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POSITIVE SOLUTIONS OF P-TYPE RETARDED FUNCTIONAL LINEAR DIFFERENTIAL EQUATIONS

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Abstract. For systems of retarded functional linear differential equations with unbounded delay and with finite memory sufficient and necessary conditions of existence of positive solutions on an interval of the form $[t_0,\infty)$ are derived. A general criterion is given together with corresponding applications. Examples are inserted to illustrate the results.

Key words. Positive solution, delayed equation, p-function.

AMS subject classifications. 34K20, 34K25

1. Introduction. In this paper are studied positive solutions (i.e. a solution with positive coordinates on a considered interval) for systems of retarded linear functional differential equations (RFDE's) with *unbounded delay* and with *finite memory*. The criterion for the existence positive solutions is given together with applications. Results are illustrated on examples too. Let us shortly describe the basic notions of theory p-type differential equations.

The basic notion of RFDE's with unbounded delay but with finite memory is a function $p \in C[\mathbb{R}\times[-1,0],\mathbb{R}]$ which is called a p*-function* and it has following properties [12, p. 8]:

(i) $p(t,0)=t$.
(ii) $p(t,-1)$ is a nondecreasing function of t.
(iii) there exists a $\sigma \ge -\infty$ such that $p(t,\vartheta)$ is an increasing function for ϑ for each $t\in(\sigma,\infty)$. (Throughout the following text we suppose $t\in(\sigma,\infty)$.)

In the theory of RFDE's with bounded delay the symbol $y_t \in C([-\tau,0],\mathbb{R}^n)$ denotes the function which expresses the history of the process $y(t)$. For RFDE's with unbounded delay the symbol y_t is defined as follows:

Definition 1. ([12, p. 8]) Let $t_0\in\mathbb{R}$, $A>0$ and $y\in C([p(t_0,-1),t_0+A),\mathbb{R}^n)$. For any $t\in[t_0,t_0+A)$, we define

$$y_t(\vartheta) := y(p(t,\vartheta)),\ -1\le\vartheta\le 0$$

and write

$$y_t \in \mathcal{C} := C[[-1,0],\mathbb{R}^n].$$

Note that the symbol "y_t" (e.g., in [13, p. 38], defined by $y_t(s) := y(t+s)$, where $-\tau\le s\le 0$, $\tau>0$, $\tau=\text{const}$) used in the theory of delayed functional differential equations for equations with bounded delays is a partial case of the above definition. Really, we can put $p(t,\vartheta) := t+\tau\vartheta$, $\vartheta\in[-1,0]$.

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In this paper we study positive solutions, especially the conditions of existence of positive solutions of the system

$$\dot{y}(t) = f(t, y_t) \tag{1.1}$$

where y_t is defined as in Definition 1 and $f \in C([t_0, t_0 + A) \times \mathcal{C}, \mathbb{R}^n)$, $A > 0$ is continuous. This system is called the system of *p-type retarded functional differential equations* (p-RFDE's) or a system *with unbounded delay with finite memory.*

Definition 2. The function $y \in C([p(t_0, -1), t_0 + A), \mathbb{R}^n) \cap C^1([t_0, t_0 + A), \mathbb{R}^n)$ satisfying (1.1) on $[t_0, t_0 + A)$ is called a *solution of* (1.1) *on* $[\,p(t_0, -1), t_0 + A)$.

Suppose that the function $f : \Omega \to \mathbb{R}^n$ is continuous on Ω and Ω is an open subset of $\mathbb{R} \times \mathcal{C}$. For any $(t_0, \phi) \in \Omega$, *there exists a solution* $y = y(t_0, \phi)$ of the system p-RFDE's (1.1) through (t_0, ϕ) (see [12, p. 25]). If $f(t, \phi)$ is locally Lipschitzian with respect to second argument then this *solution is unique* ϕ ([12, p. 30]) and moreover this solution is *continuable* in the usual sense of extended existence if f is quasibounded ([12, p. 41]). For p-RFDE's such that the solution $y = y(t_0, \phi)$ through $(t_0, \phi) \in \Omega$, defined on $[t_0, A]$, is unique the property of the *continuous dependence* holds too (see [12, p. 33]). These results holds also for $\Omega = [\,t^*, \infty) \times \mathcal{C}$ where $t^* \in \mathbb{R}$.

Let us cite some known results concerning with the problem of existence of *positive solutions* (i.e. solutions having all its coordinates positive on the intervals considered) for linear systems of RFDE's with unbounded delay but with finite memory. The scalar equation

$$\dot{x}(t) + p(t)x(t - \tau(t)) = 0 \tag{1.2}$$

with $p, \tau \in C([t_0, \infty), \mathbb{R}_+)$, $\tau(t) \le t$, $\lim\limits_{t \to \infty}(t - \tau(t)) = \infty$ and $\mathbb{R}_+ = [0, \infty)$ was studied in the book [10] and the criterion for existence of a positive solution was given. Namely, (1.2) has a positive solution with respect to t_1 if and only if there exists a continuous function $\lambda(t)$ on $[T_1, \infty)$ with $T_1 = \inf\limits_{t \ge t_1}\{t - \tau(t)\}$, such that $\lambda(t) > 0$ for $t \ge t_1$ and

$$\lambda(t) \ge p(t)\mathrm{e}^{\int_{t-\tau(t)}^{t} \lambda(s)\mathrm{d}s}, \quad t \ge t_1. \tag{1.3}$$

(A function x is called a solution of (1.2) with respect to an initial point $t_1 \ge t_0$ if x is defined and is continuous on $[T_1, \infty)$, differentiable on $[t_1, \infty)$, and satisfies (1.2) for $t \ge t_1$.) Analogous results in this direction are formulated in the book [11] and in the papers [1, 2], too. The cited criterion was generalized for nonlinear systems of RFDE's with bounded retardation in [3] and for nonlinear systems of RFDE's with unbounded delay and with finite memory in [6]. These generalizations are direct generalizations since in their formulations existence of a positive (vector) functions playing a similar role as λ in (1.3) is supposed. Many works e.g. [4]–[10] deal with positive solutions of (1.2) in the critical case.

2. General linear case. The main results are reformulated for the linear case i.e. for the system

$$\dot{y}(t) = L(t, y_t) + h(t), \tag{2.1}$$

$L \in C(\Omega \times \mathcal{C}, \mathbb{R}^n)$ is a linear functional and y_t is defined in accordance with Definition 1. We note that systems (2.1) posses the property of continuous dependence and continuation of solutions to infinity, therefore all assumptions ensuring these properties are fulfilled.

With $\mathbb{R}^n_{\geq 0}$ $(\mathbb{R}^n_{>0})$ we denote the set of all component-wise nonnegative (positive) vectors v in $\mathbb{R}^n$, i. e., $v = (v_1, \ldots, v_n) \in \mathbb{R}^n_{\geq 0}\,(\mathbb{R}^n_{>0})$ if and only if $v_i \geq 0$ $(v_i > 0)$ for $i = 1, \ldots, n$. For $u, v \in \mathbb{R}^n$ we write $u \leq v$ if $v - u \in \mathbb{R}^n_{\geq 0}$, $u \ll v$ if $v - u \in \mathbb{R}^n_{>0}$ and $u < v$ if $u \leq v$ and $u \neq v$.

Let $k = (k_1, \ldots, k_n) \gg 0$ be a constant vector. Let $\lambda(t) = (\lambda_1(t), \ldots, \lambda_n(t))$ denote a vector, defined and locally integrable on $[\,p^*, \infty)$. Let us define an auxiliary operator

$$T(k, \lambda)(t) := k\mathrm{e}^{\int_{p^*}^{t} \lambda(s)\mathrm{d}s} = \left(k_1\mathrm{e}^{\int_{p^*}^{t} \lambda_1(s)\mathrm{d}s}, \ldots, k_n\mathrm{e}^{\int_{p^*}^{t} \lambda_n(s)\mathrm{d}s}\right). \tag{2.2}$$

Definition 3. We say that the functional $L \in C(\Omega, \mathbb{R}^n)$ is *i-strongly decreasing* (or *i-strongly increasing*), $i \in \{1, 2, \ldots, n\}$ if for each $(t, \varphi) \in \Omega$ and $(t, \psi) \in \Omega$ such that

$$\varphi(p(t, \vartheta)) \ll \psi(p(t, \vartheta)), \quad \text{where } \vartheta \in [-1, 0) \ \text{ and } \ \varphi_i(p(t, 0) = \psi_i(p(t, 0))$$

the inequality

$$L_i(t, \varphi) > L_i(t, \psi) \quad (\text{or } \ L_i(t, \varphi) < L_i(t, \psi))$$

holds.

The sufficient and necessary condition of existence of positive solution is specified in the following theorem.

Theorem 1. *Suppose $L \in C(\Omega \times \mathcal{C}, \mathbb{R}^n)$ and, moreover:*

(*i*) *For $i = 1, \ldots, p$ is L i-strongly decreasing and $L_i(t, \mathbf{0}) + h_i(t) \leq 0$ if $(t, \mathbf{0}) \in \Omega$ and*

(*ii*) *for $i = p + 1, \ldots, n$ is L i-strongly increasing and $L_i(t, \mathbf{0}) + h_i(t) \geq 0$ for if $(t, \mathbf{0}) \in \Omega$.*

Then the existence of a positive solution $y(t)$ on $[\,p^*, \infty)$ of the system p-RFDE's (2.1) (*where $p^* = p(t^*, -1)$*) *is equivalent with the existence of a positive constant vector k and a locally integrable vector $\lambda : [\,p^*, \infty) \to \mathbb{R}^n$ continuous on $[\,p^*, t^*) \cup [t^*, \infty)$ satisfying the system of integral inequalities*

$$\mu_i\lambda_i(t) \geq \frac{\mu_i}{k_i} \cdot \mathrm{e}^{-\int_{p^*}^{t} \lambda_i(s)\mathrm{d}s} \cdot \left(L_i\left(t, T(k, \lambda)_t\right) + h_i(t)\right), \quad i = 1, \ldots, n \tag{2.3}$$

for $t \geq t^*$ with $\mu_i = -1$ for $i = 1, \ldots, p$ and $\mu_i = 1$ for $i = p + 1, \ldots, n$.

The proofs of this and next theorems, based on the retract method and on the Lyapunoff method, are small modification of proof of the main result in [6]. In the formulation of the above-mentioned theorem a relative restrictive condition, that L is i-strongly decreasing or i-strongly increasing, is used. By leaving this assumption it is possible to obtain only sufficient condition for the existence of positive solutions. Moreover verifying the similar inequalities as (2.3) is more difficult.

Let a constant vector $k \gg 0$ and a vector $\lambda(t)$ defined and locally integrable on $[\,p^*, \infty)$ are given. Then the operator T is well defined by (2.2). Let us define for every $i \in \{1, 2, \ldots, n\}$ two types of subsets of the set $\mathcal{C}$:

$$\mathcal{T}^{\,i}(t) := \Big\{\phi \in \mathcal{C} : 0 \ll \phi(\vartheta) \ll T(k, \lambda)_t(\vartheta),\, \vartheta \in [-1, 0] \\ \text{except for } \ \phi_i(0) = k_i\mathrm{e}^{\int_{p^*}^{t} \lambda_i(s)\mathrm{d}s}\Big\}$$

and

$$\mathcal{T}_i(t) := \{\phi \in \mathcal{C} : 0 \ll \phi(\vartheta) \ll T(k, \lambda)_t(\vartheta),\, \vartheta \in [-1, 0] \ \text{ except for } \ \phi_i(0) = 0\}.$$

THEOREM 2. *Suppose $L \in C(\Omega, \mathbb{R}^n)$ is linear. Let a constant vector $k \gg 0$ and a vector $\lambda(t)$ defined and locally integrable on $[p^*, \infty)$ are given. If, moreover, inequalities*

$$\mu_i \lambda_i(t) > \frac{\mu_i}{k_i} \cdot \mathrm{e}^{-\int_{p^*}^{t} \lambda_i(s)\mathrm{d}s} \cdot (L_i(t, \phi) + h_i(t)) \tag{2.4}$$

hold for every $i \in \{1, 2, \ldots, n\}$, $(t, \phi) \in [t^*, \infty) \times \mathcal{T}^i$ and inequalities

$$\mu_i (L_i(t, \phi) + h_i(t)) > 0 \tag{2.5}$$

hold for every $i \in \{1, 2, \ldots, n\}$, $(t, \phi) \in [t^*, \infty) \times \mathcal{T}_i$, where $\mu_i = -1$ for $i = 1, \ldots, p$ and $\mu_i = 1$ for $i = p+1, \ldots, n$, then there exists a positive solution $y = y(t)$ on $[p^*, \infty)$ of the system p-RFDE's (2.1).

We remark that monocitity of functional L is not used in formulation of THEOREM 2. In the next example the functional L_1 is neither 1-strongly increasing nor 1-strongly decreasing.

EXAMPLE 1. We show that the system

$$\begin{aligned} y_1'(t) &= -\frac{1}{t}\left[\frac{1}{8}y_1(t) + \frac{1}{8\sqrt{t}}y_1(\sqrt{t}) - \frac{1}{t}y_2(\sqrt{t}) - \frac{1}{t}\right], \\ y_2'(t) &= -\frac{1}{t}\left[\frac{1}{8t}y_1(t) + \frac{1}{8}y_2(t) + \frac{1}{4t}y_2(\sqrt{t})\right] \end{aligned} \tag{2.6}$$

has a positive solution on the interval $[16, \infty)$. The system (2.6) can be rewritten in the form (2.1):

$$\begin{aligned} y_1'(t) &= -\frac{1}{t}\left[\frac{1}{8}y_1(p(t,0)) + \frac{1}{8\sqrt{t}}y_1(p(t,-1)) - \frac{1}{t}y_2(p(t,-1)) - \frac{1}{t}\right], \\ y_2'(t) &= -\frac{1}{t}\left[\frac{1}{8t}y_1(p(t,0)) + \frac{1}{8}y_2(p(t,0)) + \frac{1}{4t}y_2(p(t,-1))\right], \end{aligned}$$

and also in the form

$$\begin{aligned} L_1(t,\phi) &= -\frac{1}{t}\left[\frac{1}{8}\phi_1(0) + \frac{1}{8\sqrt{t}}\phi_1(-1) - \frac{1}{t}\phi_2(-1)\right], \quad & h_1(t) &= -\frac{1}{t^2}, \\ L_2(t,\phi) &= -\frac{1}{t}\left[\frac{1}{8t}\phi_1(0) + \frac{1}{8}\phi_2(0) + \frac{1}{4t}\phi_2(-1)\right], & h_2(t) &= 0, \end{aligned}$$

where the p-function is defined as $p(t, \vartheta) := t + (t - \sqrt{t})\vartheta$, $\vartheta \in [-1, 0]$. To verify that the THEOREM 2 can be used, we put: $p^* = 4 = p(t^*, -1)$, $t^* = 16$, $k = (k_1, k_2) = (1/2, 1/4)$, $\lambda(t) = (\lambda_1(t), \lambda_2(t)) = (-1/(2t), -1/t)$, $\mu_1 = \mu_2 = -1$. We have

$$\begin{aligned} T(k,\lambda)(t)) &= k \exp\left(\textstyle\int_4^t \lambda(s)\mathrm{d}s\right) = \\ &\left(\frac{1}{4}\exp\left(-\textstyle\int_4^t \mathrm{d}s/(2s)\right), \frac{1}{2}\exp\left(\textstyle\int_4^t \mathrm{d}s/s\right)\right) = \left(\frac{1}{\sqrt{t}}, \frac{1}{t}\right). \end{aligned}$$

Subsets $\mathcal{T}^i(t)$ and $\mathcal{T}_i(t)$ have the form:

$$\begin{aligned} \mathcal{T}^1(t) = \{\phi = (\phi_1, \phi_2) | 0 < \phi_1(\vartheta) < (1/\sqrt{t})_t, \text{ for } -1 \le \vartheta < 0, \phi_1(0) = 1/\sqrt{t} \text{ and} \\ 0 < \phi_2(\vartheta) < (1/t)_t, \text{ for } -1 \le \vartheta \le 0\}, \end{aligned}$$

$$\mathcal{T}_1(t) = \{\phi = (\phi_1, \phi_2) | 0 < \phi_1(\vartheta) < (1/\sqrt{t})_t, \text{ for } -1 \le \vartheta < 0, \phi_1(0) = 0 \text{ and } \\ 0 < \phi_2(\vartheta) < (1/t)_t, \text{ for } -1 \le \vartheta \le 0\},$$

$$\mathcal{T}^2(t) = \{\phi = (\phi_1, \phi_2) | 0 < \phi_1(\vartheta) < (1/\sqrt{t})_t, \text{ for } -1 \le \vartheta \le 0 \text{ and } \\ 0 < \phi_2(\vartheta) < (1/t)_t, \text{ for } -1 \le \vartheta < 0, \phi_2(0) = 1/t\},$$

$$\mathcal{T}_2(t) = \{\phi = (\phi_1, \phi_2) | 0 < \phi_1(\vartheta) < (1/\sqrt{t})_t, \text{ for } -1 \le \vartheta \le 0 \text{ and } \\ 0 < \phi_2(\vartheta) < (1/t)_t, \text{ for } -1 \le \vartheta < 0, \phi_2(0) = 0\}.$$

Let us verify inequalities (2.4) and (2.5) for $i = 1$.
If $\phi \in \mathcal{T}^1(t)$ then $\exp\left(-\int_{p^*}^t \lambda_1(s)\mathrm{d}(s)\right) = \exp\left(\left[\frac{1}{2}\ln s\right]_4^t\right) = \frac{\sqrt{t}}{2}$ and

$$\frac{\mu_1}{k_1}\exp\left(-\int_{p^*}^t \lambda_1(s)\mathrm{d}s\right)(L_1(t,\phi) + h_1(t)) = \\ -2\frac{\sqrt{t}}{2}\left(-\frac{1}{8t}\frac{1}{\sqrt{t}} - \frac{1}{8t\sqrt{t}}\xi + \frac{1}{t^2}\zeta - \frac{1}{t^2}\right) \\ \left[\text{where } 0 < \xi < \frac{1}{\sqrt{p(t,-1)}} = \frac{1}{\sqrt[4]{t}} \text{ and } 0 < \zeta < \frac{1}{p(t,-1)} = \frac{1}{\sqrt{t}}\right] \\ \le \left(\frac{1}{8} + \frac{1}{8}\frac{1}{\sqrt[4]{t}} + \frac{1}{\sqrt{t}}\right)\frac{1}{t} = [\text{ for } t > 16\ ] = \frac{7}{16}\frac{1}{t} < \frac{-1}{-2t} = \mu_1\lambda_1(t).$$

If $\phi \in \mathcal{T}_1(t)$ then

$$\mu_1 L_1(\phi) = -\left(-\frac{1}{8t\sqrt{t}}\xi + \frac{1}{t^2}\zeta - \frac{1}{t^2}\right) = \frac{1}{t^2}\left(-1 - \xi\sqrt{t} + \zeta\right) \\ \left[\text{where } 0 < \xi < \frac{1}{\sqrt{p(t,-1)}} = \frac{1}{\sqrt[4]{t}} \text{ and } 0 < \zeta < \frac{1}{p(t,-1)} = \frac{1}{\sqrt{t}}\right] \\ \le \frac{1}{t^2}\left(-1 + \frac{1}{\sqrt{t}}\right) = [\text{ for } t > 16\ ] \le \frac{1}{t^2}\left(-1 + \frac{1}{4}\right) \le \frac{3}{4t^2} < 0.$$

Likewise for $i = 2$ we get analogous inequalities.
If $\phi \in \mathcal{T}^2(t)$ then $\exp\left(-\int_{p^*}^t \lambda_2(s)\mathrm{d}(s)\right) = \exp\left([\ln s]_4^t\right) = \frac{t}{4}$ and

$$\frac{\mu_2}{k_2}\exp\left(-\int_{p^*}^t \lambda_2(s)\mathrm{d}s\right)(L_2(t,\phi) + h_2(t)) = -4\frac{t}{4}\left(-\frac{1}{8t}\frac{1}{t} - \frac{1}{8t^2}\xi - \frac{1}{4t^2}\zeta\right) \\ \left[\text{where } 0 < \xi < \frac{1}{\sqrt{p(t,0)}} = \frac{1}{\sqrt{t}} \text{ and } 0 < \zeta < \frac{1}{p(t,-1)} = \frac{1}{\sqrt{t}}\right] \\ \le \frac{1}{t}\left(\frac{1}{8} + \frac{1}{8}\frac{1}{\sqrt{t}} + \frac{1}{4}\frac{1}{\sqrt{t}}\right) = [\text{ for } t > 16\ ] = \frac{1}{t}\frac{7}{32} < \frac{-1}{-t} = \mu_2\lambda_2(t).$$

If $\phi \in \mathcal{T}_2(t)$ then

$$\mu_2 L_2(\phi) = -\left(-\frac{1}{8t^2}\xi - \frac{1}{4t^2}\zeta\right) = [\text{where } 0 < \xi \text{ and } 0 < \zeta] < 0.$$

All conditions of Theorem 2 are valid. Therefore a positive solution

$$y(t) = (y_1(y), y_2(t)),$$

of system (2.6) exists on $[16, \infty)$. Since the proofs of both Theorems 1, 2 are based on Ważewski's topological method with the domain defined by inequalities $0 \ll y(t) \ll T(k, \lambda)(t)$, we get also estimations for this solution:

$$0 < y_1(t) < \frac{1}{\sqrt{t}},$$
$$0 < y_2(t) < \frac{1}{t}.$$

Now, let us give several applications.

3. A scalar equation with discrete delays. At the first we study conditions for existence of a positive solution of a scalar equation with discrete delays

$$\dot{y}(t) = -\sum_{q=1}^{m} c_q(t) y(p(t, \vartheta_q)) \tag{3.1}$$

with $-1 = \vartheta_1 < \vartheta_2 < \cdots < \vartheta_m = 0$ and $t^* \geq \sigma$ from *iii*) of the definition of the function p; the functions c_q are continuous on $[t^*, \infty)$ for $q = 1, 2, \ldots, m$, which are nonnegative if $q = 1, 2, \ldots, m-1$ and satisfy inequality $\sum\limits_{q=1}^{m-1} c_q(t) > 0$ for $t \in [t^*, \infty)$.

Theorem 3. *The existence of a positive solution $y = y(t)$ of the equation* (3.1)*) on* $[p^*, \infty)$ (*where* $p^* = p(t^*, -1)$) *is equivalent with the existence a locally integrable function* $\lambda^* : [p^*, \infty) \to \mathbb{R}$ *continuous on* $[p^*, t^*) \cup [t^*, \infty)$ *and satisfying the integral inequality*

$$\lambda^*(t) \geq \sum_{q=1}^{m} c_q(t) \exp\left(\int_{p(t,\vartheta_q)}^{t} \lambda^*(s) \mathrm{d}s\right) \tag{3.2}$$

for $t \geq t^*$.

Proof. The proof uses Theorem 1. Let us put $n = p = 1$ and define functional L corresponding to the right hand side of (3.2):

$$L(t, \varphi) := -\sum_{q=1}^{m} c_q(t) \varphi(\vartheta_q)$$

where $(t, \varphi) \in \Omega \times \mathcal{C}$. Then conditions (i), (ii) of Theorem 1 are satisfied. Note that the sign constancy of the function c_m is not necessary for verifying that L is a 1-strongly decreasing functional. Conclusion of Theorem 3 is now a consequence of scalar inequality (2.3) if $\lambda := -\lambda^*$. □

Remark 1. The result mentioned in Section 1 can be considered as a partial case of Theorem 3 if $m = 1$. Let us underline that a condition equivalent to $\lim\limits_{t\to\infty} (t - \tau(t)) = \infty$ is not involved in Theorem 3. In the next example Theorem 3 is applied for equation with two type delays.

EXAMPLE 2. We consider equation (3.1) with $m = 3$, $c_3(t) \equiv 0$. Let $c_1(t)$, $c_2(t)$ be positive continuous function, $\vartheta_1 = -1$, $\vartheta_2 = -1/2$, $\vartheta_3 = 0$ and the p-function is defined as:

$$p(t,\theta) = \begin{cases} t + 2\tau\theta & \text{for } \theta \in (-1/2, 0], \\ 2(t-\tau)(\theta+1) + \sqrt{t}(\theta + 1/2)(-2) & \text{for } \theta \in [-1, -1/2]. \end{cases}$$

Then the equation (3.1) takes the form:

$$\dot{y}(t) = -c_1(t)y(\sqrt{t}) - c_2(t)y(t-\tau), \tag{3.3}$$

where $c_1,\ c_2$ are positive continuous functions. Then the inequality (3.2) has the form:

$$\lambda(t) \geq c_1(t)\exp\left(\int_{\sqrt{t}}^{t} \lambda(s)\,\mathrm{d}s\right) + c_2(t)\exp\left(\int_{t-\tau}^{t} \lambda(s)\,\mathrm{d}s\right).$$

We put $\lambda(t) = 1/t$. Then we obtain

$$\frac{1}{t} \geq c_1(t)\frac{t}{t-\tau} + c_2(t)\frac{t}{\sqrt{t}}.$$

This inequality (on the interval $[p^*, \infty)$) is a sufficient condition for the existence of a positive solution of equation (3.3) on interval $[(p^*)^2, \infty)$. Also relations

$$c_1(t) = o\Big(\frac{1}{t}\Big) \ \text{ and } \ c_2(t) = o\Big(\frac{1}{t\sqrt{t}}\Big) \text{ for } t \to \infty$$

are sufficient conditions for the existence of an eventually positive solution of equation (3.3).

THEOREM 3 can serve as a source of various sufficient conditions including well known sufficient conditions given e.g. in [10, 11]. Let us give several concrete consequences of THEOREM 3 concerning the equation

$$\dot{y}(t) = -c(t)y(p(t,-1)) \tag{3.4}$$

with a positive continuous function c. Obviously, equation (3.4) is a partial case of (3.1) if $m = 1$.

THEOREM 4. *For positive continuous function $c(t)$ on $[\,p^*, \infty)$ satisfying inequality*

$$\mathrm{e} \cdot \int_{p(t,-1)}^{t} c(s)\mathrm{d}s \leq 1 \tag{3.5}$$

on $[t^*, \infty)$ (where $p^* = p(t^*, -1)$), the equation (3.4) has a positive solution $y = y(t)$ on $[\,p^*, \infty)$.

Proof. We define $\lambda^*(t) := c(t) \cdot \mathrm{e}$ to employ THEOREM 3. Then, due to (3.5), inequality (3.2) holds on $[t^*, \infty)$ since it turns into

$$\mathrm{e} \geq \exp\left[\mathrm{e} \cdot \int_{p(t,-1)}^{t} c(s)\mathrm{d}s\right].$$

□

EXAMPLE 3. If $\tau(t)$ in the equation (1.2) is positive and nondecreasing, then (3.5) yields

$$\mathrm{e} \cdot \limsup_{t\to\infty}\big[\tau(t) \cdot \max_{t-\tau(t)\leq s\leq t} p(t)\big] < 1. \tag{3.6}$$

This is a sufficient condition for existence positive solution of (1.2) on $[t_1,\infty)$, where t_1 is enough large.

The following corollary follows directly from (3.5) .
Corollary 1. *Let all conditions of* Theorem 4 *be valid and moreover there exists a nondecreasing function $b(t)$, $t\in[p^*,\infty)$ such that $c(t)\le b(t)$ holds on $[p^*,\infty)$and*

$$b(t)\le \frac{1}{\mathrm{e}\cdot[t-p(t,-1)]} \tag{3.7}$$

holds on $[t^*,\infty)$. Then (3.4) *has a positive solution $y=y(t)$ on $[\,p^*,\infty)$.*

Example 4. If nondecreasing function $p(t)$ and positive nondecreasing function $\tau(t)$ in the equation (1.2) satisfy

$$\mathrm{e}\cdot p(t)\tau(t)\le 1 \tag{3.8}$$

then, as it follows from (3.5) or (3.6) equation (1.2) has a positive solution.

Remark 2. Presented results are sharp. We can demonstrate it on the Example 4. If $p(t)\equiv c=\text{const}$, $p(t,-1):=t-\tau$ with a positive constant τ, then Example 4 yields a classical result ([11, Theorem 2.2.3]) ensuring existence of a positive solution:

$$c\tau\mathrm{e}\le 1.$$

4. Positive solutions of a linear system. We consider existence of positive solutions of the following linear system

$$y'(t)=-A(t)y(p(t,-1)) \tag{4.1}$$

where $A=\{a_{ij}(t)\}$ is continuous $n\times n$ matrix on $[t^*,\infty)$, such that $a_{ij}(t)$ satisfy:

$$a_{ij}(t)\ge 0,\ \text{for } i,j=1,2,\dots,n \text{ and } \sum_{j=1}^{n}a_{ij}(t)>0 \text{ for every } i=1,2,\dots,n.$$

Theorem 5. *For existence of a positive solution $y=y(t)$ of linear system* (4.1) *on $[\,p^*,\infty)$ (with $p^*=p(t^*,-1)$) the existence of a positive constant vector k and a locally integrable function $\lambda^*:[\,p^*,\infty)\to\mathbb{R}$ continuous on $[\,p^*,t^*)\cup[t^*,\infty)$ and satisfying the integral inequality*

$$\lambda^*(t)\mathrm{e}^{-\int_{p(t,-1)}^{t}\lambda^*(q)\mathrm{d}q}\ge \max_{i=1,2,\dots,n}\left\{\frac{1}{k_i}\sum_{j=1}^{n}k_ja_{ij}(t)\right\} \tag{4.2}$$

for $t\ge t^*$,is sufficient condition.

Proof. Functional $L\in C(\Omega\times\mathcal{C},\mathbb{R}^n)$, corresponding to the system (4.1) has the form

$$L(t,\varphi):=-A(t)\varphi(-1)$$

and is i-strongly decreasing if $i=1,2,\dots,n$, and $L(t,\mathbf{0})=0$ if $(t,\mathbf{0})\in\Omega$. Then, as it follows from Theorem 1, for existence of a positive solution on $[\,p^*,\infty)$ is sufficient if inequalities (2.3) with $\mu_i = -1$, $i = 1, 2, \dots, n$ hold for $t \geq t^*$. Let us suppose $\lambda_1 \equiv \lambda_2 \equiv \cdots \equiv \lambda_n \equiv -\lambda^*$. Then inequalities (2.3) turn into

$$\lambda^*(t) \geq \frac{1}{k_i} \cdot \mathrm{e}^{\int_{p(t,-1)}^{t} \lambda^*(q)\mathrm{d}q} \cdot \sum_{j=1}^{n} k_j a_{ij}(t),$$

where $i = 1, 2, \dots, n$, and hold on $[t^*, \infty)$ if inequality (4.2) is valid. □

Inequality (4.2) gives a lot of possibilities to develop concrete sufficient conditions. We consider two of them.

Theorem 6. *Suppose that a continuous nondecreasing function $\lambda^* : [\,p^*, \infty) \to \mathbb{R}$ satisfies the inequality*

$$\lambda^*(t)\mathrm{e}^{-\lambda^*(t)\cdot[t-p(t,-1)]} \geq \max_{i=1,2,\dots,n} \left\{ \frac{1}{k_i} \sum_{j=1}^{n} k_j a_{ij}(t) \right\} \tag{4.3}$$

for $t \geq t^*$, where $k = (k_1, k_2, \dots, k_n)$ is a suitable positive constant vector. Then linear system (4.1) *has a positive solution $y = y(t)$ on $[\,p^*, \infty)$ (with $p^* = p(t^*, -1)$).*

Proof. Presented result is a straightforward consequence of Theorem 5 since obviously

$$\lambda^*(t)\mathrm{e}^{-\int_{p(t,-1)}^{t} \lambda^*(q)\mathrm{d}q} \geq \lambda^*(t)\mathrm{e}^{-\lambda^*(t)\cdot[t-p(t,-1)]}.$$

Then inequality (4.2) is a consequence of inequality (4.3). □

Example 5. We consider equation (4.1) with $p(t, \vartheta) = t - (t - \tau(t))\vartheta$. Then

$$y'(t) = -A(t)y(t - \tau(t)). \tag{4.4}$$

The inequality (4.3) takes the form:

$$\lambda^*(t)\mathrm{e}^{-\lambda^*(t)\cdot[t-\tau(t)]} \geq \max_{i=1,2,\dots,n} \left\{ \frac{1}{k_i} \sum_{j=1}^{n} k_j a_{ij}(t) \right\}.$$

For $k_1 = k_2 = \cdots = k_n$ this inequality turns into

$$\lambda^*(t)\mathrm{e}^{-\lambda^*(t)\cdot[t-\tau(t)]} \geq \max_{i=1,2,\dots,n} \left\{ \sum_{j=1}^{n} a_{ij}(t) \right\} = ||A(t)||.$$

Then the sufficient condition of the existence of a positive solution of system (4.4) is the existence of a positive solution of scalar equation:

$$y'(t) = p(t)y(t - \tau(t))$$

with $p(t) = ||A(t)||$, where $||\cdot||$ is the row norm of the matrix $A(t)$. With the aid of Theorem 3 we get for the equation above the sufficient and necessary condition for existence of a positive solution in the form of inequality (3.2) with $m = 2$, $c_2(t) \equiv 0$, $c_1(t) = ||A(t)||$. This inequality assures the validity of the inequality (4.3) and the existence of a positive solution of (4.4).

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REFERENCES

[1] L. Berezanski, E. Braverman, *On oscillation of a logistic equation with several delays,* J. Comput. and Appl. Mathem. **113** (2000), 255–265.

[2] J. Čermák, *A change of variables in the asymptotic theory of differential equations with unbounded delay,* J. Comput. Appl. Mathem. **143** (2002), 81–93.

[3] J. Diblík, *A criterion for existence of positive solutions of systems of retarded functional differential equations,* Nonl. Anal., TMA **38** (1999), 327-339.

[4] J. Diblík, *Positive and oscillating solutions of differential equations with delay in critical case,* J. Comput. Appl. Mathem. **88** (1998), 185–202.

[5] J. Diblík, N. Koksch, *Positive solutions of the equation* $\dot{x}(t) = -c(t)x(t-\tau)$ *in the critical case,* J. Math. Anal. Appl. **250** (2000), 635–659.

[6] J. Diblík, Z. Svoboda, *An existence criterion of positive solutions of p-type retarded functional differential equations,* J. Comput. Appl. Mathem. **147** (2002), 315–331.

[7] Y. Domshlak, *On oscillation properties of delay differential equations with oscillating coefficients,* Funct. Diff. Equat., Israel Seminar **2** (1996), 59–68.

[8] Y. Domshlak, I. P. Stavroulakis, *Oscillation of first-order delay differential equations in a critical case*, Appl. Anal. **61** (1996), 359–371.

[9] Á. Elbert, I.P. Stavroulakis, *Oscillation and non-oscillation criteria for delay differential equations,* Proc. Amer. Math. Soc. **123** (1995), 1503–1510.

[10] L.H. Erbe, Q. Kong, B. G. Zhang, *Oscillation Theory for Functional Differential Equations.* Marcel Dekker, New York, 1995.

[11] I. Györi, G. Ladas, *Oscillation Theory of Delay Differential Equations.* Clarendon Press, Oxford, 1991.

[12] V. Lakshmikamthan, L. Wen, B. Zhang, *Theory of Differential Equations with Unbounded Delay,* Kluwer Academic Publishers, 1994.

[13] J. K. Hale, S. M. V. Lunel, *Introduction to Functional Differential Equations,* Springer–Verlag, New York, Inc., 1993.