ORDINAL INVARIANTS IN TOPOLOGY - SUMMARY

V. KANNAN; MADURAI

[Reported by M. Venkataraman]

INTRODUCTION: In this work, we show that all useful ordinal invariants in topology, studied hitherto, (such as the derived length of scattered spaces, the sequential order of sequential spaces, etc.) are closely inter-related and can be brought under the single heading of what we define as the order of a function from a topological space onto a set (equivalently, of a partition of a topological space, or of a quotient map between topological spaces). Using this, we extend many of them for arbitrary topological spaces and view them from different angles.

We start with an elementary concept that is basic for our discussions.

0. Order of a closure operation: (E. ⊂ E ⊂ H|3|) : A closure operation on a set X is a map $V : \mathcal{P}(X) \to \mathcal{P}(X)$ satisfying the following conditions:

$$V(\varphi) = 0$$

$$V(A) \supset A \quad \text{for every} \quad A \subset X$$

$$V(A \cup B) = V(A) \cup V(B) \quad \text{for every} \quad A,B \subset X .$$

If V is a closure operation on X, if α is an ordinal number and if A is a subset of X, we define

$$V^{\alpha}(A) = \begin{cases} A & \text{if } \alpha = 0 \\ V(V^{\beta}(A)) & \text{if } \alpha = \beta + 1 \\ \bigcup\limits_{\beta<\alpha} V^{\beta}(A) & \text{if } \alpha \text{ is a limit ordinal .} \end{cases}$$

This inductively defines a closure operation V^{α} on X, for every α. There always exists an ordinal number α such that $V^{\alpha}(A) = V^{\alpha+1}(A)$ for every $A \subset X$. The least such ordinal is called the order of the closure operation V. We denote it by $\eta(X,V)$. The closure operations with order ≤ 1, that is the idempotent ones, are the topological closure operations. The pair (X,V) is known as a closure space. The order of V is also called the order of this closure space.

1. Order of a function: Let X be a topological space, Y be a set and let $f : X \to Y$ be a function. If we let $V(A) = f(\overline{f^{-1}(A)})$ for every $A \subset X$, then V is a closure operation on X. The order of this closure operation is called the order of f and is denoted by $\sigma(f)$.

2. The E-order: Let $\underline{E}$ be any family of topological spaces. We denote by $D(\underline{E})$ the family of all spaces that can be obtained as quotients of sums of members of $\underline{E}$. In categorical terminology, $D(\underline{E})$ is the coreflective hull of $\underline{E}$ in the category TOP of topological spaces. For each X in $D(\underline{E})$, we now associate an

ordinal number, called the $\underline{E}$-order of X. It is defined as $\sigma_{\underline{E}} = \text{glb}\{\sigma(f) \mid f$ is a quotient map onto X from a sum of members of $\underline{E}\}$.

3. Canonical Maps: We say that $\underline{E}$ is imagive, if $\underline{E}$ is stable under the formation of continuous images. Let $\underline{E}$ be an imagive family and let X be any topological space. Let Y be the sum of all those subspaces of X that are members of $\underline{E}$. The map from Y to X, whose restriction on every summand of Y, is the inclusion map into X, is called the canonical map of X with respect to $\underline{E}$.

Secondly, let $\underline{E}$ be any set (necessarily, not imagive) of topological spaces. We denote by $C(\underline{E},X)$ the family of all continuous maps into X from members of $\underline{E}$. For each f in $C(\underline{E},X)$ we denote the domain of f by D_f. We denote by Y, the sum of all these D_f's. The map from Y onto X, whose restriction to each D_f is the same as f, is called the canonical map of X with respect to $\underline{E}$.

Our assertions are the following: Let $\underline{E}$ be either a set or an imagive family. Let X be any topological space. Let φ be the canonical map of X with respect to $\underline{E}$. Then

(i) $X \in D(\underline{E})$ if and only if φ is a quotient map.

(ii) The quotient topology of φ is the coreflection of X in $D(\underline{E})$. (See [9] or [8] for definitions.)

(iii) $\sigma_{\underline{E}}(X) = \sigma(\varphi)$.

We use this theorem later to show that the concept of $\underline{E}$-order generalizes and unifies several known ordinal invariants in topology.

4. Alternate characterizations of $\underline{E}$-order: (1) We have, just now stated a characterization of $\underline{E}$-order in terms of canonical maps, under some conditions on $\underline{E}$.

(2) For any family $\underline{E}$ of topological spaces, let G(E) be the coreflective hull of $\underline{E}$ in the category CL of closure spaces. (The morphisms are the continuous maps as defined in [3].) Let $T : CL \to TOP$ be the reflector (adjoint of the inclusion functor of TOP in CL) defined by $T(X,V) = (X,V^{\eta(X,V)})$. Let X be any topological space. Let (X,V) be the coreflection of X in $G(\underline{E})$. Then $X \in D(\underline{E})$ if and only if $X = T(X,V)$. Further, the $\underline{E}$-order of X is exactly the order of the closure space (X,V).

(3) As soon as a family $\underline{E}$ of topological spaces is given, we describe a well-ordered increasing family $\underline{E}_1 \subset \underline{E}_2 \subset \underline{E}_3 \subset \cdots \subset \underline{E}_\alpha \subset \cdots$ of coreflective subcategories of CL containing $\underline{E}$. We show that a topological space $X \in \underline{E}_\alpha$ if and only if $\sigma_{\underline{E}}(X) \leq \alpha$.

5. $\underline{E}$-Fréchet Spaces: Those spaces X in $D(\underline{E})$ for which the $\underline{E}$-order is ≤ 1 are defined as $\underline{E}$-Fréchet spaces. We denote this class by $F(\underline{E})$. We prove the following:

(i) The members of $F(\underline{E})$ are precisely the pseudo-open continuous images of sums of members of $\underline{E}$.

(ii) If every subspace of a space X belongs to $D(\underline{E})$, then $X \in F(\underline{E})$.

(iii) $F(\underline{E}) = G(\underline{E}) \cap TOP$.

(iv) As a partial converse of (iii) we have: If $\underline{E}$ is hereditary, then every subspace of every $\underline{E}$-Fréchet space belongs to $D(\underline{E})$ (and hence is $\underline{E}$-Fréchet).

(v) The full converse of (iii) is not true.

6. Equivalent Families: Two families $\underline{E}_1$ and $\underline{E}_2$ of topological spaces are said to be equivalent if

(a) they generate the same coreflective subcategory, i.e.

$$D(\underline{E}_1) = D(\underline{E}_2)$$

and

(b) $\sigma_{\underline{E}_1}(X) = \sigma_{\underline{E}_2}(X)$ for every X in $D(\underline{E}_1)$.

The assertion is that E_1 and E_2 are equivalent if and only if $F(\underline{E}_1) = F(\underline{E}_2)$. In a similar fashion, if X and Y are two topological spaces, then $F(\{X\}) = F(\{Y\})$ if and only if $\sigma_{\underline{E}}(X) = \sigma_{\underline{E}}(Y)$ for any family $\underline{E}$. This is the extent to which the $\underline{E}$-orders can distinguish spaces.

7. Particular Cases: (1) When $\underline{E}$ is the family of all metrisable spaces, then $D(\underline{E}) = \{\text{Sequential spaces}\}$. It follows from the results of [6] and [3] that the $\underline{E}$-order is the sequential order. ARHANGELSKII's characterization of Fréchet spaces [1], FRANKLIN's characterization of Fréchet spaces, [6, 5], etc. follow as corollaries.

(2) When $\underline{E} = \{\text{compact spaces}\}$, then $D(\underline{E}) = \{\text{k-spaces}\}$, $F(E) = \{\text{k'-spaces}\}$ (see [7] for definition), and $\underline{E}$-order = k-order. ARHANGELSKII's characterization of k'-spaces [1] is a corollary.

(3) If $\underline{E} = \{\text{orderable spaces}\}$, then $D(\underline{E}) = \{\text{chain-net spaces}\}$ otherwise known as {weakly sequentional spaces} and $F(\underline{E}) = \{\text{weakly Fréchet spaces}\}$. KENT's characterizations of weakly Fréchet spaces are corollaries.

(4) Let m be an infinite cardinal number. Let $\underline{E} = \{\text{spaces of local weight} \leq m\}$. Then $D(\underline{E}) = \{\text{m-Fréchet spaces}\}$ and $\underline{E}$-order = m-sequential order. MEYER's [13] characterizations of m-Fréchet spaces, follow as corollaries.

(5) Let $\underline{E}$ be any imagive family. Then $D(\underline{E}) = \{X \mid X$ receives weak topology from its subspaces belonging to $\underline{E}\}$. If we let $\Sigma_X = \{Y \subset X \mid Y \in \underline{E}\}$, then Σ is a natural cover in the sense of FRANKLIN [7]. Conversely, every natural cover arises in this way. Further $D(\underline{E}) = \{\Sigma\text{-spaces}\}$, $F(E) = \{\Sigma'\text{-spaces}\}$ and E-order = Σ-characteristic. All the results of [7] follow as corollaries.

8. Behavior of E-order: We show that the order of a quotient map has a nice behavior with respect to sums, restrictions to open (closed) subspaces and composites. Using this we show the following:

(i) The $\underline{E}$-order of a sum of spaces is exactly the supremum of the E-orders of the individual spaces.

(ii) If $f : X \to Y$ is a quotient map, then $\sigma_{\underline{E}}(Y) < \sigma(f)\sigma_{\underline{E}}(X)$.

(iii) In particular, the $\underline{E}$-order is decreased by quotient maps with order ≤ 1 (these are precisely the pseudo-open continuous maps, or equivalently the hereditarily quotient maps).

(iv) If $\underline{E}$ is open- (closed-) hereditary, then the $\underline{E}$-order is decreased by open (closed) subspaces.

9. Unification: The similarities among the theories of sequential order, k-order, and m-sequential orders were noticed and the need for their unified treatment was felt long ago. (See the introduction of [7].) It is in this background that FRANKLIN [7] studied natural covers and MEYER [9] studied convergence bases and subbases. Simultaneously, there were some generalizations and results of the same pattern proved in some other cases. But none of the earlier attempts was able to unify all these, though each of them unified several things. Our theory of $\underline{E}$-order does this job.

Besides, in each of the above-mentioned cases, an ordinal invariant was defined in a particular coreflective subcategory of TOP. It is natural to ask whether these can be generalized to arbitrary coreflective subcategories. We have achieved this also.

Moreover, we show that the theories of FRANKLIN [7] and MEYER [10] are nicely situated in the theory of $\underline{E}$-order. The first is equivalent to the case $\underline{E}$ is imagive; the second is equivalent to the case $\underline{E}$ is a family of spaces with unique accumulation points.

10. Several E-orders: Let $\underline{A}$ be any coreflective subcategory of TOP. Then there may be several families $\underline{E}$ such that $D(\underline{E}) = \underline{A}$. Each of these gives rise to an $\underline{E}$-rrder in $\underline{A}$. Some of them may be equivalent. How many equivalence classes are there?

A coreflective subcategory of CL is said to be topologically generated (abbreviation: t.g.c.s.) if it is $G(\underline{E})$ for some family $\underline{E}$ of topological spaces. We say that it is lying above $\underline{A}$ if $\underline{A} = T(G(\underline{E}))$. Our assertion is this: If $\underline{B}$ is any tgcs lying above $\underline{A}$, then $D(\underline{B} \cap \text{TOP}) = \underline{A}$ and the $\underline{B} \cap$ TOP-order is defined in $\underline{A}$. Conversely, if $\underline{E}$ is any family such that $D(E) = \underline{A}$ then $\underline{E}$ is equivalent to $\underline{B} \cap$ TOP for some tgcs $\underline{B}$ lying above $\underline{A}$. Further, if B_1 and B_2 are distinct tgcs lying above, then they induce non-equivalent $\underline{E}$-orders. Thus, we have a 1-1 correspondence between the family of all equivalence classes of $\underline{E}$-orders definable in $\underline{A}$ and the family of all tgcs lying above $\underline{A}$.

11. Distinguished E-orders: If there could be several $\underline{E}$-orders in the same coreflective subcategory $\underline{A}$ of TOP, does there exist a distinguished one among them? Equivalently, among the tgcs lying above $\underline{A}$, is there a special one? We may look for (i) the largest and (ii) the smallest of tgcs above $\underline{A}$. The largest always exists; the corresponding $\underline{E}$-order is always trivial -- 0 for diserate

spaces and 1 for others. The smallest exists in some cases. We show that if $\underline{A} = D(\underline{E})$ for some hereditary family $\underline{E}_1$, then the $\underline{E}_1$-order is the best among all $\underline{E}$-orders in $\underline{A}$. Thus the sequential order, m-sequential order, etc. (in general, the ordinal invariant arising out of convergence subbases of [9]) are the best, in their respective coreflective subcategories. If $\underline{A}$ is hereditary, then every $\underline{E}$-order in $\underline{A}$ is trivial.

12. Analogue of Fréchet spaces: Does every coreflective subcategory $\underline{A}$ of TOP possess a subcategory $\underline{A}$ which plays the role of {Fréchet spaces} in {sequential spaces}, of {m-Fréchet spaces} in {m-sequential spaces}, etc. Yes, we have. We have only to define $\underline{A}' = \{X \in \underline{A} \mid \sigma_E(X) \leq 1$ whenever $D(\underline{E}) = \underline{A}\}$. Equivalently $\underline{A}' = \{X \in \underline{A} \mid \sigma_E(X) \leq$ whenever $D(\underline{E}) = \underline{A}\}$. Always $\underline{A}'$ is closed under the formation of sums and pseudo-open continuous images. The best $\underline{E}$-order in $\underline{A}$ exists if and only if $D(\underline{A}') = \underline{A}$.

We show that if $\underline{A}$ is generated by a family of spaces with unique accumulation points, then $\underline{A}'$ can be described directly in terms of $\underline{A}$ as follows:

$$\underline{A}' = \{X \in \underline{A} \mid X_x \in \underline{A} \text{ for each } x \text{ in } X\} .$$

(Here X_x is defined as the space gotten from X, by isolating each point other than x, and by not altering the neighborhood base at x.)

13. Ordinal invariants in the whole of TOP: Next, we shall be interested in ordinal invariants that are definable on the whole of TOP. Two methods are natural:

(1) For every topological space X, associate a natural class of closure spaces. Take their orders. Take the glb or sup of this set.

(2) For every topological space, associate a natural class of continuous maps. Take their orders. Take the glb or sup of this set. In each of these methods, the glb leads to trivial invariants. Therefore we take only the supremum.

Given X, the natural classes of closure spaces are:

(i) All closure spaces lying above X.

(ii) All closure spaces finer than X. (That is, the identity map onto X is continuous.)

(iii) All closure spaces coarser than X.

(iv) All closure spaces that are coreflections of X in the tgcs.

On the other hand, given X, the natural classes of continuous maps are:

(i) All quotient maps onto X.

(ii) All continuous maps onto X.

(iii) All continuous maps from X, or all quotient maps from X.

(iv) All canonical maps of X with respect to singleton families.

We show that the two methods become equivalent, case by case. Thus we have four ordinal invariants, defined on TOP. We denote these by $\rho(X)$, $\delta(X)$, $\gamma(X)$ and $\sigma(X)$.

14. The invariant ρ: If X is a topological space, recall the definition: $\rho(X) = \mathrm{Sup}\{\sigma(f) \mid f$ is a quotient map onto $X\} = \mathrm{Sup}\{\eta(X,V) \mid T(X,V) = X\}$. We prove the following:

(a) If X is a Hausdorff sequential space, then $\rho(X)$ coincides with the sequential order of X. Thus ρ can be viewed as a natural extension of sequential order to an arbitrary topological space.

(b) ρ assigns the supremum for the sums and is decreased by open subspaces and closed subspaces.

(c) The spaces X for which $\rho(X) \leq 1$ constitute a nice class including all metrisable spaces.

15. The invariant δ: If X is any topological space recall our definition: $\delta(X) = \mathrm{Sup}\{\eta(X,V) \mid V(A) \subset A$ for each $A \subset X\}$. We prove the following:

(a) $\delta(X)$ can be equivalently described as the supremum of the orders of all continuous maps into X.

(b) We already saw that ρ extends a known ordinal invariant to the whole of TOP; we show that δ also has such a significance: If X is a scattered space, then $\delta(X)$ coincides with the derived length of X. The derived length is a well-studied invariant. See [10] for a short history.

(c) Not only does δ allow us to talk of the derived length of an arbitrary space, but it has the same behavior with respect to standard operations. It is the supremum for the sums, is decreased by subspaces, and is increased by weaker topologies.

16. The invariant γ: The result 15.(a) suggests the notion dual to that of derived length -- this is a bonus, because the usual definition of derived length does not at all suggest this. This dual invariant is defined by $\gamma(X) = \mathrm{Sup}\{\sigma(f) \mid f$ is a continuous map from $X\}$. This is also the dual of ρ. We show:

(a) $\gamma(X)$ can be described intrinsically as follows:

$$\gamma(X) = \mathrm{Sup}\{\alpha \mid \exists \text{ a well-ordered chain } A_0 < A < \cdots < A_\alpha \text{ in } X\}$$

where $A < B$ means $A \neq \overline{A} \subset B$.

(b) γ is decreased by subspaces and quotient images; if $\gamma(X)$ is a non limit ordinal, then $\gamma(X + Y) \geq \gamma(X) + \gamma(Y)$. Thus γ differs essentially from ρ and δ in its behavior.

(c) There is another peculiarity of γ among the invariants that we are considering. Even mild topological conditions imply severe restrictions on the values of γ. For example, we show:

If X is any Hausdorff space, then $\gamma(X)$ is either finite or uncountable; If X is further perfect, then $\gamma(X)$ is either a limit ordinal or the successor of a limit ordinal.

(d) We also give some characterizations of:

(i) $\{X \in \mathrm{TOP} \mid \gamma(X) < 1\}$

and

(ii) $\{X \in \text{HAUS} \mid \gamma(X) = n\}$.

17. <u>The invariant</u> σ: For any topological space X, we define $\sigma(X) = \text{Sup}\{\eta(X,V) \mid (X,V)$ is the coreflection of X in some tgcs$\}$. We prove the following:

(a) $\sigma(X)$ can alternately be described as any one of the following:

(i) $\text{Sup}\{\sigma(f) \mid f$ is the canonical map of X with respect to a singleton family$\}$.

(ii) $\text{Sup}\{\sigma_{\underline{E}}(X) \mid X \in D(\underline{E})\}$. Thus σ is significant, because it is the supremum of all the $\underline{E}$-orders.

(b) Just like ρ, the invariant σ also coincides with the sequential order, among Hausdorff sequential spaces.

(c) For any space X we have the inequalities $\sigma(X) \leq \rho(X) \leq \delta(X) \leq$ the initial ordinal of the first infinite cardinal bigger than that of X.

18. <u>The order at a point</u>: The invariants ρ, δ, and σ can be defined at each point of a topological space. For example, if (X,V) is a closure space and $x \in X$, we define $\eta(X,V,x) = \text{glb}\{\alpha \mid$ For every $A \subset X$, $x \in V^{\beta}(A)$ for some $\beta \Rightarrow x \in V^{\alpha}A\}$ and $\rho(X,x) = \text{Sup}\{\eta(X,V,x) \mid T(X,V) = X\}$.

We show that ρ, δ, and σ are of local character, under a reasonable definition of this term, but γ is not so.

Having defined ρ, δ and σ at points of a topological space, we may view them as mathematical models of measuring the badness of a person in a society. In this light, our results can be interpreted as supporting the following three principles:

(a) The goodness of a society is entirely determined by that of its individuals; if each individual is α-good, then the society is α-good and conversely. That is, $\rho(X) \leq \alpha \Leftrightarrow \rho(X,x) \leq \alpha \; \forall \, x \in S$.

(b) The character of a person is completely determined in his locality; a person is α-good in his neighborhood if and only if he is α-good in the whole society. That is $\rho(X,x) = \rho(V,x)$ whenever V is a neighborhood of x, with relative topology.

(c) A person, living away from bad persons, i.e. completely surrounded by good persons, becomes good at least to a close level; If in a neighborhood of a person, everyone is α-good, then this person is $\alpha+1$-good. That is: If $\rho(X,t) \leq \alpha \; \forall \, t \in V \setminus \{x\}$ where V is a neighborhood of x, then $\rho(X,x) \leq \alpha + 1$.

The results of this section are also to be used as tools in the proofs of the results of the next section.

19. <u>What ordinals appear?</u>: Finally, we develop a construction technique of topological spaces (which we call the construction of a brush) and use it repeatedly to obtain the answers for some natural questions concerning these ordinal invariants.

For example, consider this question: Given a family $\underline{E}$ of topological spaces,

spaces, what ordinal numbers appear as $\underline{E}$-orders of spaces? We have two types of answers for this question: (i) For some particular examples of $\underline{E}$, we actually give the set of ordinals that are $\underline{E}$-orders of spaces. (ii) We have some theorems of general pattern which relate conditions on $\underline{E}$ to conditions on the family of $\underline{E}$-orders of spaces. One such theorem is the following:

Let $\underline{E}$ be a T_2-imagive (i.e. imagive among Hausdorff spaces) closed-hereditary family of Hausdorff spaces. Then the following are equivalent:

(1) Given any ordinal, there is a bigger ordinal, which appears as the $\underline{E}$-order of some T_2 space in $D(\underline{E})$.

(2) Every ordinal appears as the $\underline{E}$-order of some T_2-space in $D(\underline{E})$.

(3) There exists no cardinal m such that $\underline{E} \subset \underline{T}_{\underline{m}}$. (Here $\underline{T}_{\underline{m}}$ is the coreflective hull of the family of all spaces of cardinality $\leq$ m.)

(4) $D(\underline{E})$ is not contained in any simply generated coreflective subcategory of TOP.

It follows that if $\underline{E}$ is the family of all well-ordered compact spaces, then every ordinal appears as $\underline{E}$-order. Another theorem asserts that the same conclusion holds, for every bigger imagive family. In particular, every ordinal number is the k-order of some k-space. This answers a question posed by ARHANGELSKII and FRANKLIN [2].

Now let us turn to the invariants ρ, δ, and σ. We show that in each case, every ordinal appears. In fact, we construct for every ordinal number α, a Hausdorff k-space X_α such that $\rho(X_\alpha) = \sigma(X_\alpha) =$ k-order of $X_\alpha = \alpha$. This improves a result of [10] and [9].

20. <u>What sets of ordinals appear</u>: The invariant ρ (or δ or σ) can be viewed as an ordinal-valued function on each topological space. One natural question is: Given a set X, what ordinal valued functions on X appear as the function ρ (or δ or σ) for a suitable topology on X? Equivalently, what sets of ordinal numbers appear as the range of ρ (or δ or σ)?

We give a partial, but elegant answer: The dense subsets of initial segments, and only these, appear as the range of ρ (or δ or σ) in scattered spaces.

21. <u>Two open problems</u>: The following are some of the problems considered by us, incidentally: -

(a) For which topological spaces do there exist finest closure operations above them?

(b) Which topological spaces have a unique closure operation on them?

(c) For which coreflective subcategories of TOP is there a unique tgcs above them?

We give a partial answer to (a) by showing that every Hausdorff sequential space has this property; a partial answer to (c) by showing that every hereditary coreflective subcategory of TOP has this property. We give a complete answer for (b). It will be interesting to obtain complete answers for (a) and (c).

22. <u>One more result on the invariant</u> ρ: We finish this summary by giving the full statement of a theorem:

<u>Theorem</u>: The following are equivalent for a topological space (X,τ):

(1) $\rho(X) \leq 1$, i.e. every quotient map onto X is hereditarily quotient.

(2) Every two-to-two quotient map onto X is hereditarily quotient.

(3) Every locally injective quotient map onto X is hereditarily quotient.

(4) Every closure operation inducing the topology on X is idempotent; that is, there is a unique closure space over X.

(5) In any pull-back diagram of the following kind, if e is extremal epi and m is extremal mono, then e' is extremely epi and m' is extremely mono. (For definitions, refer to [8].)

(6) Whenever $\tau = \tau_1 \wedge \tau_2$ and $A \subset X$, we have $\overline{A}^{\tau} = \overline{A}^{\tau_1} \cup \overline{A}^{\tau_2}$.

(7) τ can be <u>uniquely</u> expressed in the form $\tau = \wedge_{\alpha\in J}\, \tau_\alpha$ where (i) each τ_α has exactly one non-isolated point and (ii) distinct τ_α's have no common accumulation point.

(8) Let for each $A \subset X$, τ_A be the smallest topology finer than τ such that all points outside A are isolated. Then for each $A \subset X$, the topology τ_A has a <u>unique</u> complement in the lattice $[\tau)$ of all topologies finer than or equal to τ.

(9) For each x in X, the topology $\tau_{\{x\}}$ has a unique complement in $[\tau)$.

(10) For each $A \subset X$, τ_A is a maximal complement of $\tau_{X\setminus A}$ in $[\tau)$.

(11) For every x in X, $\tau_{\{x\}}$ is a maximal complement of $\tau_{X\setminus\{x\}}$ in $[\tau)$.

(12) Let $x \in A \subset X$. If A is not a neighborhood of x, there exists a neighborhood N of x and a set $W \supset A \cap N$ such that W is not a neighborhood of x, but W iw a neighborhood of all its other points.

(13) Whenever $A \subset X$ and $x \in A\setminus A^0$, there exists a set W such that

(i) W is not open in X, but $W \cap A$ is open in A

and (ii) W is a neighborhood (in X) of each of its points except x.

(14) Whenever $A \subset X$ and $x \in \overline{A}\setminus A$, $\exists$ a set F such that

(i) $\overline{F} = F \cup \{x\} \neq F$

and (ii) $x \notin \overline{F\setminus A}$.

(15) Whenever $A \subset X$ and $x \in \overline{A}\setminus A$, $\exists$ a closed (in A) subset F of A such that

(i) x is a limit point of $F \cap A$

but (ii) x is not a limit point of $F\setminus A$.

<u>REMARKS</u>: (a) The problem of characterizing spaces onto which every quotient map is hereditarily quotient, was first considered by G.T. Whyburn [15]. He proves the equivalence of (1) and (14), assuming a mild condition on the space X. We show that such a condition is not necessary.

(b) E.D. Shirley [14] has obtained two intrinsic characterizations of spaces with (1); they are too complex to be given here; they can be directly proved to be equivalent to conditions (13) and (15).

(c) The problem of characterizing spaces with (4) is completely analogous to that of DOSS [4] and LINDGREN [11].

23. Conclusion: The proofs of these results will appear elsewhere. Some more natural open problems have also been given there.

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MADURAI UNIVERSITY

MADURAI, 625021,
INDIA