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THE NOTION OF $\mathcal{K}$-SHAPE

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The notion of shape was introduced by Borsuk [1] for metric compacta. Further this notion was extended to the metric spaces by Borsuk [2] himself and Fox [6] in two different ways, to the compact Hausdorff spaces by Mardešić and Segal [8] , to the Hausdorff spaces by Rubin and Sanders [9] , finally to the arbitrary topological spaces by Mardešić [7]. The shape equivalence between two spaces meant in the sense of either of these notions is a weaker relation than the homotopy equivalence.

In the paper [3] I introduced a notion of uniform shape equivalence for metric spaces which on one hand is waker than the uniform homotopy equivalence and on the other hand is stronger than the shape equivalence in the sense of Fox. In order to come to its definition one considers every metric space as uniformly embedded in some complete metric space which is an absolute neighbourhood extensor for metric spaces and then one follows the idea of Fox [6] .

It is not hard to see that in the same way, but by means of embedding in spaces which are absolute neighbourhood extensor for uniform spaces, one can get a notion of uniform shape equivalence for arbitrary uniform spaces (its construction is given more detailed in [5]) which is also waker than the uniform homotopy equivalence. But it seems quite unlikely that it is stronger than, for instance, the shape equivalence in the sense of Mardešić [7] or in the sense of Rubin - Sanders [9] .

At the same time an uniform shape equivalence should be a stronger relation than a simple (i.e. non-uniform) shape equivalence. Here a notion of shape equivalence having some relative character is proposed, which, in the case of the uniform spaces, satisfies this requirement in respect to the uniform shape equivalence just mentioned above. A notion of shape of this kind was introduced earlier in [4] . In its definition Fox's notion of mutation is taken as basic.

When P is a topological space and $X \subset P$, then $\underline{U}(X,P)$ will denote in the sequel the family of all (open) neighbourhoods of X in P. The mutation $\underline{f}:\underline{U}(X,P)\to\underline{U}(Y,Q)$ is understood in the meaning of Fox [6] with homotopy as basic equivalence relation. Also in the manner of Fox one defines the composition $\underline{g}\ \underline{f}$ of two mutations $\underline{f}:\underline{U}(X,P)\to\underline{U}(Y,Q)$ and $\underline{g}:\underline{U}(Y,Q)\to\underline{U}(Z,R)$, the identity mutation $\underline{i}_{(X,P)}:\underline{U}(X,P)\to\underline{U}(X,P)$, as well as the homotopy relation $\underline{f}\simeq\underline{g}$ between two mutations $\underline{f},\underline{g}:\underline{U}(X,P)\to\underline{U}(Y,Q)$. Finally, one writes $\underline{U}(X,P)\simeq\underline{U}(Y,Q)$ if there exist two mutations

$\underline{f}:\underline{U}(X,P)\to\underline{U}(Y,Q)$ and $\underline{g}:\underline{U}(Y,Q)\to\underline{U}(X,P)$ such that $\underline{g}\ \underline{f}\simeq\underline{i}_{(X,P)}$ and $\underline{f}\ \underline{g}\simeq\underline{i}_{(Y,Q)}$, and thus an equivalence relation is defined.

Let now $\mathcal{K}$ be a class of topological spaces. Two spaces X and Y of the class $\mathcal{K}$ are called $\mathcal{K}$-shape similar to one another if they can be homeomorphically embedded in some, belonging to $\mathcal{K}$, spaces P and Q, respectively, in such a manner that $\underline{U}(X,P)\simeq\underline{U}(Y,Q)$ holds. X and Y are said to be of the same shape in respect to the class $\mathcal{K}$ or, briefly, $\mathcal{K}$-shape equivalent to one another if there exists a finite system $X_1, X_2, \dots, X_k$ of spaces in $\mathcal{K}$ with $X_1 = X$, $X_k = Y$ and such that X_{i-1} is $\mathcal{K}$-shape similar to X_i for $i=2,3,\dots,k$. Evidently the $\mathcal{K}$-shape equivalence is a proper equivalence relation which is weaker than the homotopy equivalence.

The class of spaces in $\mathcal{K}$ which are $\mathcal{K}$-shape equivalent to a given space X is called $\mathcal{K}$-shape of X and is denoted by $sh_{\mathcal{K}}X$. Thus a notion of shape is defined which depends on the given class $\mathcal{K}$ of topological spaces and, consequently, may be called relative shape.

It is clear that if $\mathcal{K}'\subset\mathcal{K}$ and $X,Y\in\mathcal{K}'$, then $sh_{\mathcal{K}'}X= sh_{\mathcal{K}'}Y$ implies $sh_{\mathcal{K}}X = sh_{\mathcal{K}}Y$ but there are examples [4] showing that the inverse is generally not true.

If one considers only closed embedding, i.e. if only neighbourhood systems $\underline{U}(X,P)$ are considered in which X is a closed subset of P, then the given above construction leads to another notion of shape - let us call it $\mathcal{K}$-shape in narrow sense and denote by $Sh_{\mathcal{K}}X$. (it is just this notion which was introduced in [4])

On the other hand it is not hard to get in the same way the notion of uniform $\mathcal{K}$-shape. Let $\mathcal{K}$ be a class of uniform spaces. Denoting by $\underline{U}(X,P)$ also the family of all open neighbourhoods of X in P, where $X\in\mathcal{K}$, $P\in\mathcal{K}$ and $X\subset P$, one defines the uniform mutation $\underline{f}:\underline{U}(X,P)\to\underline{U}(Y,Q)$ as a collection of uniformly continuous mapping satisfying the Fox's conditions for mutation with uniform homotopy as basic equivalence relation. Analogously one defines the uniform homotopy between uniform mutations. In this way one gets the notion of uniform $\mathcal{K}$-shape equivalence - an equivalence relation weaker than the uniform homotopy equivalence between uniform spaces. The uniform $\mathcal{K}$-shape of a given space X will be denoted by $ush_{\mathcal{K}}X$.

In the sequel the following notations are used: $\mathcal{M}$ - the class of all metric spaces, $\mathcal{C}$ - the class of all compact Hausdorff spaces, $\mathcal{P}$ - the class of all paracompact spaces, $\mathcal{B}$ - the class of all binormal spaces, $\mathcal{CR}$ - the class of all completely regular spaces, $\mathcal{U}$ - the class of all uniform spaces.

Now one can see that the following statements are true.

Proposition 1. If $X \in \mathcal{M}$, then $sh_{\mathcal{M}} X = Sh_{\mathcal{M}} X = ShX$, where ShX is the shape of X in the sense of Fox [6] .

Proposition 2. If $X \in \mathcal{C}$, then $sh_{\mathcal{C}} X = Sh_{\mathcal{C}} X = ShX$, where ShX is the shape of X in the sense of Mardešić - Segal [8] .

Proposition 3. If $X \in \mathcal{M}$, then $ush_{\mathcal{M}} X = ushX$, where ushX is the uniform shape of X in the sense of [3] .

Proposition 4. If $X \in \mathcal{U}$, then $ush_{\mathcal{U}} X = ushX$, where ushX is the uniform shape of X in the sense of [5] .

Besides that, directly from the corresponding definitions follows

Proposition 5. If X and Y are uniform spaces, considered also as completely regular topological spaces, then $ush_{\mathcal{U}} X = ush_{\mathcal{U}} Y$ implies $sh_{\mathcal{CR}} X = sh_{\mathcal{CR}} Y$.

By means of some results of Morita [11] one can get also

Proposition 6. If $X, Y \in \mathcal{P}$, then $Sh_{\mathcal{P}} X = Sh_{\mathcal{P}} Y$ implies Sh X = Sh Y, where ShX and ShY are meant in the sense of Mardešić [7] .

However, I do not know if the inverse statement is true.

In spite of the great generality of the notion of the relative shape, it allows, at least when it is taken in the narrow sense, to get some assertions about certain classical topological notions. For example, the following two theorems are true.

Theorem 1. If $X, Y \in \mathcal{P}$, and $Sh_{\mathcal{P}} X = Sh_{\mathcal{P}} Y$, then for their Čech homology and cohomology groups over any abelian group G it is $H_n(X;G) = H_n(Y;G)$ and $H^n(X;G) = H^n(Y;G)$ for every n.

Theorem 2. If $X, Y \in \mathcal{B}$, $\dim X < 2n-1$, $\dim Y < 2n-1$, and $Sh_{\mathcal{B}} X = Sh_{\mathcal{B}} Y$, then for the n-th cohomotopy groups it is $\pi^n(X) = \pi^n(Y)$.

The first of these theorems can be derived from Proposition 6 and some results of Morita [11], but its direct proof seems simpler. It is based essentially on the existence of similar extensions of the open coverings of any closed set in a paracompact space [10] . What concerns the second Theorem, it also allows a direct proof based on the fact that the homotopy extension Theorem is true for binormal spaces in respect to the n-sphere, this sphere being an absolute neighbourhood extensor for normal spaces.

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