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ON THE NEHARI SOLUTIONS*

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Abstract. We show that there exist equations of the Emden-Fowler type which have multiple Nehari solutions.

Key words. Nehari solutions, characteristic numbers, Emden-Fowler equation

AMS subject classifications. 34B15

1. Introduction.

We consider the Emden-Fowler type equations

$$x'' = -q(t)|x|^{2\varepsilon}x, \quad ' = \frac{\mathrm{d}}{\mathrm{d}t}, \quad \varepsilon > 0, \quad q \in C(R, (0, +\infty)), \tag{1.1}$$

of which the typical representative is equation

$$x'' = -q(t)x^3.$$

Behavior of solutions of equations (1.1) can be highly irregular if the coefficient $q(t)$ is non-monotone. One might expect that some regularity to the theory of the Emden-Fowler superlinear equations is brought by the results of Nehari ([3], [4]), which are variational in nature. Brief description of the Nehari theory is given below.

1.1. Brief overview of the Nehari theory.

Consider differential equation

$$x'' + xF(t, x^2) = 0,$$

where
(A1) $F(t, s) \in C((0, +\infty) \times [0, +\infty), R)$;
(A2) $F(t, s) > 0$ for $t > 0,\ s > 0$;
(A3) $t_2^{-\varepsilon}F(t_2, s) > t_1^{-\varepsilon}F(t_1, s)$ for $0 \leq t_1 < t_2 < \infty$, fixed $s > 0$ and some $\varepsilon > 0$.
The general theorem was proved in [4, Theorem 3.2].

THEOREM 1.1. *Let Γ_n denote the class of functions $x(t)$ with the following properties: $x(t)$ is continuous and piecewise differentiable in $[a, b]$, $x(a_\nu) = 0$ $(\nu = 0, 1 \ldots, n,\ n \geq 1)$, where the a_ν are numbers such that $a = a_0 < a_1 < \ldots < a_n = b$ for $\nu = 1, \ldots, n$, but $x(t) \not\equiv 0$ in any interval $[a_{\nu-1}, a_\nu]$ and*

$$\int_{a_{\nu-1}}^{a_\nu} x'^2(t)\ \mathrm{d}t = \int_{a_{\nu-1}}^{a_\nu} x^2(t)F(t, x^2(t))\ \mathrm{d}t,$$

where F is subject to the conditions **(A1)–(A3)**. *Set $G(t, y) = \int_0^y F(t, s)\ \mathrm{d}s$.*

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The extremal problem

$$\int_a^b [x'^2 - G(t, x^2)]\ \mathrm{d}t = \min = \lambda_n, \qquad x(t) \in \Gamma_n \tag{1.2}$$

has a solution $x_n(t)$ whose derivative is continuous throughout $[a, b]$, and the characteristic values λ_n are strictly increasing with n. The function $x_n(t)$ has precisely $n-1$ zeros in (a, b), and it is a solution of the differential system

$$x'' + xF(t, x^2) = 0, \qquad x(a) = x(b) = 0.$$

1.2. The Nehari numbers for the Emden-Fowler type equations. The Nehari theory applies to the Emden-Fowler type equations of the form

$$x'' = -q(t)|x|^{2\varepsilon}x, \qquad \varepsilon > 0, \quad q \in C(R, (0, +\infty)). \tag{1.3}$$

The extremal problem (1.2) for the case of equation (1.3) takes the form:

$$H(x) = \int_a^b \left[x'^2 - (1+\varepsilon)^{-1} q(t) x^{2+2\varepsilon}\right]\ \mathrm{d}t \to \inf \tag{1.4}$$

over all functions $x(t)$, which are continuous and piece-wise continuously differentiable in $[a, b]$; there exist numbers a_ν such that

$$a = a_0 < a_1 < \ldots < a_n = b;$$

$x(a_0) = 0$ and $x(a_\nu) = 0$ for $\nu = 1, \ldots, n$, but $x \not\equiv 0$ in any $[a_{\nu-1}, a_\nu]$ and

$$\int_{a_{\nu-1}}^{a_\nu} x'^2(t)\ \mathrm{d}t = \int_{a_{\nu-1}}^{a_\nu} q(t) x^2 |x|^{2\varepsilon}\ \mathrm{d}t.$$

The respective extremal functions $x_n(t)$ are solutions of equation (1.3), vanish at the points $t = a$ and $t = b$, have exactly $n-1$ zeros in (a, b) and satisfy the condition

$$\int_a^b x'^2(t)\, dt = \int_a^b q(t) x^2 |x|^{2\varepsilon}\ \mathrm{d}t. \tag{1.5}$$

By combining (1.4) with (1.5) one gets

$$\lambda_n(a, b) = \min_{x \in \Gamma_n} H(x) = \frac{\varepsilon}{1+\varepsilon} \int_a^b q(t) x_n^{2+2\varepsilon}\ \mathrm{d}t = \frac{\varepsilon}{1+\varepsilon} \int_a^b x_n'^2(t)\ \mathrm{d}t.$$

Thus the characteristic number $\lambda_n(a, b)$ is up to a constant the minimal value of $\int_a^b x'^2(t)\ \mathrm{d}t$ over the set of solutions of the boundary value problem

$$x'' = -q(t)|x|^{2\varepsilon}x, \qquad x(a) = x(b) = 0, \qquad x(t) \text{ has } n-1 \text{ zeros in } (a, b).$$

Definition 1.2. We will call the characteristic numbers λ_n by the Nehari numbers and the respective solutions $x_n(t)$ by the Nehari solutions.

It was asked in the paper [3] whether the Nehari solutions are unique for n and (a, b) given. The answer is no, as was shown theoretically in [5]. There exist equations of the type (1.3), which have more than one Nehari solution for certain a and b.

In order to get the constructive proof let us consider the example below.

2. Example: nonuniqueness of the Nehari solutions. We construct the Emden-Fowler equation of the form (2.2) which possesses three solutions which obey the conditions (2.3). Two of those three solutions are the Nehari solutions.

2.1. Lemniscatic functions. We use in our considerations the so called lemniscatic functions which can be defined as solutions of the equation

$$x'' = -2x^3. \tag{2.1}$$

The functions $\operatorname{sl} t$ and $\operatorname{cl} t$ ([6, § 22.8]) solve equation (2.1) and satisfy respectively the initial conditions

$$x(0) = 0, \quad x'(0) = 1 \quad \text{and} \quad x(0) = 1, \quad x'(0) = 0.$$

The lemniscatic sine and cosine functions are periodic with a minimal period of $4A$, where $A = \int\limits_0^1 \frac{\mathrm{d}s}{\sqrt{1-s^4}} \approx 1.311$. For convenient reference we mention that

$$\begin{aligned}
&\operatorname{sl} 0 = \operatorname{sl} 2A = 0, \quad \operatorname{sl} A = 1, \quad \operatorname{cl} 0 = 1, \quad \operatorname{cl} A = 0, \quad \operatorname{cl} 2A = -1,\\
&\operatorname{sl}' t = \operatorname{cl} t\,(1 + \operatorname{sl}^2 t), \quad \operatorname{cl}' t = -\operatorname{sl} t(1 + \operatorname{cl}^2 t),\\
&\lim_{t\to 0} \frac{\operatorname{sl} t}{t} = 1.
\end{aligned}$$

The interested reader may consult the paper [1] for more properties and useful formulas of these functions.

2.2. Equation. Consider the boundary value problem

$$x'' = -q(t)\, x^3, \tag{2.2}$$

$$x(-1) = 0, \quad x(1) = 0, \qquad x(t) > 0, \quad t \in (-1, 1). \tag{2.3}$$

Let $q(t) = \dfrac{2}{\xi^6(t)}$, where

$$\xi(t) = \begin{cases} \xi_1(t), & -1 \le t \le 0,\\ \xi_2(t), & 0 \le t \le 1 \end{cases}$$

and

$$\begin{aligned}
\xi_1(t) &= ht + \eta, & -1 \le t \le 0,\\
\xi_2(t) &= -ht + \eta, & 0 \le t \le 1.
\end{aligned}$$

Thus $\xi(t)$ is a "Λ-shaped" piece-wise linear function, which depends on a positive valued parameter h, $\eta := h + 1$.

2.3. Solutions. Solution (solutions) of the problem (2.2), (2.3) can be composed of solutions of the problems

$$x_1'' = -\frac{2}{(ht+\eta)^6}\, x_1^3, \qquad x_1(-1) = 0,\ x_1(0) = \tau, \qquad x_1(t) > 0,\ t \in (-1, 0); \tag{2.4}$$

$$x_2'' = -\frac{2}{(-ht+\eta)^6}\, x_2^3, \qquad x_2(0) = \tau,\ x_2(1) = 0, \qquad x_2(t) > 0,\ t \in (0, 1), \tag{2.5}$$

where $\tau > 0$. The function

$$x(t) = \begin{cases} x_1(t), & \text{if} \quad -1 \le t \le 0, \\ x_2(t), & \text{if} \quad 0 \le t \le 1, \end{cases}$$

is a C^2-solution of the problem (2.2), (2.3) if additionally the smoothness condition

$$x_1'(0) = x_2'(0)$$

is satisfied. The problem (2.4) has a solution

$$x_1(t;\beta_1) = \beta_1^{\frac{1}{2}}(ht+\eta) \cdot \operatorname{sl}\left(\beta_1^{\frac{1}{2}} \frac{t+1}{ht+\eta}\right),$$

where $\beta_1 = x_1'(-1) > 0$ is such that $x_1(0;\beta_1) = \tau$. The derivative is given by

$$x_1'(t;\beta_1) = \beta_1^{\frac{1}{2}} h \cdot \operatorname{sl}\left(\beta_1^{\frac{1}{2}} \frac{t+1}{ht+\eta}\right) + \frac{\beta_1}{ht+\eta} \cdot \operatorname{sl}'\left(\beta_1^{\frac{1}{2}} \frac{t+1}{ht+\eta}\right).$$

Similar formulas are valid for $x_2(t)$. Notice that $x_2'(1) = -\beta_2 < 0$. In order to get an explicit formula for a solution of the BVP (2.2), (2.3) one has to solve a system of two equations with respect to (β_1, β_2)

$$\begin{cases} x_1(0;\beta_1) = x_2(0;\beta_2), \\ x_1'(0;\beta_1) = x_2'(0;\beta_2). \end{cases}$$

This system after replacements and simplifications looks as

$$\begin{cases} \beta_1^{\frac{1}{2}} \cdot \operatorname{sl}\left(\frac{\beta_1^{\frac{1}{2}}}{\eta}\right) = \beta_2^{\frac{1}{2}} \cdot \operatorname{sl}\left(\frac{\beta_2^{\frac{1}{2}}}{\eta}\right), \\ \beta_1^{\frac{1}{2}} h \cdot \operatorname{sl}\left(\frac{\beta_1^{\frac{1}{2}}}{\eta}\right) + \frac{\beta_1}{\eta} \cdot \operatorname{sl}'\left(\frac{\beta_1^{\frac{1}{2}}}{\eta}\right) = -\beta_2^{\frac{1}{2}} h \cdot \operatorname{sl}\left(\frac{\beta_2^{\frac{1}{2}}}{\eta}\right) - \frac{\beta_2}{\eta} \cdot \operatorname{sl}'\left(\frac{\beta_2^{\frac{1}{2}}}{\eta}\right), \end{cases} \tag{2.6}$$

where $0 < \frac{\beta_1^{\frac{1}{2}}}{\eta}, \frac{\beta_2^{\frac{1}{2}}}{\eta} < 2A$. In new variables $u := \frac{\beta_1^{\frac{1}{2}}}{\eta}, \quad v := \frac{\beta_2^{\frac{1}{2}}}{\eta}$ the system takes the form

$$\begin{cases} \Phi(u) &=& \Phi(v), & 0 < u,\ v < 2A, \\ \Psi_h(u) &=& -\Psi_h(v), & h > 0, \end{cases} \tag{2.7}$$

where $\Phi(z) := z \operatorname{sl} z$ and $\Psi_h(z) := hz \operatorname{sl} z + z^2 \operatorname{sl}' z$. Notice that if a solution $(\bar{u}, \bar{v})$ of the system (2.7) exists, then a solution $x(t)$ of the BVP (2.2), (2.3) can be constructed such that

$$x'(-1) = \beta_1 = \bar{u}^2(h+1)^2, \quad x'(1) = -\beta_2 = -\bar{v}^2(h+1)^2.$$

Proposition 2.1. *For h large enough the system (2.7) has exactly three solutions:*

1. *There exists a unique solution of the form (u_0, u_0). One has that $(u_0, u_0) \to (2A, 2A)$ as $h \to +\infty$.*
2. *There exists a unique solution (u_1, v_1) in the triangle $\{0 < u, v < 2A,\ v > u)\}$ for h large. Moreover, $(u_1, v_1) \to (0, 2A)$ as $h \to +\infty$.*
3. *There exists a unique solution (u_2, v_2) in the triangle $\{0 < u, v < 2A,\ v < u)\}$ for h large. Solutions (u_1, v_1) and (u_2, v_2) are symmetric, that is, $(v_2, u_2) = (u_1, v_1)$.*

2.4. Investigation of a system. Standard analysis shows that the function $\Phi(z) = z \,\mathrm{sl}\, z$ has the following properties (see FIG. 2.1):

$$\Phi(0) = \Phi(2A) = 0, \qquad \Phi(z) > 0 \quad \forall z \in (0, 2A),$$
$$\Phi'(z) = \mathrm{sl}\, z + z \,\mathrm{sl}'\, z,$$

$\Phi_{\max} = \Phi(z_0) \approx 1.47233$ at the unique point of maximum $z_0 \approx 1.61879$.

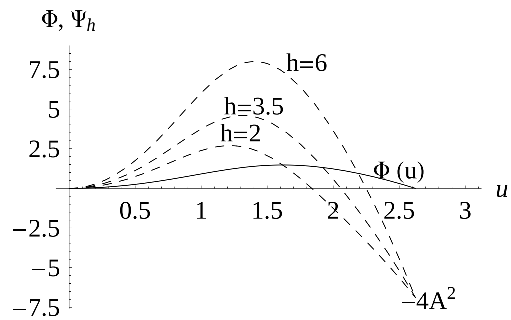

FIG. 2.1. *Functions $\Phi(u)$ (solid line) and $\Psi_h(u)$ (dashed lines).*

Consider a set of zeros of a function $\Phi(u) - \Phi(v)$ in the square $Q = \{(u,v) : 0 \le u, v \le 2A\}$. It consists of the diagonal segment Γ_0 $(u = v)$ and two symmetric branches Γ_+ and Γ_-, which are shown in FIG. 2.2.

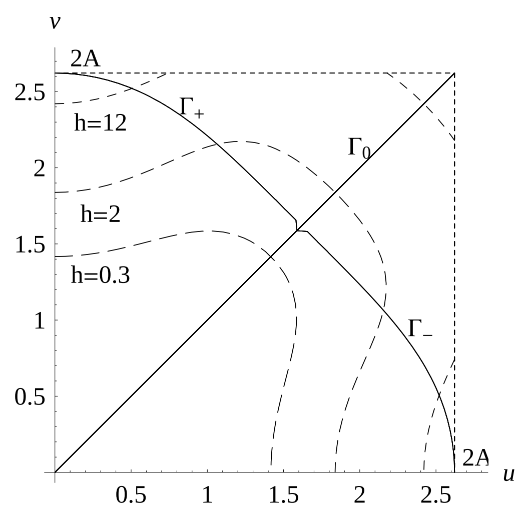

FIG. 2.2. *Zeros of $\Phi(u) - \Phi(v) = 0$ (solid line) and $\Psi_h(u) + \Psi_h(v) = 0$ (dashed line), $h = 0.3$, $h = 2$, $h = 12$.*

LEMMA 2.2. *The relation $F(u,v) = \Phi(u) - \Phi(v) = 0$ defines a function $v = f(u)$ for $u \in [0, 2A]$. One has that $f(0) = 2A$, $f(2A) = 0$, $f'(u) = \left.\frac{\Phi(u)}{\Phi(v)}\right|_{v=f(u)} < 0$ for $u \in [0, 2A]$.*

Proof. A set of zeros of $F(u,v)$ for $v > u$ (branch Γ_+ in FIG. 2.2) can be parametrized by the equalities $\Phi(u) = p$, $\Phi(v) = p$, where $p \in [0, \Phi_{\max}]$. If p changes from 0 to $\Phi_{\max}$, variables u and v respectively increase from 0 to z_0 and decrease from $2A$ to z_0. One gets by using Implicit Function Theorem that there exists function $v = f(u),\ u \in (0, z_0)$ such that $F\big(u, f(u)\big) = 0$ for $u \in (0, z_0)$ and

$$f'(u) = -\left.\frac{\dfrac{\partial F}{\partial u}(u;v)}{\dfrac{\partial F}{\partial v}(u;v)}\right|_{v=f(u)} = \left.\frac{\Phi'(u)}{\Phi'(v)}\right|_{v=f(u)}.$$

Since $\Phi'(u) > 0$ for $u \in (0, z_0)$ and $\Phi'(v) < 0$ for $v \in (z_0, 2A)$ one has that $f'(u) < 0$. The graph of $f(u)$ is the set Γ_+. The same type argument can be applied for (u, v) in the lower triangle, $u > v$. Thus a decreasing function $v = f(u)$ exists for $u \in [0, 2A]$. The graph of this function is the union of Γ_+ and Γ_-. □

Consider the functions $\Psi_h(z)$. We mention the following properties:

$$\Psi_h(0) = 0, \qquad \Psi_h(2A) = -4A^2 \qquad \forall h > 0.$$
$$\Psi_h'(z) = h\Phi'(z) + z\Phi''(z).$$

The function $\Psi_h(z)$ increases for $z \in \big(0, z_{\max}(h)\big)$ and decreases for $z \in \big(z_{\max}(h), 2A\big)$. It is easy to show that $(\Psi_h)_{\max} = \Psi_h(z_{\max}) \to +\infty$ and $z_{\max}(h) \to z_0$ as $h \to +\infty$.

Lemma 2.3. *A set Z of zeros of the function $\Psi_h(u) + \Psi_h(v)$ in the square Q consists of three mutually disjointed sets*

$$\begin{aligned} Z_+ &\subset \big\{(x,y):\ 0 \le x \le \delta,\ 2A - \delta \le y \le 2A\big\}, \\ Z_0 &\subset \big\{(x,y):\ 2A - \delta \le x \le 2A,\ 2A - \delta \le y \le 2A\big\}, \\ Z_- &\subset \big\{(x,y):\ 2A - \delta \le x \le 2A,\ 0 \le y \le \delta\big\}. \end{aligned}$$

It is true that $\delta \to 0$ as $h \to +\infty$.

Proof. Let z_1 be a unique zero of $\Psi_h(z)$ in the interval $(0, 2A)$. Let z_* and z^* be the level points defined by the relations $\Psi_h(z_*) = \Psi_h(z^*) = 4A^2,\ z_* < z^*$. It is clear that the equality $\Psi_h(x) + \Psi_h(y) = 0$ implies the inclusion $(x, y) \in (0, z_*) \bigcup (z^*, 2A)$. Indeed, if $x \in (z_*, z^*)$ then $\Psi_h(x) > 4A^2$ and $\Psi_h(x) + \Psi_h(y) > 0$ for any y. If $\Psi_h(x) + \Psi_h(y) = 0$ then either x or y belongs to $(z_1, 2A)$.

Consider the case $x \in (z_1, 2A)$. Then there are two values of y, say, y_1 and y_2, such that $\Psi_h(x) + \Psi_h(y) = 0,\ y_1 \in (0, z_*)$ and $y_2 \in (z^*, z_1)$. Similarly, if $y \in (z_1, 2A)$, then there are two values of x, x_1 and x_2, such that $x_1 \in (0, z_*)$ and $x_2 \in (z^*, z_1)$. Therefore any point $(x, y) \in Q$ such that $\Psi_h(x) + \Psi_h(y) = 0$ belongs to one of the sets Z_+, Z_0 or Z_-.

Let us show that $z_* \to 0$ and $z^* \to 2A$ as $h \to +\infty$. Both values of z satisfy the relation $\Psi_h(z) = hz\,\mathrm{sl}\,z + z^2\,\mathrm{sl}'\,z = 4A^2$. Then $z\,\mathrm{sl}\,z + \dfrac{1}{h}z^2\,\mathrm{sl}'\,z = \dfrac{1}{h}4A^2$. If $h \to +\infty$ then $z\,\mathrm{sl}\,z \to 0$ and the level points $z_*(h)$ and $z^*(h)$ tend respectively to 0 and $2A$. □

Lemma 2.4. *The relation $G_h(u, v) = \Psi_h(u) + \Psi_h(v) = 0$ defines a function $v = g(u)$ for $u \in [0, \delta]$. One has that $g'(u) = -\left.\frac{\Psi_h'(u)}{\Psi_h'(v)}\right|_{v=g(u)} > 0$ for $u \in [0, \delta]$.*

Proof. [Proof of Proposition 2.1] Consider the set Z_+. The function $v = f(u)$ strictly decreases and satisfies the relation $f(0) = 2A$. The function $v = g(u)$ strictly increases and satisfies the relations $g(0) < 2A$, $g(u_*) = 2A$ for some $u_* \in (0, \delta)$. Therefore there exists a unique point of intersection of the graphs of both functions in Z_+.

By symmetry with respect to the diagonal, the same ir true for the set Z_-. Thus two solutions of the system (2.7).

For $u = v$ the system (2.7) reduces to a single equation $\Psi_h(z) = 0$, which has a unique solution, tending to $2A$ as $h \to +\infty$. Thus exactly free solutions of the system (2.7). $\square$

We give also the alternative proof.

Proof. [Alternative proof of Proposition 2.1] Let us parametrize the upper (the left) branch Γ_+ (for this branch $v > u$) by $\Phi(u) = \Phi(v) = p$, where $0 < p < p_*$, $p_* = \max\limits_{[0,2A]} \Phi(u)$. The function $\Phi(u)$ attains its maximal value $p_* \approx 1.47233$ at the point $m_* \approx 1.61879$.

This branch is then defined parametrically as $u = u(p)$, $v = v(p)$. Notice that $(u(0), v(0)) = (0, 2A)$ and $\big(u(p_*), v(p_*)\big) = (m_*, m_*)$.

Suppose that $h > 1$ and consider the one argument function

$$\omega(p) := hu(p)\ \mathrm{sl}\, u(p) + u^2(p)\, \mathrm{sl}'\, u(p) + hv(p)\ \mathrm{sl}\, v(p) + v^2(p)\, \mathrm{sl}'\, v(p)$$

in the interval $[0, p_*]$. Our intent is to show that this function changes sign only once. Then there exists a unique solution of the system (2.7) on Γ_+ and as a consequence, there exist exactly three solutions of the system (2.7) for $0 < u, v < 2A$.

Since $u(p)\, \mathrm{sl}\, u(p) = v(p)\, \mathrm{sl}\, v(p) = p$, the function ω takes the form

$$\omega(p) = 2hp + u^2(p)\ \mathrm{sl}'\, u(p) + v^2(p)\ \mathrm{sl}'\, v(p).$$

The problem is to show that the function $\omega(p)$ strictly increases on the interval $(0, p_*)$, where $u(p)$ is defined parametrically as $u\ \mathrm{sl}\, u = p, \quad u \in (0, m_*)$, and $v(p)$ is defined by $v\ \mathrm{sl}\, v = p, \quad v \in (m_*, 2A)$.

Consider the first equation in (2.7). Define two functions $x(p)$ and $y(p)$ parametrically using the equalities

$$x\, \mathrm{sl}\, x = y\, \mathrm{sl}\, y = p, \tag{2.8}$$

where $p \in [0, p_*]$, $x : [0, p_*] \to [0, m_*]$, $y : [0, p_*] \to [m_*, 2A]$. The functions $x(p)$ and $y(p)$ are well defined, continuous, but may have infinite derivatives. One has from (2.8) that

$$\frac{\mathrm{d}x}{\mathrm{d}p} = \frac{1}{\mathrm{sl}\, x + x\, \mathrm{sl}'\, x}, \quad \frac{\mathrm{d}y}{\mathrm{d}p} = \frac{1}{\mathrm{sl}\, y + y\, \mathrm{sl}'\, y}.$$

Thus $x(p)$ has infinite derivatives at $p = 0$ and $p = p_*$, and $y(p)$ has infinite derivative at $p = p_*$.

Consider now the second equation in (2.7). We will show that the function

$$\begin{aligned} \omega(p) \quad &= h\, x\, \mathrm{sl}\, x + x^2\, \mathrm{sl}'\, x + h\, y\, \mathrm{sl}\, y + y^2\, \mathrm{sl}'\, y \\ &= h\, (x\, \mathrm{sl}\, x + y\, \mathrm{sl}\, y) + x^2\, \mathrm{sl}'\, x + +y^2\, \mathrm{sl}'\, y \\ &= 2hp + x^2\, \mathrm{sl}'\, x + +y^2\, \mathrm{sl}'\, y \end{aligned}$$

is strictly increasing in p for h large enough.

One has that

$$\begin{aligned} \frac{\mathrm{d}\,\omega(p)}{\mathrm{d}p} \quad &= 2h + 2xx'\ \mathrm{sl}'\, x + x^2\, \mathrm{sl}''\, x\, x' + 2yy'\ \mathrm{sl}'\, y + y^2\, \mathrm{sl}''\, y\, y' \\ &= 2h + xx'(2\, \mathrm{sl}'\, x + x\, \mathrm{sl}''\, x) + yy'(2\, \mathrm{sl}'\, y + y\, \mathrm{sl}''\, y). \end{aligned} \tag{2.9}$$

Since $\mathrm{sl}\, x$ is a bounded periodic function together with the derivatives $\mathrm{sl}'\, x$ and $\mathrm{sl}''\, x$, the expressions in parentheses are bounded.

Let us evaluate the products $x(p)\, x'(p)$ and $y(p)\, y'(p)$.

One has for the second one that $\frac{dy}{\mathrm{d}p} = \frac{1}{\operatorname{sl} y + y \operatorname{sl}' y}$ at $p = 0$ is $\frac{dy}{\mathrm{d}p} = \frac{1}{\operatorname{sl} 2A + 2A \operatorname{sl}' 2A} = -\frac{1}{2A}$ and $y(0)\frac{dy}{\mathrm{d}p}(0) = 2A \cdot \left(-\frac{1}{2A}\right) = -1$.

Evaluation of $x(p)\, x'(p)$ follows. One gets by using the l'Hospital's rule

$$\left(x' = \frac{1}{\operatorname{sl} x + x \operatorname{sl}' x} \Rightarrow x'' = -x' \frac{\operatorname{sl}' x x' + x' \operatorname{sl}' x + x x' \operatorname{sl}'' x}{(\operatorname{sl} x + x \operatorname{sl}' x)^2} = -x' \frac{2 \operatorname{sl}' x + x \operatorname{sl}'' x}{(\operatorname{sl} x + x \operatorname{sl}' x)^2}\right)$$

$$\begin{aligned}
\lim_{p\to 0} x(p)\, x'(p) &= \lim_{p\to 0} \frac{x'(p)}{\frac{1}{x(p)}} = \lim_{p\to 0} \frac{x''(p)}{\frac{-1}{x^2(p)}\, x'(p)} \\
&= \lim_{p\to 0} \frac{2 \operatorname{sl}' x + x \operatorname{sl}'' x}{(\operatorname{sl}^2 x + 2 \operatorname{sl} x \cdot \operatorname{sl}' x \cdot x + x^2 \operatorname{sl}' x) \cdot \frac{1}{x^2}} \\
&= \lim_{p\to 0} \frac{2 \operatorname{sl}' x - 2x \operatorname{sl}^3 x}{\left(\frac{\operatorname{sl} x}{x}\right)^2 + 2\left(\frac{\operatorname{sl} x}{x}\right) \operatorname{sl}' x + \operatorname{sl}' x} = \frac{2 \operatorname{sl}' 0}{1 + 2 \operatorname{sl}' 0 + \operatorname{sl}' 0} = \frac{1}{2}.
\end{aligned}$$

Similarly can be shown that $\lim\limits_{p\to 0} y(p)\, y'(p)$, $\lim\limits_{p\to p_*} x(p)\, x'(p)$ and $\lim\limits_{p\to p_*} y(p)\, y'(p)$ are finite. Then the last two addends in (2.9) are finite in the interval $[0, p_*]$ and for h large enough $\dfrac{\mathrm{d}\,\omega(p)}{\mathrm{d}p}$ is positive. Since $\omega(0) < 0$ and $\omega(p_*) > 0$, this function can change sign only once. Thus only one zero of the system (2.7) in the upper triangle. Totally exactly three solutions. ∎

2.5. Integrals. The Nehari number $\lambda\,(-1, 1)$ we are looking for is a minimal value of the functional

$$H(x) = \frac{1}{2} \int_{-1}^{1} x'^2(t)\ \mathrm{d}t \quad \text{or that of} \quad H(x) = \frac{1}{2} \int_{-1}^{1} q(t) x^4(t)\ \mathrm{d}t$$

over all solutions of the BVP.

Notice that

$$H(x) = \frac{1}{2} \int_{-1}^{0} q_1(t) x_1^4(t)\ \mathrm{d}t + \frac{1}{2} \int_{0}^{1} q_2(t) x_2^4(t)\ \mathrm{d}t = J_1 + J_2,$$

where $q_i(t) = \dfrac{2}{\xi_i^6(t)}$, $i = 1, 2$.

Computation yields

$$H(x) = \frac{1}{3}\, \beta_1^{\frac{3}{2}} \left[\frac{\beta_1^{\frac{1}{2}}}{\eta} - \operatorname{sl}'\left(\frac{\beta_1^{\frac{1}{2}}}{\eta}\right) \operatorname{sl}\left(\frac{\beta_1^{\frac{1}{2}}}{\eta}\right)\right] + \frac{1}{3}\, \beta_2^{\frac{3}{2}} \left[\frac{\beta_2^{\frac{1}{2}}}{\eta} - \operatorname{sl}'\left(\frac{\beta_2^{\frac{1}{2}}}{\eta}\right) \operatorname{sl}\left(\frac{\beta_2^{\frac{1}{2}}}{\eta}\right)\right],$$

where β_1 and β_2 solves the system (2.6), $\eta = h + 1$.

If $x(t)$ is a "symmetric" solution, then $\beta_1 = \beta_2 =: \beta_0$, and the above formula looks as

$$H(x) = 2\, \beta_0^{\frac{3}{2}} \int_{0}^{\beta_0^{\frac{1}{2}}/\eta} \operatorname{sl}^4(z)\, dz = \frac{2}{3}\, \beta_0^{\frac{3}{2}} \left[\frac{\beta_0^{\frac{1}{2}}}{\eta} - \operatorname{sl}'\left(\frac{\beta_0^{\frac{1}{2}}}{\eta}\right) \operatorname{sl}\left(\frac{\beta_0^{\frac{1}{2}}}{\eta}\right)\right].$$

One has that

$$\lim_{h\to+\infty}\frac{\beta_1^{\frac12}}{\eta}=0,\quad \lim_{h\to+\infty}\frac{\beta_0^{\frac12}}{\eta}=2A,\quad \lim_{h\to+\infty}\frac{\beta_2^{\frac12}}{\eta}=2A,$$

$$\begin{aligned}
&\lim_{h\to+\infty}\frac{H_{\rm sym}}{H_{\rm asym}}\\
&=\lim_{h\to+\infty}\frac{\frac23\beta_0^{\frac32}\left[\frac{\beta_0^{\frac12}}{\eta}-\mathrm{sl}'\left(\frac{\beta_0^{\frac12}}{\eta}\right)\mathrm{sl}\left(\frac{\beta_0^{\frac12}}{\eta}\right)\right]}{\frac13\beta_1^{\frac32}\left[\frac{\beta_1^{\frac12}}{\eta}-\mathrm{sl}'\left(\frac{\beta_1^{\frac12}}{\eta}\right)\mathrm{sl}\left(\frac{\beta_1^{\frac12}}{\eta}\right)\right]+\frac13\beta_2^{\frac32}\left[\frac{\beta_2^{\frac12}}{\eta}-\mathrm{sl}'\left(\frac{\beta_2^{\frac12}}{\eta}\right)\mathrm{sl}\left(\frac{\beta_2^{\frac12}}{\eta}\right)\right]}\\
&=2\lim_{h\to+\infty}\frac{\left(\frac{\beta_0^{\frac12}}{\eta}\right)^3\left[\frac{\beta_0^{\frac12}}{\eta}-\mathrm{sl}'\left(\frac{\beta_0^{\frac12}}{\eta}\right)\mathrm{sl}\left(\frac{\beta_0^{\frac12}}{\eta}\right)\right]}{\left(\frac{\beta_2^{\frac12}}{\eta}\right)^3\left[\frac{\beta_2^{\frac12}}{\eta}-\mathrm{sl}'\left(\frac{\beta_2^{\frac12}}{\eta}\right)\mathrm{sl}\left(\frac{\beta_2^{\frac12}}{\eta}\right)\right]}=2.
\end{aligned}$$

3. Conclusion. We have shown that the boundary value problem (2.2), (2.3) for sufficiently large values of parameter h has exactly three nontrivial solutions. One of those solutions is symmetric with respect to $t=0$ and two others are asymmetric as shown in FIG. 3.1. Both asymmetric solutions are the Nehari solutions.

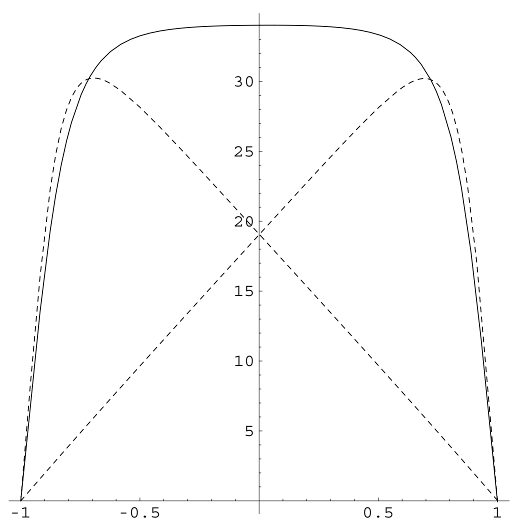

FIG. 3.1.

Computations in the TABLE 3.1 show that in fact the system (2.7) has exactly three solutions for $h>1$. The respective three solutions of the boundary value problem looks like shown in FIG. 3.1. The value of the functional $H(x)$ for any of two asymmetric solutions is less than that of the symmetric solution.

TABLE 3.1
The results of computation for different values of h.

h	β_0	β_1	β_2	H_{sym}	J_1	J_2	$H_{\text{asym}} = J_1 + J_2$
0	1.72	–	–	1.97	–	–	–
0.5	4.93	–	–	13.12	–	–	–
1	10.48	–	–	48.19	–	–	–
2	30.41	10.20	41.79	274.92	41.04	200.73	241.77
3	63.79	13.15	83.53	871.82	75.93	595.70	671.63
4	111.22	16.34	138.76	2034.08	122.82	1309.47	1432.29
5	172.68	19.62	207.64	3952.96	181.61	2436.69	2618.29
6	248.05	22.95	290.22	6817.91	252.26	4071.93	4324.18
7	337.27	26.31	386.53	10817.92	334.76	6309.75	6644.51
8	440.29	29.69	496.57	16141.90	429.12	9244.71	9673.83
9	557.10	33.08	620.35	22978.82	535.31	12971.36	13506.67
10	687.68	36.48	757.88	31517.65	653.34	17584.24	18237.58
20	2750.10	70.67	2889.26	252098.71	2484.21	133255.04	135739.25

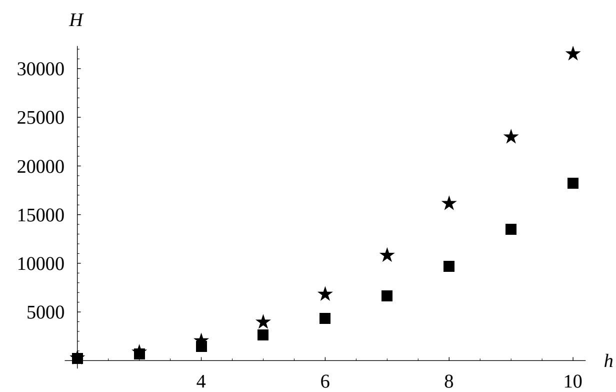

FIG. 3.2. *Values of H_{sym} (stars) and H_{asym} (boxes) for $h = 2, 3, \dots, 10$.*

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