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Semiuniform spaces, and seminorms, semimetrics, semiecarts in apo-semigroups

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SEMIUNIFORM SPACES, AND SEMINORMS, SEMIMETRICS, SEMIECARTS IN APO-SEMIGROUPS

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The uniform space is one generalization of the metric space; another is the space with ecart in a ordered set considered by Colmez. This is really equivalent to the uniform space. Intermediate types can be obtained by restricting the values of the ecart or metric to lie in ordered groups or semigroups of a suitable type. We do this here, with the additional relaxation of the symmetry condition for these.

The treatment in the first part of the (non symmetric) semiuniform spaces sketches the process of completing such a space. This is based on an earlier paper of the author [1]. For the (Abelian) semigroups with order or semiuniformity the possibilities of immersion in similar groups are also worked out. The Aposemigroup is an Abelian ordered semigroup with zero, with special properties for its positive part. This class includes all totally ordered Abelian groups in which the elements strictly greater than 0 form a down-directed weakly divisible set. Lexicographic products of such semigroups are also in the class. These have a natural or intrinsic semiuniformity which behaves nicely relative to $+$, $\leqq$. And their canonical completions are also Apo-semigroups with intrinsic semiuniformities. A seminorm, semiecart or semimetric for a set S in such a Apo-semigroup or group leads to induced semiuniformities for the set S. And then the canonical completion of this semiuniform space has its semiuniformity similarly derivable by means of a seminorm, semiecart or semimetric in the Apo-semigroup which is the completion of the original. This is the basic theorem of the paper. Among consequences, it is seen that if a space has its semiuniformity defined by a seminorm, semiecart or semimetric in an Abelian totally ordered group without a first Archimedean class, then its canonical completion has also its semiuniformity so defined.

As a somewhat trivial example, note that the rational numbers have a seminorm in the Apo-semigroup of the positive rationals, while the real numbers have a seminorm in the Apo-semigroup of the positive reals.

1. The Semiuniform Space and Its Canonical Completion

A *semiuniform space* (S, U, J) consists of a set of points S and a monotone mapping U of a down directed ordered set J (or $(J, \leqq)$) in the ordered set $(S \times S, \subseteq)$ satisfying the following conditions:

(SU 1) For each j in J, $U(j)$ contains the set $\Delta = [(x, x) : x \text{ in } S]$;

(SU 2) For each j in J, there exists a j' in J such that $U(j)$ contains the relational product $U(j') \circ U(j')$ [that is: if (x, y) and (y, z) are both in $U(j')$ then (x, z) is in $U(j)$]; (U, J) is called a semiuniformity (or semiuniform structure) for S, when (S, U, J) is a semiuniform space. The semiuniform space is called a symmetric semiuniform space, and (U, J) is called a symmetric semiuniformity if it satisfies the condition:

(SU 3) For each j in J, and each pair of points x, y from S, (x, y) is in $U(j)$ if, and only if, (y, x) is in $U(j)$.

The semiuniformity is called a T_0-semiuniformity, and the space (S, U, J) is called a T_0-semiuniform space if the following condition holds:

(SU 4) If x, y are points of S such that, for each j in J, both (x, y) and (y, x) belong to $U(j)$, then $x = y$.

A uniform space is a symmetric, T_0-semiuniform space.

A semiuniformity (U, J) for S determines a *conjugate semiuniformity* (U_c, J) and a *symmetric associate semiuniformity* (U_s, J) defined by:

(x, y) is in $U_c(j)$ if and only if (y, x) is in $U(j)$, and

(x, y) is in $U_s(j)$ if and only if both (x, y) and (y, x) are in $U(j)$ [for each j in J].

Given semiuniform spaces (S, U, J) and (T, V, K) and a mapping f of S in T, we say that (i) f is a *uniformly continuous mapping* of (S, U, J) in (T, V, K) if, for each k in K there can be found a j in J such that $(x, y) \in U(j)$ implies $(f(x), f(y)) \in V(k)$; (ii) f is a *unimorphism* of (S, U, J) on (T, V, K) if f is a bijective map of S on T, and f and its inverse f^{-1} are both uniformly continuous mappings.

A semiuniformity (U, J) for S is said to be (i) finer than, (ii) equivalent to, the semiuniformity (V, K) for S, if the identity mapping I_S of S on itself is a (i) uniformly continuous mapping, (ii) unimorphism, of (S, U, J) in (on) (S, V, K).

A semiuniformity (U, J) for S determines a topology $\Gamma = \Gamma(U, J)$ when we take the family of sets $[U(j)](x) = [y \in S : (x, y) \in U(j)]$, as j ranges over J, as a filter-base of neighbourhoods at the point x, [for each x in S]. Clearly this topology is a T_0 topology for S if and only if the semiuniformity is T_0.

Let $(D, \leqq)$ be a down directed ordered set. By $I(D)$ or I we denote the family of nonnull initial subsets of $(D, \leqq)$ [that is subsets A such that, $a \in A$, $d \in D$ and $d \leqq a$ imply $d \in A$]. It is easily seen that finite intersections of sets in I belong to I. Given $(D, \leqq)$ any mapping X of D in the space S is called a $(D, \leqq)$-sequence of S. The $(D, \leqq)$-sequence X of S is said to converge to a point x in the topological space (S, Γ), if, for each neighbourhood $U(x)$ of x there exists an A in $I(D)$ such that $X(d)$ belongs to $U(x)$ for each d in A.

If now (S, U, J) is a semiuniform space, (V, J) is the symmetric associate of (U, J), and X is a $(D, \leqq)$-sequence of S, we say that:

(i) X converges to x in (S, U, J) if X converges to x in $[S, \Gamma(U, J)]$;

(ii) X star converges to x in (S, U, J) if X converges to x in $[S, \Gamma(V, J)]$; and

(iii) X is a Cauchy $(D, \leqq)$-sequence of (S, U, J) if, for each j in J, there exists a B in $I(D)$ such that, for all d, e from B, $[X(d), X(e)]$ belongs to $U(j)$.

We note that: (a) X is a Cauchy sequence in (S, U, J) if and only if it is a Cauchy sequence in (S, V, J); (b) if X star converges in (S, U, J) to some point x, then X is a Cauchy sequence; and (c) if (SU 4) is assumed, X can star converge to at most one point in S.

We now define (S, U, J) to be a *complete semiuniform space* if, for any down directed set $(D, \leqq)$ and any Cauchy $(D, \leqq)$-sequence X in (S, U, J), a point x can be found in S such that X star converges to x in (S, U, J).

Lemma 1. (a) *The semiuniform space (S, U, J) is complete if and only if (S, V, J) is complete.* (b) *The space (S, U, J) is complete if and only if each Cauchy $(J, \leqq)$-sequence X in it star converges to some point x in the space.*

Proof. The part (a) follows easily from the definitions and the earlier remarks. The "only if" part being obvious in part (b) we prove only the "if" part. Let $(D, \leqq)$ be a down directed set and X a $(D, \leqq)$-sequence in S. If X is a Cauchy $(D, \leqq)$-sequence in (S, U, J), we can determine a Cauchy $(I, \leqq)$-sequence Y in the space such that $Y(I)$ is contained in $X(D)$ and when Y star converges to some x in the space, then X also star converges to x. Given j in J we choose a B in $I(D)$ such that $[X(d), X(e)]$ is in $U(J)$ for d, e from B. And choosing a fixed b in B, we set $Y(j) = X(b)$. That this Y satisfies our requirements can be verified. This proves the "if" part of (b).

Theorem 1. (a) *Given a semiuniform space (S, U, J) there is an associated complete T_0 semiuniform space (S^*, U^*, J), called the canonical completion of (S, U, J) satisfying the conditions*: (i) *there is a uniformly continuous mapping h of (S, U, J) in (S^*, U^*, J), such that $(x, y) \in U(j)$ if and only if $(h(x), h(y)) \in U^*(j)$*; (ii) *if there is a uniformly continuous mapping g of (S, U, J) in a complete T_0 semiuniform space (T, V, K) then there is a (unique) uniformly continuous mapping g^* of (S^*, U^*, J^*) in (T, V, K) such that $g^* . h = g$.* (b) *If in the above, the space (S, U, J) is itself T_0 then the mapping h is a unimorphism of (S, U, J) on a subspace of (S^*, U^*, J).*

Proof. (a) Let S' denote the set of Cauchy $(J, \leqq)$-sequences of (S, U, J). For each j in J, we define $U'(j) = [(X, Y) \in S' \times S'$: there is a B in $I(J)$ such that, for each l in B, $(X(l), Y(l)) \in U(j)]$. Then (S', U', J) becomes a complete semiuniform space: from $U(j) \cap U(k) \subseteq U(l)$ we can deduce that $U'(j) \cap U'(k) \subseteq U'(l)$; and from $U(j') \circ U(j') \subseteq U(j)$ we can deduce that $U'(j') \circ U'(j') \subseteq U'(j)$; also for any X in S', $(X, X) \in$ each $U'(j)$, since X is Cauchy. Thus (U', J) gives a semiuniformity for S'. To prove completeness, given a Cauchy $(J, \leqq)$-sequence $\mathscr{X}$ of (S', U', J) and an element j of J, we choose in order: (using (SU 2)) a j'' in J such that $U(j'') \circ U(j'') \circ U(j'') \circ U(j'') \subseteq U(j)$; a B_1 in $I(J)$ such that $[\mathscr{X}(k), \mathscr{X}(l)]$ is in $U'(j'')$ for k, l in B_1 (using the fact that $\mathscr{X}$ is Cauchy); an element b_1 in B_1; for the Cauchy sequence $\mathscr{X}(b_1)$ of (S, U, J) a B_2 in $I(J)$ such that $\{[\mathscr{X}(b_1)](k), [\mathscr{X}(b_1)](l)\} \in U(j'')$ for k, l in B_2; and an element b_2 in B_2. We set then $X(j) = [\mathscr{X}(b_1)](b_2)$. This gives us a X of S' and it can be shown that $\mathscr{X}$ star converges in (S', U', J) to this point X.

To get S^*, we use the basic T_0-equivalence E on $S' : (X, Y) \in E$ meaning both (X, Y) and (Y, X) belong to every $U'(j)$, j in J. Then $S^* = S'/E$, with $U^*(j) = [(E(X), E(Y)) : (X, Y) \in U'(j)]$ gives the desired complete T_0 semiuniform space (S^*, U^*, J). The mapping h is defined by: $h(x) = E(x')$, where x' in S' is the "repeated sequence of x" defined by: $(x')(j) = x$ for each j in J. This gives (i). For (ii) given the mapping g, we define g^* by: $g^*[E(X)] =$ the unique limit y in (T, V, K) to which the Cauchy $(J, \leqq)$ sequence hX of (T, V, K) star converges. This definition is really independent of the choice of the X (of S') from $E(X)$. The details can be verified. (b) It is almost evident that the mapping h will be injective if, and only if (S, U, J) is itself a T_0 semiuniform space. Then since $(x, y) \in U(j)$ is equivalent to $(h(x), h(y)) \in U^*(j)$ the unimorphism property of h follows, for the T_0 case.

2. Cancellative, Ordered, Semiuniform Semigroups and Groups

All semigroups and groups are Abelian, written additively and have a zero element. A semigroup $(H, +)$ together with an order relation $\leqq$ on H gives an ordered semigroup $(H, +, \leqq)$ provided the following condition is satisfied: for all x, y, z from $H, x \leqq y \Rightarrow (x + z) \leqq (y + z)$.

A semigroup $(H, +)$ with a semiuniformity (U, J) gives a semiuniform semigroup $(H, +, U, J)$ provided the following condition is satisfied; for all j in J and for all x, y, z from H, $(x, y) \in U(j) \Rightarrow [(x + z), (y + z)] \in U(j)$.

Lemma 2. (a) *A semigroup $(H, +)$ is isomorphic with a subsemigroup of a (Abelian) group $(G, +)$ if, and only if, it satisfies the cancellation law: for all x, y, z from H, $(x + z) = (y + z) \Rightarrow x = y$;*

(b) *An ordered semigroup $(H, +, \leqq)$ is isomorphic (for + and $\leqq$) with an ordered subsemigroup of an ordered (Abelian) group $(G, +, \leqq)$ if, and only if, it satisfies the order-cancellation law: for all x, y, z from H, $(x + z) \leqq (y + z) \Rightarrow x \leqq y$;*

(c) *A semiuniform semigroup $(H, +, U, J)$ is isomorphic and unimorphic with a subsemigroup, with its relative semiuniformity, of a semiuniform (Abelian) group $(G, +, V, K)$ if and only if it satisfies the cancellation law and the following* *homogeneity condition for the semiuniformity: for all j in J, and for all x, y, z from H,* $[(x + z), (y + z)] \in U(j) \Rightarrow (x, y) \in U(j)$.

Proof. It is easy to see that in the case of a group the conditions asserted in (a), (b) or (c) are true and then they are easily deduced for a subsemigroup. Thus the "only if" part follows for all three.

(a) Assuming that the semigroup satisfies the cancellation law, the group $(G, +)$ of "differences" over S is constructed as usual: G is the quotient of the semigroup $H \times H$ by the congruence C on it given by: $[(x, y), (x', y')] \in C \Leftrightarrow [x + y' = x' + y]$. It is seen that h is an isomorphism of $(H, +)$ in $(G, +)$ if $h(x) = C(x, 0)$, and then $C(x, y) = h(x) - h(y)$ in G.

(b) The order cancellation law implies the cancellation law and, so as in (a), the group $(G, +)$ can be constructed from $(H, +)$. To define $\leqq$ for G, we set: $C(x, y) \leqq C(x', y') \Leftrightarrow (x + y') \leqq (x' + y)$, and note that this is really independent of the choice of the pairs from the classes $C(x, y)$ and $C(x', y')$. The other properties can be easily verified now.

(c) If the semigroup $(H, +)$ is cancellative, we define $(G, +)$ as in (*a*). Given the homogeneity condition for (U, J), we now define a semiuniformity (V, J) for $(G, +)$ by: $[C(x, y), C(x', y')] \in V(j) \Leftrightarrow [(x + y'), (x' + y)] \in U(j)$. That this satisfies all our requirements can be proved without any trouble. In the above cases we call the group, ordered group or semiuniform group, the group completion of the semigroup in question.

Let $(H, +, \leqq)$ be an ordered semigroup with order cancellation law; a subset J of H is called a permissible subset if it satisfies the following conditions:

(i) if j is in J, $j > 0$; if $x > y$ in H, there is a j in J such that $x > y + j$;

(ii) $(J, \leqq)$ is down directed (any two elements j, k of J are $\geqq$ some l in J);

(iii) $(J, +, \leqq)$ is weakly divisible: that is, each j in J is $> j_1 + j_1$, for some j_1 in J; and

(iv) if $x \neq y$ in H, there is a j in J such that either $y \nless x + j$ or $x \nless y + j$.

Lemma 3. *The semigroup $(H, +, \leqq)$ with order cancellation law has a permissible subset if and only if $P(H) = [x \in H : x > 0]$ is a permissible set. Then J is a permissible set if, and only if, J is coinitial in $[P(H), \leqq]$.*

The proof follows easily from the definitions. An ordered semigroup with order cancellation law and a permissible set J [or with $P(H)$ as a permissible set] is called an Apo-semigroup (short for: Abelian perfectly ordered semigroup). A permissible set J in an Apo-semigroup $(H, +, \leqq)$ defines a semiuniformity (U, J) when we set $U(j) = [(x, y) \in H \times H : y < x + j]$. Clearly different permissible sets, being all coinitial in $P(H)$, define equivalent semiuniformities. This semiuniformity (U, J), unique up to equivalence, defined by any permissible set is called the intrinsic semiuniformity of the Apo-semigroup. Note that the condition (iv) for the Apo-semigroup is used just for ensuring that the semiuniformity is T_0; we call it thus the T_0 condition.

Theorem 2. (a) *If (U, J) is the semiuniformity of the Apo-semigroup $(H, +, \leqq)$ defined by the permissible set J, $(G, +, \leqq)$ and (G, V, J) are the group completions of $(H, +, \leqq)$ and (H, U, J), then $(G, +, \leqq)$ is an Apo-group with $h(J)$ a permissible set, and the intrinsic semiuniformity of this group is equivalent to (V, J).*

(b) *If (H^*, U^*, J) is the canonical completion of the semiuniform Apo-semigroup $(H, +, \leqq, U, J)$ an addition $+^*$ and an order $\leqq^*$ can be so defined for H^* that $(H^*, +^*, \leqq^*)$ is an Apo-semigroup with $h^*(J)$ as permissible set, and the intrinsic semiuniformity equivalent to (U^*, J).*

Proof. (a) This part is easy to verify using the definitions. (b) We defined (H^*, U^*, J) as the quotient of (H', U', J) by its T_0 equivalence. We define a $+'$ and a $<'$ for the set H' by: $(X +' Y)(j) = X(j) + Y(j)$, for each j in J, and $X <' Y \Leftrightarrow$ $\Leftrightarrow$ (there is a k in J and a B in $I(J)$ such that $X(l) + k \leqq Y(l)$ for all l in B). We deduce a number of consequences; the first three below are easily proved:

(i) $(H', +')$ is an Abelian semigroup with zero, and E is a congruence on this.

(ii) For all X, Y, Z from H', $X <' Y \Leftrightarrow (X +' Z) <' (Y +' Z)$.

(iii) $<' \circ <' \subseteq <'$.

(iv) $E \circ <' \circ E \subseteq <'$: if (X', X) and (Y, Y') are in E (the basic T_0 equivalence in (H', U')) and $X <' Y$, there exist k, k_1 in J and B, B_1 in $I(J)$ such that $X(l) +$ $+ k \leqq Y(l)$ for all l in B, $k_1 + k_1 + k_1 \leqq k$, and $X'(l) \leqq X(l) + k_1$, $Y(l) \leqq$ $\leqq Y'(l) + k_1$ for all l in B_1. Then for all l in $B \cap B_1$, which also belongs to $I(J)$, we can deduce that $X'(l) + k_1 + k_1 \leqq Y'(l) + k_1$ and so $X'(l) + k_1 \leqq Y'(l)$, proving that $X' <' Y'$.

(v) For any x, y of H if x', y' denote the elements of H' such that $x'(j) =$ $= x$, $y'(j) = y$ for all j in J, then $(x + y)' = x' +' y'$, and $x < y \Leftrightarrow x' <' y'$. Note that from $x < y$ we have $x + k \leqq y$ for some k in J by condition (i) on the permissible set J. Hence $x' <' y'$.

(vi) If $j_1 + j_1 \leqq j$ in J, for any X, Y of H', $(X, Y) \in U'(j_1) \Rightarrow Y <' X +' j' \Rightarrow$ $\Rightarrow (X', Y') \in V(j)$, when (X', X), (Y, Y') are in E. Since $(X, Y) \in U'(j_1)$ gives, for all l in a suitable B of $I(J)$, $Y(l) \leqq X(l) + j_1$, we can deduce that $Y(l) + j_1 \leqq$ $\leqq X(l) + j = (X +' j')(l)$, so that $Y <' X +' j'$ follows. The other part is easy to prove.

(vii) If $(H, \leqq)$ is totally ordered, and X, Y are elements of H' such that $X(l) \leqq$ $\leqq Y(l)$ for all l in a suitable B of $I(J)$, then either $X <' Y$ or $(X, Y) \in E$. Assuming $X(l) \leqq Y(l)$ for almost all l in J, and $(X, Y) \notin E$, we can prove that there exist j, j_1 in J such that $(X, Y) \notin U'(j)$, and $j_1 + j_1 \leqq j$; for (Y, X) is in each $U'(j)$. If the set of k in J for which $Y(k) < X(k) + j_1$ were coinitial in $(J, \leqq)$, then such a k

will belong to the B_1 in $I(J)$, which we can so choose that $[Y(l), Y(k)]$ and $[X(k), X(l)]$ belong to $U(j_1)$ for all k, l of B_1. It would then follow that $Y(l) < X(l) + j$ for all l in B_1, contradicting the fact that $(X, Y) \notin U'(j)$. Hence the set of these k in J for which $Y(k) < X(k) + j_1$ cannot be coinitial in $(J, \leqq)$, which means that for a B_2 in $I(J)$ one has $Y(l) \nless X(l) + j_1$ for each l in B_2. If $(H, \leqq)$ were totally ordered then $\nless$ can be replaced by $\geqq$, so that it follows then that $X <' Y$.

We can now define $+^*$ and $\leqq^*$ in $H^* = H'/E$ by: $E(X) +^* E(Y) = E(X +' Y)$; $E(X) \leqq^* E(Y) \Leftrightarrow [X <' Y$ or $(X, Y) \in E]$. Using (i) (ii) (iii) and (iv) we deduce that $(H^*, +^*, \leqq^*)$ is an ordered semiuniform semigroup with order cancellation law. (Note that $[(X + Z), (Y + Z)]$ is in E implies that (X, Y) is in E). The mapping $h : H \to H^*$ given by $h(X) = E(x')$ is an isomorphism as well as unimorphism of $(H, +, \leqq, U, J)$ in $(H^*, +^*, \leqq^*, U^*, J)$. Thence the conditions (i), (ii), (iii) for $(H^*, +^*, \leqq^*)$ to have $h(J)$ as permissible set are easily deduced. To deduce (iv), it suffices to show that the J^* (or $h^*(J)$) semiuniformity for $(H^*, +^*, \leqq^*)$ is a T_0 semiuniformity; this would follow if we prove that this semiuniformity is equivalent to the semiuniformity (U^*, J) of the canonical completion, which is known to be T_0. But result (vi) proved earlier gives this equivalence, since U^* and V^* are equivalent where $[E(X), E(Y)] \in U^*(j) \in [V^*(j)]$ if, and only if, $(X', Y') \in U'(j)$ for some [for all] X', Y' from $E(X)$, $E(Y)$.

3. Seminorm, Semiecart, Semimetric in Apo-Semigroups

Let $(H, +, \leqq)$ be an Apo-semigroup, with J as a permissible set. The typical method of defining the intrinsic semiuniformity (U, J) for H can be extended further:

(i) A mapping n of a set S in H is called a seminorm for S in H; this determines a semiuniformity (V, J) for S as follows: for each j in J, we set $V(j) = [(x, y) \in S \times S : n(y) < n(x) + j]$;

(ii) A mapping d of $S \times S$ in H is called a semimetric for S in H, if, for all x, y, z from S, $d(x, y) + d(y, z) \geqq d(x, z)$; then a semiuniformity (V, J) is determined for S if we set $V(j) = [(x, y) \in S \times S : d(x, y) < j]$, for each j in J;

(iii) A mapping d of $S \times S$ in H is called a semiecart for S in H if, for each j in J there is a j_1 in J such that, whatever be x, y, z from S, $d(x, y) < j_1$ and $d(y, z) < j_1$ imply $d(x, z) < j$; this also determines a semiuniformity (V, J) in the same way as for the semimetric.

Note that, if $(G, +, \leqq)$ is the group completion of $(H, +, \leqq)$, a seminorm n for S in H gives rise to a semimetric d for S in G such that both these give rise to the same semiuniformity for S: we have only to set $d(x, y) = h\, n(y) - h\, n(x)$, where h is the isomorphism of H in G. And again, any semimetric d is also a semiecart, since J is weakly divisible.

Theorem 3. (a) *If* (S^*, V^*, J) *is the canonical completion of the semiuniform space* (S, V, J) *where the semiuniformity* (V, J) *is derived from a seminorm* n *for* S *in* H, *a seminorm* N^* *for* S^* *in* H^* *can be found such that* (V^*, J) *is the semiuniformity derived from this,* (H^*, U^*, J) *being the canonical completion of* H *with its intrinsic semiuniformity* (U, J).

(b) *If in the above,* (V, J) *is derived from a semiecart* d *for* S *in* H, *then* (V^*, J) *can be derived from a semiecart* d^* *defined for* S^* *in* H^*.

(c) *If in* (a), $(H, \leqq)$ *is totally ordered, and the semiuniformity* (V, J) *for* S *is derived from a semimetric* d *for* S *in* H, *then the semiuniformity* (V^*, J) *for* S^* *can be derived from a semimetric* d^* *for* S^* *in* H^*.

Proof. (a) We define n^* as follows: $n^*[E(X)] = E(nX)$, for any X from S', that is any Cauchy $(J, \leqq)$-sequence of (S, V, J); then nX is a Cauchy $(J, \leqq)$-sequence of (H, U, J). The details are not hard to verify.

(b) and (c). We define d^* by: $d^*[E(X), E(Y)] = E[d(X, Y)]$, where for Cauchy sequences X, Y from (S, V, J) we set $d(X, Y)(j) = d[X(j), Y(j)]$, for each j in J, so that $d(X, Y)$ is a Cauchy sequence of (H, U, J). Again the proofs are not hard; for part (*c*) we have to use the result (vii) proved in the course of proving part (b) of theorem 2, to show that the semiecart d^* is a semimetric when d is a semimetric.

4. Some Examples

If $(L, \leqq)$ is an inversely well ordered set, and for each l in L, we have an Apo-semigroup $(H_1, +, \leqq)$ with J_l a permissible set in it, we can form the (inverse) lexicographic product $P^*[(H_l, +, \leqq) : l \text{ in } (L, \leqq)]$; the elements are those X of the direct product $P[(H_1, +) : l \text{ in } L]$ of the semigroups for which $X(l)$ is zero except for atmost a finite number of values of l; addition is componentwise; and order $\leqq$ is defined by last differences: $X \leqq Y$ if either $X = Y$ or for the last place l at which $X(l) \neq Y(l)$, we have $X(l) < Y(l)$. It is not hard to see that this lexicographic product semigroup is also an Apo-semigroup, with a permissible set J given by: J = union of the $i_l(J_l)$, where i_l denotes the injective homomorphism of $(H_l, +, \leqq)$ in the lexicographic product ($i_l(x) = X$, with $X(m)$ = zero except for $m = l$, and $X(l) = x$).

It can be verified that the lexicographic product is complete in its intrinsic semiuniformity if either (i) $(L, \leqq)$ has no least element or (ii) there is a least element m in $(L, \leqq)$ and $(H_m, +, \leqq)$ is complete in its semiuniformity. Thus using the last theorem and the well known Hahn's theorem that any Abelian ordered group is isomorphic with an ordered subgroup of a lexicographic product of ordered groups each isomorphic with the group of reals, we have the following corollary.

Corollary. *The canonical completion of a semiuniform space, whose semiuniformity is derived from a seminorm, semimetric or semiecart in an ordered*

Abelian group without a first Archimedean class, has also its semiuniformity derivable from a similar seminorm, semimetric or semiecart in a similar totally ordered Abelian group.

References

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