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ON REFLECTIVE AND COREFLECTIVE SUBCATEGORIES OF LIM, LUN, AND LVEC

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There are many important reflective and coreflective subcategories of the categories LIM, LUN, and LVEC of all limit spaces, all limit-uniform spaces, and all limit vector spaces, respectively.

We define limit spaces as convergence spaces in the sense of Kowalsky [10] and Fischer [2] but we omit the condition $[x] \longrightarrow x$. Analogously, we define limit-uniform spaces as uniform convergence spaces in the sense of Cook and Fischer [1] but we do not demand that the principal filter $[\triangle]$ generated by the diagonal $\triangle$ belongs to the limit-uniform structure. Convergence spaces in the sense of Kowalsky and Fischer we also call pseudo-topological spaces and uniform convergence spaces pseudo-uniform spaces. In an obvious manner we define the limit vector spaces and the more special pseudo-topological vector spaces. There is a functor λ: LUN $\longrightarrow$ LIM with the following properties: For every limit-uniform space X with limit-uniform structure $\mathfrak{u}$, $\lambda(X)$ is the limit space with the same underlying set as X and the limit structure τ defined by $\mathcal{F} \in \tau(x) \Longleftrightarrow [x] \times \mathcal{F} \in \mathfrak{u}$.

A limit space X is called limit-uniformizable (pseudo-uniformizable, uniformizable) if there exists a limit-uniform (pseudo-uniform, uniform) structure $\mathfrak{u}$ of the underlying set which generates the limit structure τ of X ($\lambda(\mathfrak{u}) = \tau$). In [8] it is shown that limit vector spaces in general are not limit-uniformizable. It is well known that every pseudo-topological vector space is pseudo-uniformizable. To every pseudo-topological vector space X there is a natural pseudo-uniform structure of the underlying set, which we call canonical pseudo-uniform structure of X (see [6]). It is defined as the finest pseudo-uniform structure $\mathfrak{u}$ of the underlying set with the properties

1. $\mathfrak{u}$ generates the limit structure of X.
2. $\mathcal{F}$ is a Cauchy filter iff $\mathcal{F} - \mathcal{F} \longrightarrow 0$.
3. $\mathcal{F}, \mathcal{G} \in \mathfrak{u} \Longrightarrow \mathcal{F} - \mathcal{G} \in \mathfrak{u}$.

If w is the mapping $(x,y) \longmapsto x - y$ of $X \times X$ into X, then the canonical pseudo-uniform structure has the base $\{w^{-1}(\mathcal{F}) \mid \mathcal{F} \rightarrow 0\}$. All notions for pseudo-topological vector spaces which depend on pseudo-uniform structures are understood with respect to the canonical pseudo-uniform structures.

A subcategory $\mathcal{D}$ of a category $\mathcal{C}$ is called reflective in $\mathcal{C}$ if the embedding functor of $\mathcal{D}$ into $\mathcal{C}$ has a left adjoint, that means that to every object X of $\mathcal{C}$ there correspond an object $X^{\mathcal{D}}$ of $\mathcal{D}$ and a morphism $\iota: X \longrightarrow X^{\mathcal{D}}$, the reflection of X, with the property: To each morphism f of X to an object Y of $\mathcal{D}$ there exists one and only one morphism $g: X^{\mathcal{D}} \longrightarrow Y$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\iota} & X^{\mathcal{D}} \\ & \searrow_{f} & \downarrow g \\ & & Y \end{array}$$

commutes. $\mathcal{D}$ is called epireflective in $\mathcal{C}$ if $\mathcal{D}$ is reflective in and the reflections are epimorphisms.

The categories PTOP, MTOP, and TOP of all pseudo-topological spaces, all pretopological spaces (mehrstufig topologische Räume), and all topological spaces, respectively, are simple examples of epireflective subcategories of LIM. The categories PUN and UN of all pseudo-uniform spaces and all uniform spaces, respectively, are simple examples of epireflective subcategories of LUN. Finally, the categories PVEC and TVEC of all pseudo-topological vector spaces and all topological vector spaces, respectively, are simple examples of epireflective subcategories of LVEC. In all these cases the reflections are identical mappings. In the first three cases we denote these reflections by $X \longrightarrow p(X)$, $X \longrightarrow m(X)$, and $X \longrightarrow t(X)$, respectively.

A morphism f of a category $\mathcal{C}$ is called an extreme monomorphism if it is a monomorphism and if for each factorization $h \circ g$ of f with an epimorphism g it follows that g is an isomorphism. Every equalizer is an extreme monomorphism. In certain cases the converse statement also holds.

Proposition 1. If every morphism of a category $\mathcal{C}$ has a factorization $h \circ g$ with an epimorphism g and an equalizer h, then every extreme monomorphism of $\mathcal{C}$ is an equalizer.

In [7] and [8] it is proved that in LIM, LUN, and LVEC equalizers characterize subspaces. Because for these categories the assumption in Proposition 1 is fullfilled, in LIM, LUN, and LVEC extreme monomorphisms characterize subspaces.

We note a useful criterion for subcategories to be epireflective (see e.g. [12] and proposition 1).

Theorem 1. Let $\mathcal{C}$ be a locally cosmall category with products. Suppose that every morphism of $\mathcal{C}$ has a factorization $h \circ g$ with an epimorphism g and an equalizer h. Let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$ which is closed in $\mathcal{C}$ with respect to isomorphisms. Then $\mathcal{D}$ is epireflective in $\mathcal{C}$ iff $\mathcal{D}$ is closed in $\mathcal{C}$ with respect to products and equalizers.

LIM, LUN, and LVEC are complete and cocomplete categories ([7], [8]). They are also locally small and cosmall.

Let T be one of the following separation axioms

T_{1w} (weak T_1): $[x] \to y \Longrightarrow ([x],\ [y] \to x$ and $[x], [y] \to y)$
T_1 : $[x] \to y \Longrightarrow x = y$
T_{2w} (weak T_2, weakly separated): $\exists \mathcal{F} \neq PX,\ \mathcal{F} \to x, y \Longrightarrow (\mathcal{G} \to x \Longleftrightarrow \mathcal{G} \to y)$
T_2(separated): $\exists \mathcal{F} \neq PX,\ \mathcal{F} \to x, y \Longrightarrow x = y$
T_3 (regular) : $\mathcal{F} \to x \Longrightarrow \mathcal{F}^-$ with base $\{\overline{F} \mid F \in \mathcal{F}\}$ converges to x
T_{3s} (strictly regular): $\mathcal{F} \to x \Longrightarrow \mathcal{F}^\sim$ with base $\{F \in \mathcal{F} \mid F = \overline{F}\}$ converges to x.

In [7] it is proved, that then the product of T-limit spaces X_i $(i \in I)$ is a T-limit space and every subspace of a T-limit space is a T-limit space.

From Theorem 1, therefore it follows

Proposition 2. For every quoted separation axiom T the full subcategory LIM^T of LIM of all T-limit spaces is epireflective in LIM.

A limit space is called a pseudo-topological space in the wide sense, if for every proper filter $\mathcal{F} \to x$ it follows $[x] \to x$. In [6] it is shown that a limit space is limit-uniformizable iff it is a pseudo-topological space in the wide sense which is T_{1w} and T_{2w}.

Because products and subspaces of pseudo-topological spaces in the wide sense are pseudo-topological in the wide sense, by Theorem 1 we obtain

Proposition 3. The full subcategory LIM^{lun} of LIM of all limit-uniformizable limit spaces is epireflective in LIM.

By means of a result of Keller [9] we analogously get, that the full subcategory $PTOP^{pun}$ of LIM of all pseudo-uniformizable limit spaces is epireflective in LIM. By a classical result we have, that the full subcategory TOP^{un} of LIM of all uniformizable limit spaces is epireflective in LIM.

By Proposition 3 and Theorem 4.4.3 of [7] we can prove that the functor λ: LUN $\to$ LIM has a left adjoint. Analogously we can show that the restrictions λ^{PUN}: PUN $\to$ LIM and λ^{UN}: UN $\to$ LIM of λ have left adjoints (see for the last case [4]).

Ramaley and Wyler have proved in [13] that the category LIM^c of all regular compact T_1-limit spaces is reflective in PTOP and therefore also in LIM. In [14], Wyler has solved the problem of completion of pseudo-uniform spaces. He has proved that the category $PUN^{\sim}$ of all separated complete pseudo-uniform spaces is reflective in PUN and therefore in LUN. The corresponding reflections ι: $X \to X^{\sim}$ have the following properties:

1. $\iota[X]$ is dense in $X^{\sim}$.
2. ι is an embedding iff X is a separated pseudo-uniform space.

In [7] the problem of completion of the more general limit-uniform spaces is solved with analogues results.

In [5] , by S.Gähler, G.Kneis, and the author, the problem of completion of pseudo-topological vector spaces is solved. There are two important properties of pseudo-topological vector spaces, the saturation and the Cq-property. A pseudo-topological vector space X is saturated if the principal filter [N] with $N = \{x \in X \mid [x] \to 0\}$ converges to 0. We say that a pseudo-topological vector space X has the Cq-property if every Cauchy filter $\mathcal{F}$ is quasi-bounded, i.e. the product $\mathbb{V}\mathcal{F}$ converges to 0 where $\mathbb{V}$ is the neigborhood filter of 0 in $\mathbb{R}$. We construct our completion in three steps. In the first and second step we prove that the category $PVEC^{sat}$ of all saturated pseudo-topological vector spaces and the category $PVEC^{Cq}$ of all pseudo-topological vector spaces with the Cq-property are reflective in PVEC. The corresponding reflections, in the second case, we denote by $X \to X^{Cq}$. Because X^{Cq} is saturated if X is saturated, the category $PVEC^{Cq\,sat}$ of all saturated pseudo-topological vector spaces with the Cq-property is reflective in PVEC, too. In the third step we prove that the category $PVEC^{\sim}$ of all separated complete pseudo-topological vector spaces is reflective in $PVEC^{Cq\,sat}$ and therefore in LVEC. In the last case, the corresponding reflections j: $X \to X^{\sim}$ have the following properties:

1. j[X] is dense in $X^{\sim}$.
2. j is an embedding iff X is a separated pseudo-topological vector space with the Cq-property.

If X is a topological vector space and ι: $X \to c(X)$ is the classical completion, then it is $\iota = j$ and c(X) is the vector-topological modification of X .

In the following we note some coreflective subcategories of LIM and LVEC. Of course, "coreflective" is defined dually to "reflective". The category of all locally compact limit spaces is an example of a coreflective subcategory of LIM. The corresponding coreflections $X^1 \to X$ are identical mappings. If τ is the limit structure of X, the limit structure τ^1 of X^1 is defined by $\tau^1(x) = \{ \mathcal{F} \in \tau(x) \mid \mathcal{F}$ contains a compact subset of $X\}$. As it is shown in [3], a limit space X is a compactly generated topological space iff $X = t(X^1)$. The category of all compactly generated topological spaces is coreflective in TOP, the category of all pretopological spaces X with $X = m(X^1)$ is coreflective in MTOP, and the category of all locally compact pseudo-topological spaces is coreflective in PTOP.

We denote by $PVEC^{\#}$, $PVEC^{\varepsilon}$, and $PVEC^{\delta}$ the full subcategories of PVEC which are defined as follows: The equable pseudo-topological vector spaces are the objects of $PVEC^{\#}$. They are defined as the pseudo-topological vector spaces for which to every filter $\mathcal{F} \to 0$ there exists a filter $\mathcal{G}$ with $\mathcal{F} \geqq V\mathcal{G} \to 0$. The pseudotopological vector spaces X such that for every filter $\mathcal{F} \to 0$ on X there exists a dual filter $\mathcal{G}$ on X with $\mathcal{F} \geqq \bigwedge_{G \in \mathcal{G}} VG \to 0$ are the objects of $PVEC^{\varepsilon}$. A topological vector space is of this kind iff it is bornologic with respect to TVEC. The equable and locally bounded pseudo-topological vector spaces are the objects of $PVEC^{\delta}$. They are the pseudo-topological vector spaces X for which to every filter $\mathcal{F} \to 0$ on X there exists a subset B of X with $\mathcal{F} \geqq VB \to 0$. The Mackey convergence with respect to bounded sets is the filter convergence in these spaces.

The categories $PVEC^{\#}$, $PVEC^{\varepsilon}$, and $PVEC^{\delta}$ are examples of coreflective subcategories of LVEC. They are useful in a general differential calculus and in a theory of generalized bornologic spaces (see [8]).

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