each fixed point of T, belonging to $\langle v_0, w_0 \rangle$, is contained in $\langle \bar{v}, \bar{w} \rangle$. Moreover, if T is isotone, then both points $\bar{v}, \bar{w}$ are its fixed points.

With help of this consideration the following theorem will be proved. For the sake of simplicity denote $G^+(x, t) = \frac{1}{2}(G(x, t) + |G(x, t)|)$, $G^-(x, t) =$ $= \frac{1}{2}(G(x, t) - |G(x, t)|)$. Then $G(x, t) = G^+(x, t) + G^-(x, t)$.

Theorem 5. *Let Assumption* 1 *hold. Let there exist a matrix function* $P_1(x)$ ($P_2(x)$) *defined on* $D \cup S$ *with the properties:*

a. The operator $L_1(u) = L(u) - P_1(x)\, u$ ($L_2(u) = L(u) - P_2(x)\, u$), *as well as* $M(u)$, *satisfies Assumption* 2 *with the Green's function* $G_1(x, t)$ ($G_2(x, t)$).

b. The function $f_1(x, u) = f(x, u) - P_1(x)\, u$ ($f_2(x, u) = f(x, u) - P_2(x)\, u$) *satisfies Assumption* 3.

c. The function $f_1(x, u)$ ($f_2(x, u)$) *is nondecreasing in* u *on* E (*nonincreasing in* u *on* E).

d. There exists a pair of functions v_0, $w_0 \in U$, $v_0 \leqq w_0$, *such that for* $n = 0$ *the functions* v_{n+1}, w_{n+1} *defined by the relations*

$$
\begin{aligned}
&(17) \quad & v_{n+1}(x) &= v(x) + \int_D G_1^+(x, t) f_1(t, v_n(t))\,\mathrm{d}t + \int_D G_1^-(x, t) f_1(t, w_n(t))\,\mathrm{d}t \\
& & w_{n+1}(x) &= v(x) + \int_D G_1^+(x, t) f_1(t, w_n(t))\,\mathrm{d}t + \int_D G_1^-(x, t) f_1(t, v_n(t))\,\mathrm{d}t \\
&(17') \quad & v_{n+1}(x) &= v(x) + \int_D G_2^-(x, t) f_2(t, v_n(t))\,\mathrm{d}t + \int_D G_2^-(x, t) f_2(t, w_n(t))\,\mathrm{d}t \\
& & w_{n+1}(x) &= v(x) + \int_D G_2^-(x, t) f_2(t, w_n(t))\,\mathrm{d}t + \int_D G_2^-(x, t) f_2(t, v_n(t))\,\mathrm{d}t
\end{aligned}
$$

satisfy the inequality (16).

Then the following assertions are true:

1. *The functions* $v_n(x)$, $w_n(x)$ *given by the recirsive relations* (17) ((17')) *fulfil the inequalities* (16) *for every* $n \geqq 0$ *and there exists* $\lim_{n \to \infty} v_n(x) = \bar{v}(x)$ $\lim_{n \to \infty} w_n(x) = \bar{w}(x)$, $\bar{v}(x) \leqq \bar{w}(x)$.

2. *The problem* (1), (2) *has at least one solution in the interval* $\langle \bar{v}, \bar{w} \rangle$.

3. *Each solution of the problem* (1), (2) *belonging to* $\langle v_0, w_0 \rangle$ *is contained in* $\langle \bar{v}, \bar{w} \rangle$.

4. *If* $G_1(x, t) \geqq 0J$ ($G_2(x, t) \leqq 0J$), *then both functions* $\bar{v}(x)$, $\bar{w}(x)$ *are solutions of* (1), (2).

Proof. In the sense of Lemma 2 the problem (1), (2) is equivalent to the equation

$$u(x) = \left(v(x) + \int_D G_1^+(x, t) f_1(t, u(t))\,\mathrm{d}t\right) + \int_D G_1^-(x, t) f_1(t, u(t))\mathrm{d}t = T_1 u + T_2 u,$$

where T_1 is an isotone and T_2 an antitone operator. Analogous result is obtained in the second case.

Remark 3. Theorem 5 represents a generalization of Theorem 1 in the paper [5].

Remark 4. More general results could be obtained using a Schröder's theorem ([1], p. 293).

The theory of pseudometric spaces yields great consequences for the theorems on existence and uniqueness of fixed points of functional operators. The basic facts of that theory are mentioned in [1], p. 40—44. A very general theorem on existence and uniqueness of the solutions of operator equations in pseudometric space was proved by a German mathematician J. Schröder ([1], p. 164—269). This theorem comprises Banach's Theorem and, slightly modified, the Kantorovič fixed point theorem ([6], p. 358). For the sake of simplicity, it will be mentioned here in a weaker form (the operator P will be supposed to be linear).

Schröder's fixed point theorem in a weaker form. *Let equation* (9) *be given and assume the following conditions hold*:

1. *The domain X of the operator T is contained in a complete pseudometric space R with the associated partially ordered linear space H. $TX \subset R$.*

2. *The operator T is bounded, that is, there exists a linear, continuous, and positive operator P defined on H, $PH \subset H$, with the property*

$$\varrho(Tu, Tw) \leqq P\varrho(u, w) \qquad \text{for each pair } u, w \in X. \tag{18}$$

3. *If $u_0 \in X$ is given, then the sequence σ_n defined by*

$$\sigma_n = P\sigma_{n-1} + \varrho(u_0, Tu_0), \qquad \sigma_0 = 0$$

converges. Its limit will be denoted by σ.

4. *The sphere γ of elements $w \in R$ satisfying the inequality*

$$\varrho(w, Tu_0) \leqq \sigma - \varrho(u_0, Tu_0) \tag{19}$$

is contained in X, or

4.′ *X is complete and all u_n given recursively by*

$$u_n = Tu_{n-1}, \qquad n = 1, 2, \ldots, \tag{20}$$

are contained in X.

Then there exists at least one solution of the equation (9) *and the sequence u_n, given by* (20), *converges to such a solution. All u_n and u are contained in γ and the following estimate*

$$\varrho(u, u_n) \leqq \sigma - \sigma_n$$

holds.

Remark 5. The conditions **3** any **4** can be replaced by stronger conditions

3.′ $\sum_{j=0}^{\infty} p^j f$ exists for each $f \in H$. ($P^0 = I$ means the identity operator.) (It suffices to consider only $f \geqq 0$.)

4.″ The sphere γ of elements $w \in R$ satisfying the inequality

$$\varrho(w, Tu_0) \leqq (I - P)^{-1}\varrho(u_0, Tu_0) - \varrho(u_0, Tu_0)$$

as well as u_0, are contained in X.

Theorem on uniqueness. *Under the assumptions* 1 *through* 4 *of the last theorem the sphere γ given by* (19) *contains at most one solution of the equation* (9).

Lemma 4. *If the assumptions* 1, 2 *and* 3′ *of the weakened Schröder's fixed point theorem hold, whereby R need not be complete and P continuous, then there exists at most one solution of* (9) *in X.*

Proof. Obviously P is isotone. Assume $w_1 = Tw_1$, $w_2 = Tw_2$. By (18), then it is $\varrho(w_1, w_2) = f \leqq Pf$ and further, $f \leqq Pf \leqq P^2 f \leqq \ldots$. Hence ($0 \leqq$) $f \leqq \frac{1}{n+1}\sum_{j=0}^{n} P^j f$. Since $\lim_{n\to\infty} \frac{1}{n+1}\sum_{j=0}^{n} P^j f = 0$, it follows that $f = 0$.

As an application of the Schröder's theorem the following theorem will be mentioned here (compare with an analogous Schröder's theorem in [1], p. 202).

Theorem 6. *Let Assumptions* 1, 2 *and* 3 *hold. Let there exist a matrix function* $N(x)$, *bounded and measurable on* D *such that for every* (x, u_1) *and* (x, u_2) *in* E^0

$$|f(x, u_1) - f(x, u_2)| \leqq N(x) \, |u_1 - u_2|.$$

Let the greatest positive eigenvalue λ *(provided positive eigenvalues exist) of the operator* P *defined by*

$$Pu = \int_D |G(x, t)| \, N(t) \, u(t) \, \mathrm{d}t$$

satisfy the inequality $\lambda < 1$.

Then the following is true:

1. *There exists at most one solution of the problem* (1), (2).

2. *If* $u_0 \in U$ *is chosen, the sequence* u_n *defined for* $n = 1, 2, \ldots,$ *by*

$$L(u_n) = f(x, u_{n-1}(x)), \qquad x \in D$$
$$M(u_n) = g(x), \qquad x \in S$$

converges to the solution u *of the problem* (1), (2), *whereby all* u_n *and* u *are contained in the sphere* γ *of the elements* w *satisfying the inequality*

$$|w(x) - u_1(x)| \leqq \sigma(x) - |u_0(x) - u_1(x)|, \qquad x \in D \cup S$$

where $\sigma(x)$ *is a solution of the equation*

$$\sigma(x) = |u_0(x) - u_1(x)| + P\sigma(x).$$

Here the solution of the problem (1), (2) *satisfying a Hölder's condition is dealt with if in Assumption* 2 *the second part of the alternative holds.*

Proof. Consider the pseudometric space V of all vector functions $f \in C_0(D \cup S)$ with the pseudometric $\varrho(f, g) = |f(x) - g(x)|$. By the convergence in this space is understood the uniform convergence on $D \cup S$. V, as well as each interval contained in it, are complete. The operator P is linear, positive and compact. From the inequality $\lambda < 1$, by the Theorem on alternative ([1], p. 244), it follows that for every $f \geqq 0j, f \in V$, there exists a unique solution $u_f \geqq 0j$ of $u = Pu + f$. Define the sequence σ_n by $\sigma_n = P\sigma_{n-1} + f$, $\sigma_0 = 0$. Then $\sigma_n = \sum_{j=0}^{n-1} P^j f$ and $\sigma_{n-1} \leqq \sigma_n$, $\sigma_n \leqq u_f$ for every $n \geqq 1$. At the same time σ_n form an equicontinuous set of functions. By Ascoli's Theorem there exists their uniform limit σ. From Schröder's theorem and Lemma 4 the assertion of the theorem follows.

As an illustration of possibilities of this theory the Rozenblatt—Nagumo theorem will be generalized. By the PERRON method ([7], p. 216—217) the following theorem can be proved.

Theorem 7. *Let Assumptions* 1, 2 *and* 3 *be satisfied, whereby let* S *be the* *boundary of D and $G(x, t)$ need not have the property 2 from Assumption 2. Further assume that:*

a. There exists a constant $N > 0$ such that

$$|f(x, u_1) - f(x, u_2)| \leqq \frac{N}{|x, S|} |u_1 - u_2|$$

for every (x, u_1), $(x, u_2) \in E^0$.

$$b.\ N \int_D |G(x, t)| j \, dt \leqq |x, S| j.$$

c. For any two solutions u_1, u_2 of the problem (1), (2) *there exists*

$$\lim_{y \to x} \frac{|u_1(y) - u_2(y)|}{|y, S|} = 0, \qquad \textit{for each } x \in S.$$

Then there exists at most one solution of the problem (1), (2) *(satisfying a Hölder's condition if the second part of alternative in Assumption 2 is valid).*

Proof. For any two solutions u_1, u_2 of the problem (1), (2) (satisfying a Hölder's condition if need be) the inequality

$$|u_1(x) - u_2(x)| \leqq N \int_D |G(x, t)| \frac{|u_1(t) - u_2(t)|}{|t, S|} \, dt$$

holds. The function $p(x) = \dfrac{|u_1(x) - u_2(x)|}{|x, S|}$, $x \in D$, $p(x) = 0$, $x \in S$, is continuous on $D \cup S$. If $p(x) \not\equiv 0j$, then $||p(x)|| = p > 0$. Moreover,

$$N \int_D |G(x, t)| \, p(t) \, dt < Np \int_D |G(x, t)| j \, dt \leqq p \, |x, S| j.$$

Combining the last inequality with the foregoing one, there results finally $p(x) < pj$ for each $x \in D$ but this leads to a contradiction.

Remark 6. The assertion of the theorem remains valid if the points b. and c. are replaced by the points:

b.' $$N \int_D |G(x, t)| j \, dt < |x, S| j.$$

c.' For any two solutions u_1, u_2 of the problem (1), (2) there exists a finite $\lim\limits_{\substack{y \to x \\ y \in D}} \dfrac{|u_1(y) - u_2(y)|}{|y, S|}$ for each $x \in S$, which is continuous on S.

The developed theory will be illustrated on the following example.

Let $f(x, u) = (f_1(x, u), \ldots, f_m(x, u)) \in C_0(\langle 0, 1 \rangle \times R^m)$ be a vector function of the variables x, $u = (u_1, \ldots, u_m)$, let it be periodic in x of period 1, $f(x, u) \equiv \equiv f(x + 1, u)$. Consider the periodic boundary-value problem

(21) $$u' = f(x, u)$$

(22) $$u(0) - u(1) = 0.$$

By [8], p. 718, the problem is equivalent to the integral equation

$$u(x) = \int_0^1 G(x, t)\,[f(t, u(t)) - u(t)]\,\mathrm{d}t$$

where the matrix function $G(x, t)$ is of the form

$$G(x, t) = \begin{cases} \dfrac{1}{1-e}\, e^{x-t} J_0, & 0 \leqq t \leqq x \leqq 1 \\ \dfrac{e}{1-e}\, e^{x-t} J_0, & 0 \leqq x < t \leqq 1 \end{cases}$$

It is easy to see that Assumptions 1, 2 and 3, as well as the assumptions of Lemma 1, are satisfied. Further $G(x, t) \leqq 0J$. The function $H(x) = \int_0^1 |G(x, t)|\,\mathrm{d}t = J_0$. Hence $C = \|H(x)\| = 1$. Let $K_b = \max_{k=1,\ldots,m} \max_{\substack{0 \leqq x \leqq 1 \\ |u| \leqq bj}} |f_k(x, u) - u_k|$.

Consider the operator P_j, $j = 1, 2$, given by the relation

$$P_1 u = \int_0^1 G(x, t)\, F(t)\, u(t)\,\mathrm{d}t, \qquad P_2 u = \int_0^1 |G(x, t)|\, N(t)\, u(t)\,\mathrm{d}t,$$

where $F(x)$ is a matrix function satisfying the conditions mentioned in Theorem 2 on $\langle 0, 1\rangle$ and $N(x) \in C_0(\langle 0, 1\rangle)$ is a matrix function, $u(x) \in C_0(\langle 0, 1\rangle)$ is any vector function. Then $\|P_1\| \leqq (L_1 + \ldots + L_m)$, $\|P_2\| \leqq \|N\|\, m$.

From Theorems 1, 2 and 6 these sufficient conditions for the existence of the solution of the problem (21), (22) follow.

Theorem 8. *The following statements hold*:

1. *If there exists a* $b > 0$, *for which* $m\, K_b \leqq b$ *(especially, if* $f(x, u) - u$ *is bounded on* $\langle 0, 1\rangle \times R^m$*), then there exists at least one solution of the problem* (21), (22) *in the interval* $\langle -bj, bj\rangle$.

2. *If* $|f_k(x, u) - u_k| \leqq (N + \sum_{l=1}^{m} + L_l|u_l|)$, $k = 1, \ldots, m$, $x \in \langle 0, 1\rangle$, $u \in R^m$, $N \geqq 0$, $L_l \geqq 0$, $l = 1, \ldots, m$ *are constants and*

$$\sum_{l=1}^{m} L_l < 1,$$

then there exists at least one solution of the problem (21), (22).

3. *If* $|f(x, u_1) - u_1 - f(x, u_2) + u_2| \leqq N(x)\,|u_1 - u_2|$, *where for the matrix function* $N(x) \in C_0(\langle 0, 1\rangle)$ *the inequality* $\|N(x)\| < \dfrac{1}{m}$ *holds, then there exists a unique solution of* (21), (22).

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