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It is shown that if I is the unit interval and X a set then there exists a filtertheory in the lattice I^X which deviates in a remarkable way from the theory of filters on X, but which nevertheless makes it possible to define a concept of convergence in a fuzzy topological space similar to convergence of filters in topology. If a fuzzy topological space is topologically generated [L1] relations are found between fuzzy convergence and topological convergence.

Using this notion of convergence characterizations are given of fuzzy compactness [L1] and of fuzzy continuity.

We shall only give a summary of the most important results, an extended version with proofs will appear elsewhere.

1. Filtertheory in I^X.

The definition of a *filter*, a *filterbase* or a *generating family for a filter in the lattice* I^X can be found f.i. in [Bo]. Let us recall that a *prime filter* in I^X is a filter $\mathfrak{F}$ such that if $\mu, \nu \in I^X$ and $\mu \vee \nu \in \mathfrak{F}$ then $\mu \in \mathfrak{F}$ or $\nu \in \mathfrak{F}$.

For any $a \in I$ and $A \subset X$ the function in I^X which assigns the value a to x if $x \in A$ and 0 if $x \notin A$ is denoted by $a\chi_A$.

The definition of infimum and supremum of a family of filters in I^X, and the definition of a coarser or a finer filter are straightforward generalizations of those for filters on X and can also be found f.i. in [Bo].

Given a filter $\mathfrak{F}$ in I^X we shall want to know to what extend $\mathfrak{F}$ is "uniformly bounded away from 0". To make this precise we introduce the characteristic set of $\mathfrak{F}$

$$C(\mathfrak{F}) = \{a \in I : \forall \nu \in \mathfrak{F} \quad \exists x \in X \text{ s.t. } \nu(x) > a\}$$

and characteristic value $c_{\mathfrak{F}} = \sup C(\mathfrak{F})$.

$C(\mathfrak{F})$ can be any of the following ϕ, $\{0\}$, $[0,c[$ for some $c \in I$ or $[0,c]$ for some $c \in I\setminus\{1\}$.

If $\mathbb{F}(X)$ denotes the family of all filters on X and $\mathfrak{F}_K(X)$ the family of all filters in I^X with characteristic set K then we define the following mappings

$$\omega_K : \mathbb{F}(X) \to \mathfrak{I}_K(X)$$

$$F \to \{\mu : \forall k \in K,\ \overset{-1}{\mu}]k,1] \in F\}$$

$$\iota_K : \bigcup_{K' \supset K} \mathfrak{I}_{K'}(X) \to \mathbb{F}(X)$$

$$\mathfrak{J} \longrightarrow \{\overset{-1}{\mu}]k,1] : k \in K,\ \mu \in \mathfrak{J}\}$$

It is easily verified that $\iota_K(\mathfrak{J})$ is indeed a filter on X and that $\omega_K(F)$ is indeed a filter in I^X with characteristic set K. These functions establish a natural relationship between ultrafilters on X and prime filters in I^X as is shown in the next theorem.

Theorem 1.1.

If U is an ultrafilter on X then $\omega_K(U)$ is a prime filter in I^X and moreover it is maximal in $\mathfrak{I}_K(X)$, and if $\mathfrak{U}$ is a prime filter in I^X then, for all $K \subset C(\mathfrak{U})$, $\iota_K(\mathfrak{U})$ is an ultrafilter on X.

In the same way as for filters on X a filter $\mathfrak{J}$ in I^X is completely determined by the family $P(\mathfrak{J})$ of primefilters in I^X which are finer than $\mathfrak{J}$, i.e.

$$\mathfrak{J} = \bigcap_{\mathfrak{U} \in P(\mathfrak{J})} \mathfrak{U}.$$

Contrary to the situation for filters on X, $P(\mathfrak{J})$ can be reduced to a smaller subfamily with analogous properties. Indeed the next theorem can be shown

Proposition 1.2.

The family $P(\mathfrak{J})$ is inductive in the sense that each descending chain in it has a lower bound.

This result enables us to replace $P(\mathfrak{J})$ by

$$P_m(\mathfrak{J}) = \{\mathfrak{U} : \mathfrak{U} \in P(\mathfrak{J}),\ \mathfrak{U}\ \text{minimal}\}$$

while maintaining the fact that

$$\mathfrak{J} = \bigcap_{\mathfrak{U} \in P_m(\mathfrak{J})} \mathfrak{U}.$$

It is the family $P_m(\mathfrak{J})$ rather than $P(\mathfrak{J})$ which with regard to $\mathfrak{J}$ plays the role of the family of ultrafilters finer than F with regard to F.

This is made precise in the following theorem, but first we need to introduce another concept.
Let $\mathfrak{F}$ be a filter in I^X and F a filter on X. Then $\mathfrak{F}$ and F are said to be compatible iff for all $\mu \in \mathfrak{F}$ and $F \in F$ μ does not vanish everywhere on F.
It is clear that if $\mathfrak{F}$ and F are compatible then

$$(\mathfrak{F},F) = \{\mu \in I^X : \exists \nu \in \mathfrak{F},\ A \in F \quad \mu(x) \geq \nu(x)\ \forall x \in A\}$$

is a filter in I^X.
We now come to the important result.

Theorem 1.3.

Let $\mathfrak{F}$ be a filter in I^X then $\mathfrak{U} \in P_m(\mathfrak{F})$ iff there exists an ultrafilter U on X, compatible with $\mathfrak{F}$ such that $(\mathfrak{F},U) = \mathfrak{U}$, i.e.

$$P_m(\mathfrak{F}) = \{(\mathfrak{F},U) : U \text{ ultrafilter compatible with } \mathfrak{F}\}.$$

2. Convergence in fuzzy topological spaces.

Let us recall that a fuzzy topological space, FTS for short, is a set X together with a family Δ of functions from X to I (so called fuzzy subsets of X) which fulfills properties, similar to those of the open sets in a topological space.
For a precise definition of a FTS and related notions which are used in what follows we refer to [L1] and [L2].
Let now $\mathfrak{F}$ be a filter in I^X, which for consistency we shall call a fuzzy filter then we define the *adherence of* $\mathfrak{F}$ to be the fuzzy set

$$\operatorname{adh}\mathfrak{F} : X \to I$$
$$x \to \inf_{\mu \in \mathfrak{F}} \bar{\mu}(x)$$

where $\bar{\mu}$ is the fuzzy closure [L1] of μ.
The *limit of* $\mathfrak{F}$ is the fuzzy set

$$\lim \mathfrak{F} : X \to I$$
$$x \to \inf_{\mathfrak{U} \in P_m(\mathfrak{F})} \operatorname{adh}\mathfrak{U}(x)$$

Remark that if one had taken $P(\mathfrak{F})$ instead of $P_m(\mathfrak{F})$ in this definition the limit of any fuzzy filter in any fuzzy topological space would be zero.

Proposition 2.1.

Let $\mathfrak{F}$ and $\mathfrak{G}$ be fuzzy filters then

(i) if $\mathfrak{F} \supset \mathfrak{G}$, $\operatorname{adh}\mathfrak{F} \leq \operatorname{adh}\mathfrak{G}$

(ii) $\lim \mathfrak{F} \leqslant \operatorname{adh} \mathfrak{F}$

(iii) if $\mathfrak{G}$ is prime, $\lim \mathfrak{G} = \operatorname{adh} \mathfrak{G}$

Remark that there need be no relation between the limits of comparable fuzzy filters. Anticipating on the result of the next theorem this is shown by the following counterexample.
First though we recall that given a topology $\mathcal{T}$ on X the associated fuzzy topology on X consists of the family of all lower semicontinuous functions (open fuzzy sets) from $(X,\mathcal{T})$ to the unit interval equipped with the usual topology. This associated fuzzy topology is denoted $\omega(\mathcal{T})$. By means of the identification $(X,\mathcal{T}) \approx (X,\omega(\mathcal{T}))$ the category of topological spaces becomes a full subcategory of that of fuzzy topological spaces[L1].

Counterexample

Let $(X,\mathcal{T})$ be a non-Hausdorff topological space and let $(X,\omega(\mathcal{T}))$ be the associated FTS
Let F and G be filters on X such that $F \supsetneqq G$ and $\lim F \supsetneqq \lim G \neq \phi$. Finally let $K \subsetneqq K'$ be characteristic sets. It is easily seen that $\omega_K(F)$ is finer than $\omega_{K'}(G)$ but it follows from the next theorem that their limits are incomparable.

Theorem 2.2.

If $(X,\mathcal{T})$ is a topological space, F a filter on X and K some characteristic set then in the associated FTS $(X,\omega(\mathcal{T}))$ we have

(i) $\operatorname{adh} \omega_K(F) = (\sup K) \cdot \chi_{\operatorname{adh} F}$

(ii) $\lim \omega_K(F) = (\sup K) \cdot \chi_{\lim F}$

3. Characterization of fuzzy compactness and of fuzzy continuity.

A FTS (X,Δ) is fuzzy compact [L1] iff for all family of open fuzzy sets $\Lambda \subset \Delta$, for all $a \in I$ such that $\sup_{\mu\in\Lambda} \mu \geqslant a$, and for all $b < a$, there exists a finite subfamily $\Lambda_0 \subset \Lambda$ such that $\sup_{\mu\in\Lambda_0} \mu \geqslant b$.
It was shown in [L1] and [L3] that this definition is a good extension of the notion of compactness. Indeed a topological space is compact iff the associated FTS is fuzzy compact.
Fuzzy compactness, in a similar but more elaborate way as compactness can be characterized by means of convergence of fuzzy filters. The following theorem can be shown.

Theorem 3.1.

The fuzzy topological space (X,Δ) is fuzzy compact iff one of the following equivalent properties holds

(i) for each fuzzy filter $\mathfrak{F}$ for which the characteristic value $c_{\mathfrak{F}}$ is strictly positive

$$\sup_{x\in X} \operatorname{adh}\mathfrak{F}(x) \geq c_{\mathfrak{F}}$$

(ii) for each prime fuzzy filter $\mathfrak{U}$ for which the characteristic value $c_{\mathfrak{U}}$ is strictly positive

$$\sup_{x\in X} \lim \mathfrak{U}(x) \geq c_{\mathfrak{U}}$$

A function f from a FTS (X,Δ) to a FTS (Y,Ω) is fuzzy continuous [W] iff for all $\mu \in \Omega$ $f^{-1}(\mu) \in \Delta$ where $f^{-1}(\mu)$ is defined as $\mu\circ f$.

Given a fuzzy filter $\mathfrak{F}$ on X, its image through f is defined $f(\mathfrak{F}) = \{\mu : \exists\, \nu \in \mathfrak{F} \text{ s.t. } \mu \geq f(\nu)\}$.
The image of a fuzzy set is defined as follows, let $\nu \in I^X$ then $f(\nu) = \min\{\xi \in I^Y : \xi\circ f \geq \nu\}$.
Both image and primage of fuzzy sets are straightforward generalizations of the corresponding notions for sets.
The following results are obtained.

Theorem 3.2.

A function $f : X,\Delta \to Y,\Omega$ is fuzzy continuous iff one of the following equivalent conditions holds

(i) for each fuzzy filter $\mathfrak{F}$ on X
$$\operatorname{adh} f(\mathfrak{F}) \geq f(\operatorname{adh}\mathfrak{F})$$

(ii) for each prime fuzzy filter $\mathfrak{U}$ on X
$$\lim f(\mathfrak{U}) \geq f(\lim\mathfrak{U})$$

It is worthwhile to remark that whereas in topology the result (ii) is trivial, here it is not and the technique of the proof is entirely different. It rests among other things, on the important result which says that if $\mathfrak{F}$ is a fuzzy filter then

$$\operatorname{adh}\mathfrak{F} = \sup_{\mathfrak{U}\in P_m(\mathfrak{F})} \operatorname{adh}\mathfrak{U}.$$

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