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ON THIRD ORDER ADVANCED NONLINEAR DIFFERENTIAL EQUATIONS*

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Abstract. The aim of our report is to present some results concerning the oscillatory and asymptotic properties of solutions of nonlinear differential equations of the third order with deviating argument. In particular, comparison results for properties A and B are stated. Obtained results extend some other ones known for nonlinear differential equations without deviating argument.

Key words. oscillation theory, nonlinear equation, deviating argument, quasiderivatives.

AMS subject classifications. 34K11

1. Introduction. We consider the third-order nonlinear differential equations with deviating argument of the form

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}x'(t)\right)'\right)' + q(t)f(x(h(t))) = 0\,, \quad t \geq 0 \tag{N,h}$$

and

$$\left(\frac{1}{r(t)}\left(\frac{1}{p(t)}z'(t)\right)'\right)' - q(t)f(z(h(t))) = 0\,, \quad t \geq 0 \tag{N$^{\mathcal{A}}$,h}$$

where

$$r, p, q, h \in C([0,\infty), R),\ r(t) > 0,\ p(t) > 0,\ q(t) > 0 \text{ on } [0,\infty), \tag{H1}$$

$$f \in C(R,R), \quad f(u)u > 0 \quad \text{for} \quad u \neq 0, \tag{H2}$$

$$\int_0^\infty r(t)\ \mathrm{d}t = \int_0^\infty p(t)\ \mathrm{d}t = \infty, \tag{H3}$$

$$\lim_{t\to\infty} h(t) = \infty. \tag{H4}$$

Without mentioning them again, we shall assume the validity of conditions (H1)–(H4) throughout the paper.

The notation (N$^{\mathcal{A}}$,h) is suggested by the fact that for linear equation without deviating argument, i.e., for the equation

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}x'(t)\right)'\right)' + q(t)x(t) = 0, \tag{L}$$

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adjoint equation is

$$\left(\frac{1}{r(t)}\left(\frac{1}{p(t)}z'(t)\right)'\right)' - q(t)z(t) = 0\,. \tag{$\mathrm{L}^{\mathcal{A}}$}$$

If x is a solution of (N,h), then the functions

$$x^{[0]} = x, \qquad\qquad x^{[1]} = \frac{1}{r}x',$$

$$x^{[2]} = \frac{1}{p}\left(\frac{1}{r}x'\right)' = \frac{1}{p}\left(x^{[1]}\right)', \qquad x^{[3]} = \frac{1}{q}\left(\frac{1}{p}\left(\frac{1}{r}x'\right)'\right)' = \frac{1}{q}\left(x^{[2]}\right)'$$

are called the *quasiderivatives* of x. For ($\mathrm{N}^{\mathcal{A}}$,h) we can proceed in a similar way. The linear case of equations (N,h), ($\mathrm{N}^{\mathcal{A}}$,h) denote by (L,h), ($\mathrm{L}^{\mathcal{A}}$,h), respectively. For simplicity, when $h(t) \equiv t$, we will denote (N,h) and ($\mathrm{N}^{\mathcal{A}}$,h) with (N) and ($\mathrm{N}^{\mathcal{A}}$), respectively. In addition to (H1)–(H4), we sometimes assume

$$\liminf_{|u|\to\infty} \frac{f(u)}{u} > 0\,, \tag{H5}$$

$$\limsup_{u\to 0} \frac{f(u)}{u} < \infty\,. \tag{H6}$$

By a *solution* of an equation of the form (N,h) [($\mathrm{N}^{\mathcal{A}}$,h)] we mean a function $w \in C^1([0,\infty),\mathbb{R})$ such that $w^{[1]}(t)$, $w^{[2]}(t) \in C^1([0,\infty),\mathbb{R})$ satisfying equation (N,h) [($\mathrm{N}^{\mathcal{A}}$,h)] for all $t \geq 0$. Any solution of (N,h) or ($\mathrm{N}^{\mathcal{A}}$,h) is said to be *proper* if it is defined on the interval $[0,\infty)$ and is nontrivial in any neighborhood of infinity. A proper solution is said to be *oscillatory* (*nonoscillatory*) if it has (has not) a sequence of zeros converging to ∞. In addition, (N,h) [($\mathrm{N}^{\mathcal{A}}$,h)] is called *oscillatory* if it has at least one nontrivial oscillatory solution and *nonoscillatory* if all its solutions are nonoscillatory. The study of asymptotic behavior of solutions, in the ordinary case as well as in the case with deviating argument, is often connected by introducing the concepts of *equation with property A* and *equation with property B*.

Definition 1.1. Equation (N,h) is said to have *property A* if any proper solution x of (N,h) is either oscillatory or satisfies

$$|x^{[i]}(t)| \downarrow 0 \quad \text{as} \quad t \to \infty \qquad \text{for} \quad i = 0,1,2$$

and equation ($\mathrm{N}^{\mathcal{A}}$,h) is said to have *property B* if any proper solution z of ($\mathrm{N}^{\mathcal{A}}$,h) is either oscillatory or satisfies

$$|z^{[i]}(t)| \uparrow \infty \quad \text{as} \quad t \to \infty \qquad \text{for} \quad i = 0,1,2.$$

The notations $u(t) \downarrow 0$ and $u(t) \uparrow \infty$ mean that function u monotonically decreases to zero as $t \to \infty$ or monotonically increases to infinity as $t \to \infty$, respectively.

From a slight modification of the well-known lemma of Kiguradze (see, e.g., [7]) it follows that the set $\mathcal{N}$ [(N,h)] of all proper nonoscillatory solutions of equation (N,h) can be divided into the following two classes in the same way as in [4]:

$$\mathcal{N}_0 = \{x \in \mathcal{N}\,[(\mathrm{N,h})],\ \exists\, T_x \,:\, x(t)x^{[1]}(t) < 0,\ x(t)x^{[2]}(t) > 0 \ \text{ for } t \geq T_x\}$$

$$\mathcal{N}_2 = \{x \in \mathcal{N}\,[(\mathrm{N,h})],\ \exists\, T_x \,:\, x(t)x^{[1]}(t) > 0,\ x(t)x^{[2]}(t) > 0 \ \text{ for } t \geq T_x\}$$

Similarly, the set $\mathcal{N}$ ($\mathrm{N}^{\mathcal{A}}$,h) of all proper nonoscillatory solutions of equation ($\mathrm{N}^{\mathcal{A}}$,h) can be divided into the following two classes:

$$\mathcal{M}_1 = \{z \in \mathcal{N}\,[(\mathrm{N}^{\mathcal{A}},\mathrm{h})],\, \exists\, T_z \,:\, z(t)z^{[1]}(t) > 0,\, z(t)z^{[2]}(t) < 0 \text{ for } t \geq T_z\}$$

$$\mathcal{M}_3 = \{z \in \mathcal{N}\,[(\mathrm{N}^{\mathcal{A}},\mathrm{h})],\, \exists\, T_z \,:\, z(t)z^{[1]}(t) > 0,\;\, z(t)z^{[2]}(t) > 0 \text{ for } t \geq T_z\}$$

It is clear that (N,h) has property A if and only if all nonoscillatory solutions x of (N,h) belong to the class $\mathcal{N}_0$ and $\lim_{t\to\infty} x^{[i]}(t) = 0$ for $i = 0, 1, 2$. Similarly ($\mathrm{N}^{\mathcal{A}}$,h) has property B if and only if all nonoscillatory solutions z of ($\mathrm{N}^{\mathcal{A}}$,h) belong to the class $\mathcal{M}_3$ and $\lim_{t\to\infty} |z^{[i]}(t)| = \infty$ for $i = 0, 1, 2$. We recall that solutions in the class $\mathcal{N}_0$ are called *Kneser solutions* and solutions belong to the class $\mathcal{M}_3$ are called *strongly monotone solutions*.

Note that the classification of nonoscillatory solutions of equation (N,h) [($\mathrm{N}^{\mathcal{A}}$,h)] is different if we consider differential equation in so called non-canonical form, i.e. in the case $\int^\infty r(t)\;\mathrm{d}t < \infty$ or $\int^\infty p(t)\;\mathrm{d}t < \infty$ (see [2, 3, 6]).

The oscillatory and asymptotic properties of solutions of differential equations of the third order with quasiderivatives (linear, nonlinear and with delay) have been largely investigated in [2]–[6], [8], [9].

The aim of this paper is to continue in study of such equations with deviating argument and with advanced argument. We want to complete relationships between mentioned equations which did not investigate yet. Our research is based on a study of asymptotic behavior of nonoscillatory solutions of (N,h) and ($\mathrm{N}^{\mathcal{A}}$,h), on a linearization device as well as on a comparison result between equations with different deviating arguments. The paper is organized as follows: Section 2 summarizes results which will be useful in the sequel. In the Section 3 we give a comparison theorem for properties A and B, which is more suitable for application than others existing in the literature. This theorem extends Theorem 4 in [6]. As consequence we obtain sufficient conditions ensuring property A for (N,h) and property B for ($\mathrm{N}^{\mathcal{A}}$,h) as well as a comparison result on property A between nonlinear equations without and with deviating argument. In the last section we introduce some interesting open problems.

We point out that our assumptions on nonlinearity f are related with its behavior only in a neighbourhood of zero and/or of infinity. No monotonicity conditions are required as well as no assumptions involving the behavior of f in $\mathbb{R}$ are supposed.

2. Preliminary results.

We introduce the following notation:

$$I(u_i) = \int_0^\infty u_i(t)\;\mathrm{d}t, \qquad I(u_i, u_j) = \int_0^\infty u_i(t) \int_0^t u_j(s)\,ds\;\mathrm{d}t, \quad i, j = 1, 2$$

$$I(u_i, u_j, u_k) = \int_0^\infty u_i(t) \int_0^t u_j(s) \int_0^s u_k(b)\;\mathrm{d}b\;\mathrm{d}s\;\mathrm{d}t, \quad i, j, k = 1, 2, 3,$$

where u_i, $i = 1, 2, 3$ are continuous positive functions on $[0, \infty)$.
For simplicity, sometimes we will write $u(\infty)$ instead of $\lim_{t\to\infty} u(t)$.

In the recent papers [2, 3, 6] authors have studied relationships among properties A and B and both the oscillation and the asymptotic behavior of nonoscillatory solutions for linear equations without deviating argument. We recall some of these results which will be useful in the sequel.

Theorem 2.1. [2, Theorem 2.2] *The following assertions are equivalent:*

(i) (L) *has property A.*

(i′) ($\mathrm{L}^{\mathcal{A}}$) *has property B.*

(ii) (L) *is oscillatory and* $I(q,p,r)=\infty$*.*

(ii') ($\mathrm{L}^{\mathcal{A}}$) *is oscillatory and* $I(q,p,r)=\infty$*.*

LEMMA 2.2. [2, Lemma 2.1] *If there exists a Kneser solution x of equation* (L) *such that* $\lim_{t\to\infty} x^{[i]}(t)=0$ *for $i=0$, 1, 2, then* $I(q,p,r)=\infty$*.*

The following comparison theorem and a result on Kneser solutions we will use in our consideration.

THEOREM 2.3. [3, Theorem 1] *Let the following condition be satisfied:*

$$\begin{aligned} &\text{either} && I(q,r)=\infty \\ &\text{or} && \limsup_{t\to\infty}\int_0^t p(s)\,ds\int_t^\infty q(s)\,\frac{\int_0^s r(u)\int_0^u p(v)\,dv\,du}{\int_0^s p(u)\,du}\,ds=\infty \end{aligned} \tag{2.1}$$

If for some $K>0$ the equation

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}x'(t)\right)'\right)'+Kq(t)x(t)=0 \tag{$\mathrm{L_K}$}$$

has property A, then the equation

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}x'(t)\right)'\right)'+kq(t)x(t)=0 \tag{$\mathrm{L_k}$}$$

has property A for every $k>0$.

PROPOSITION 2.4. [3, Proposition 6] *Every Kneser solution of* (L) *tends to zero for* $t\to\infty$ *if and only if* $I(q,p,r)=\infty$*.*

REMARK 1. *From* THEOREM 2.1 *and* PROPOSITION 2.4 *it follows the following statement: If* (L) *is oscillatory and it has not property A, then* (L) *has Kneser solution tending to nonzero limit and* $I(q,p,r)<\infty$*.*

To extend known results to differential equations with deviating argument we will use the following comparison criterion. It is a particular case of a more general theorem which is stated in [7] for functional differential equations of higher order.

THEOREM 2.5. [7, Theorem 1] *Consider the differential equations* $(i=1,2)$

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}x'(t)\right)'\right)'+q_i(t)x(h_i(t))=0 \tag{$\mathrm{L,h_i}$)$_\mathrm{i}$}$$

$$\left(\frac{1}{r(t)}\left(\frac{1}{p(t)}z'(t)\right)'\right)'-q_i(t)z(h_i(t))=0 \tag{$\mathrm{L^{\mathcal{A}},h_i}$)$_\mathrm{i}$}$$

where $q_i,h_i\in C([0,\infty),R)$, $q_i(t)>0$, $\lim\limits_{t\to\infty}h_i(t)=\infty$ *and*

$$h_1(t)\le h_2(t),\quad q_1(t)\le q_2(t),\quad for\ t>t_0\ge 0.$$

If $(\mathrm{L},\mathrm{h}_1)_1$ *has property A then* $(\mathrm{L},\mathrm{h}_2)_2$ *has property A.*
If $(\mathrm{L}^{\mathcal{A}},\mathrm{h}_1)_1$ *has property B then* $(\mathrm{L}^{\mathcal{A}},\mathrm{h}_2)_2$ *has property B.*

Independently on properties A and B, it is easy to show the following:

LEMMA 2.6. [4, Lemma 1.1] *It holds:*

i) *Any solution* x *of* (L,h) [(N,h)] *from* $\mathcal{N}_0$ *satisfies* $\lim_{t\to\infty} x^{[i]}(t) = 0,\ i = 1, 2.$

ii) *Any solution* z *of* ($\mathrm{L}^{\mathcal{A}}$,h) [($\mathrm{N}^{\mathcal{A}}$,h)] *from* $\mathcal{M}_3$ *satisfies* $\lim_{t\to\infty} |z^{[i]}(t)| = \infty,\ i = 0, 1.$

3. Main results. We begin by introducing the following comparison theorem.

THEOREM 3.1. *Assume* (H5) *and* $h(t) \geq t$. *If equation* (L_k) *has property A for every* $k > 0$, *then* (N,h) *has property A and* ($\mathrm{N}^{\mathcal{A}}$,h) *has property B.*

Proof. a) Let us prove that (N,h) has property A.
Let x be a proper nonoscillatory solution of (N,h). We may assume that there exists $T \geq 0$ such that $x(t) > 0$ for all $t \geq T$. The case $x(t) < 0$ for all $t \geq T^*$ may be proved by using similar arguments. We know that $x \in \mathcal{N}_0 \cup \mathcal{N}_2$. Now we assume that (N,h) has not property A. By LEMMA 2.6 there are two possibilities:

I. $x \in \mathcal{N}_2$,

II. $x \in \mathcal{N}_0$ such that $\lim_{t\to\infty} x(t) = l > 0$.

<u>Case I.</u> Let $x \in \mathcal{N}_2$. We consider the linearized differential equation with deviating argument

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}w'(t)\right)'\right)' + q(t)F_1(t)w(h(t)) = 0\,, \tag{$\mathrm{L}_{\mathrm{F}_1}$,h}$$

where $F_1(t) = \dfrac{f(x(h(t)))}{x(h(t))}$. Then $w \equiv x$ is a nonoscillatory solution. In view of the fact $x \in \mathcal{N}_2$ we have that ($\mathrm{L}_{\mathrm{F}_1}$,h) has not property A.
As $x^{[1]}$ is a positive increasing function, there exists $T \geq 0$ such that $x^{[1]}(t) \geq x^{[1]}(T)$ for all $t \geq T$. Integrating this inequality in (T, t) we get

$$x(t) \geq x(T) + x^{[1]}(T) \int_T^t r(s)\, ds.$$

As $t \to \infty$ we get that function $x(t)$ is unbounded.
In view of the facts $x(\infty) = \infty$ and assumption (H5), there exist a positive constant k_1 and $T_1 \geq 0$ such that $F_1(t) > k_1$ for all $t \geq T_1$. Hence by THEOREM 2.5 for $q_1(t) = q(t)k_1$, $q_2(t) = q(t)F_1(t)$, $h_1(t) = t$, $h_2(t) = h(t)$ we obtain that linear differential equation

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}w'(t)\right)'\right)' + k_1 q(t)w(t) = 0 \tag{$\mathrm{L}_{\mathrm{k}_1}$}$$

has not property A, which is a contradiction with the assumption that (L_{k}) has property A for all $k > 0$.

<u>Case II.</u> Let $x \in \mathcal{N}_0$ and $\lim_{t\to\infty} x(t) = l > 0$. Hence, there exists a positive constant c such that

$$x(t) \geq c > 0 \qquad \text{for } t \text{ sufficiently large.} \tag{3.1}$$

We consider the linearized differential equation

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}w'(t)\right)'\right)' + q(t)F_2(t)w(t) = 0\,, \tag{$\mathrm{L_{F_2}}$}$$

where $F_2(t) = \dfrac{f(x(h(t)))}{x(t)}$. As $w \equiv x$ is a nonoscillatory solution such that $x \in \mathcal{N}_0$ and $x(\infty) > 0$, ($\mathrm{L_{F_2}}$) has not property A. In view of continuity of function f and (3.1), there exist a positive constant k_2 and $T_2 \geq 0$ such that $F_2(t) > k_2$ for all $t \geq T_2$. Hence by Theorem 2.5 for $q_1(t) = q(t)k_2$, $q_2(t) = q(t)F_2(t)$, $h_1(t) = h_2(t) = t$ we obtain that linear differential equation

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}w'(t)\right)'\right)' + k_2 q(t)w(t) = 0 \tag{$\mathrm{L_{k_2}}$}$$

has not property A, which is a contradiction with the assumption that ($\mathrm{L_k}$) has property A for all $k > 0$.

b) Let us prove that ($\mathrm{N}^{\mathcal{A}}$,h) has property B.
Let z be a proper nonoscillatory solution of ($\mathrm{N}^{\mathcal{A}}$,h). We may assume that there exists $T \geq 0$ such that $z(t) > 0$ for all $t \geq T$. The case $z(t) < 0$ for all $t \geq T^*$ may be proved by using similar arguments. We know that $z \in \mathcal{M}_1 \cup \mathcal{M}_3$. Now we assume that ($\mathrm{N}^{\mathcal{A}}$,h) has not property B. By Lemma 2.6 there are two possibilities:

I. $z \in \mathcal{M}_3$ such that $\lim\limits_{t\to\infty} z^{[2]}(t) \neq \infty$,

II. $z \in \mathcal{M}_1$.

Case I. We consider, for sufficiently large t, the linearized differential equation with deviating argument

$$\left(\frac{1}{r(t)}\left(\frac{1}{p(t)}w'(t)\right)'\right)' - q(t)F_3(t)w(h(t)) = 0\,, \tag{$\mathrm{L^{\mathcal{A}}_{F_3}}$,h}$$

where $F_3(t) = \dfrac{f(z(h(t)))}{z(h(t))}$. As $w \equiv z$ is a nonoscillatory solution such that ($\mathrm{L^{\mathcal{A}}_{F_3}}$,h) and $\lim\limits_{t\to\infty} z^{[2]}(t) \neq \infty$, ($\mathrm{L^{\mathcal{A}}_{F_3}}$,h) has not property B. Taking into account that $z(\infty) = \infty$ and assumption (H5), there exist a positive constant k_3 and $T_3 \geq 0$ such that $F_3(t) > k_3$ for all $t \geq T_3$. Hence by Theorem 2.5 for $q_1(t) = q(t)k_3$, $q_2(t) = q(t)F_3(t)$, $h_1(t) = t$, $h_2(t) = h(t)$ we obtain that linear differential equation

$$\left(\frac{1}{r(t)}\left(\frac{1}{p(t)}w'(t)\right)'\right)' - q(t)k_3 w(t) = 0 \tag{$\mathrm{L^{\mathcal{A}}_{k_3}}$}$$

has not property B. On the other hand, because equation ($\mathrm{L_k}$) has property A for all $k > 0$ and thus by Theorem 2.1 equation ($\mathrm{L^{\mathcal{A}}_k}$) has property B for all $k > 0$, which is a contradiction.

Case II. Let $x \in \mathcal{M}_1$. Because z is a positive increasing function, there are two possibilities: $z(\infty) = \infty$ or $z(\infty) < \infty$.

If $z(\infty) = \infty$, the proof proceeds as in the *case I* and hence omitted.
Now, we suppose that $z(\infty) < \infty$ and consider the linearized differential equation

$$\left(\frac{1}{r(t)}\left(\frac{1}{p(t)}w'(t)\right)'\right)' - q(t)F_4(t)w(t) = 0\,, \tag{$\mathrm{L}^{\mathcal{A}}_{\mathrm{F}_4}$}$$

where $F_4(t) = \dfrac{f(z(h(t)))}{z(t)}$. As $w \equiv z$ is a nonoscillatory solution such that $z \in \mathcal{M}_1$, ($\mathrm{L}^{\mathcal{A}}_{\mathrm{F}_4}$) has not property B. In view of continuity of function f and $z(\infty) < \infty$, there exist a positive constant k_4 and $T_4 \geq 0$ such that $F_4(t) > k_4$ for all $t \geq T_4$. Hence by Theorem 2.5 for $q_1(t) = q(t)k_4$, $q_2(t) = q(t)F_4(t)$, $h_1(t) = h_2(t) = t$ we obtain that linear differential equation

$$\left(\frac{1}{r(t)}\left(\frac{1}{p(t)}w'(t)\right)'\right)' - q(t)k_4(t)w(t) = 0 \tag{$\mathrm{L}^{\mathcal{A}}_{\mathrm{k}_4}$}$$

has not property B. On the other hand, because equation (L_k) has property A for all $k > 0$ and thus by Theorem 2.1 equation ($\mathrm{L}^{\mathcal{A}}_{\mathrm{k}}$) has property B for all $k > 0$, which is a contradiction. The proof is now complete. □

Remark 2. Unlike other comparison results (see e.g., in [7, Theorem 1]), Theorem 3.1 does not require neither monotonicity assumptions of the nonlinearity in $\mathbb{R}$ nor the domination of the nonlinearity $|f(u)|$ over the linear term $|u|$ in $\mathbb{R}$. Theorem 3.1 will be valid even in the case of the substitution of the assumption (L_k) has property A for all $k > 0$ for the assumptions (2.1) and (L_K) has property A for some $K > 0$ (see Theorem 2.3 and the proof of Theorem 3.1). The identity $h(t) \equiv t$ in Theorem 3.1 both gives [6, Theorem 4] and extends [3, Theorem 3].

Theorem 3.1 together with integral criteria ensuring property A for (L_k) gives the following result.

Corollary 3.2. *Assume $h(t) \geq t$,* (H5) *and one of the following conditions hold:*

(i) $I(q,r) = I(q,p) = \infty$,

(ii) $I(q) = \infty$,

(iii) $I(q,p) < \infty$ *and* $\displaystyle\int_0^\infty r(t)\left(\int_t^\infty q(s)\,ds\right)\left(\int_t^\infty p(s)\int_s^\infty q(a)\,da\,ds\right)\mathrm{d}t = \infty.$

Then (N,h) *has property A and* ($\mathrm{N}^{\mathcal{A}}$,h) *has property B.*

Proof. From [5, Theorems 4 and 5] and [5, Proposition 1] it follows that (L_k) has property A for all $k > 0$. Now, we get the assertion from Theorem 3.1. The proof is finished. □

The following result also holds:

Corollary 3.3. *Assume* (H5) *and $h(t) \geq t$. If every nonoscillatory solution of* (L_k) *is a Kneser solution for any $k > 0$ and $I(q,p,r) = \infty$, then* (N,h) *has property A and* ($\mathrm{N}^{\mathcal{A}}$,h) *has property B.*

Proof. First let us remark that if $I(q,p,r) = \infty$, then $I(kq,p,r) = \infty$ for any positive constant k. By Proposition 2.4 and Lemma 2.6, every Kneser solution x of (L_k) satisfies $\lim_{t\to\infty} x^{[i]}(t) = 0$, $i = 0, 1, 2$. Taking into account that every nonoscillatory solution of (L_k) is a Kneser one, we get that (L_k) has property A for any $k > 0$. Now, Theorem 3.1 yields the assertion. This completes the proof. □

THEOREM 3.1 yields the following comparison result between nonlinear equations without and with deviating argument.

THEOREM 3.4. *Assume* (H5), (H6), $h(t) \geq t$ *and* ($\mathrm{L_k}$) *is oscillatory for all* $k > 0$. *If equation* (*N*) *has property A, then* (N,h) *has property A and* ($\mathrm{N}^{\mathcal{A}}$,h) *has property B.*

Proof. To prove this assertion we will show that *a)* if (N) has property A, then ($\mathrm{L_k}$) has property A for all $k > 0$ and *b)* if ($\mathrm{L_k}$) has property A for all $k > 0$, then (N,h) has property A and ($\mathrm{N}^{\mathcal{A}}$,h) has property B.

a) This part can be proved in the same way as the claim (a) in the proof of [2, Theorem 4.1] is done and thus we omit it.

b) Let ($\mathrm{L_k}$) has property A for all $k > 0$. From THEOREM 3.1 we immediately get that (N,h) has property A and ($\mathrm{N}^{\mathcal{A}}$,h) has property B. Now part *b)* is proved. This completes the proof. □

REMARK 3. If $h(t) \equiv t$ in THEOREM 3.4, we obtain known result concerning property A for (N) and property B for ($\mathrm{N}^{\mathcal{A}}$), see [2, Theorem 4.1].

4. Conclusion and open problems. The canonical form of investigated equations makes the proofs more easily than non-canonical one. For example, in our research we often apply that in the canonical case Kiguradze lemma holds and so we consider only two classes of nonoscillatory solutions of equation (N,h)[($\mathrm{N}^{\mathcal{A}}$,h)]. Some asymptotic properties of these solutions are known (see LEMMA 2.6) and we can use a comparison result between equations with different deviating arguments (THEOREM 2.5) etc. And so a lot of questions and new problems arise.

THE FOLLOWING PROBLEMS REMAIN OPEN:

1. To research the non-canonical case. Does Theorem 3.1 (THEOREM 3.4) hold in this case? We have already a partial answer. THEOREM 3.1 holds in the non-canonical case, if we consider nonlinear differential equation without deviating argument, that is if $h(t) \equiv t$ in equations (N,h), ($\mathrm{N}^{\mathcal{A}}$,h). This will be given elsewhere.
2. To state sufficient conditions for the existence of nonoscillatory solutions from some class $\mathcal{N}_0 - \mathcal{N}_3$, $\mathcal{M}_0 - \mathcal{M}_3$. In the literature there are only results ensuring the existence of nonoscillatory solutions from the class $\mathcal{N}_0$ for linear (see [6] and the references obtained therein) and nonlinear (see [1]) equations without deviating argument. In the linear case similar result is also stated for the class $\mathcal{M}_3$ (see [6]).
3. To obtain also necessary conditions for property A (B) or oscillation.

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