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A NARROW VIEW OF SET THEORETIC TOPOLOGY

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Topology today is many different subjects. Even leaving aside algebraic and differential topology, general topology is hardly one topic.

For example, Starbird [1] recently proved the following: Suppose that f is a piecewise linear homeomorphism of a 3-simplex onto itself which is fixed on the boundary. Then there is a triangulation T of σ and a continuous family $h_t: \sigma \to \sigma$ ($t \in [0,1]$) of linear-with-respect-to- T homeomorphisms such that h_0 is the identity, h_1 is f, and h_t is the identity on the boundary of σ for all t. Although one is interested in producing continuum many homeomorphisms, this is obviously not a set theoretic problem. Starbird's proof involves a combination of finite combinatories and purely geometric pushing and pulling. Geometric topology is a difficult branch of general topology with highly developed techniques and broad applications throughout mathematics.

The geometric techniques are the ones which have proved most effective in handling problems in infinite dimensional topology(by which I mean the study of Hilbert space manifolds). The space of all closed subsets of the closed unit interval may sound set theoretic but Shori and West [2] proved that this space is the Hilbert cube and their solution as well as those solving related problems is very geometric.

Another area in general topology which is not set theoretic is one which I call "continua theory." An example of a fundamental unsolved problem in this area is: does every compact connected set in the plane which does separate the plane have the fixed point property? Without knowing the answer, one still speculates that set theoretic considerations will not play a major role in its solution. In all of the above areas the spaces in question are separable and metric, usually manifolds with a linear or geometric structure as well. The desired constructions are achieved by finite or countable processes and the interplay between cardinals is not a factor.

However, problems involving compactifications, Baire or Borel sets, metrizability, paracompactness, normality, and more generally comparison problems between closely related topological properties are almost all really problems in set theory. The most obvious such problems involve the comparison of cardinal functions on a topological space; but many more subtle problems also have set theoretic translations.

This area blossoms at the moment due to the recognition of the set theoretic nature of the problems, the availability of an effective collection of set theoretic tools, and the increased acquaintance of topologists with these tools. Almost every day someone tells me a new result answering a question posed in the literature. The area is almost impossible to survey because too much is going on; I find it extremely difficult to distinguish which results are most important and will be lasting. I tried two years ago to survey [3] primarily those results gotten by Wisconsin visitors. A list of over 100 problems was given; perhaps 2/3 of these have now been at least partially solved. Of course, some papers have particularly wide influence. Ostaszewski's construction [4] has been frequently used. Juhász's book [5] is basic. Šapirovskii's unification [6] of the proofs of so many theorems is often used to simplify and clarify; Hodel [7] and Pol [15] have nice write ups of this. Efimov's broad paper [8] on extremal disconnectedness contains a wealth of material. The recent paper by Chaber [9] proving that countably compact spaces with a G_δ diagonal are metric is brief but enlightening. The even more recent paper of Przymusinski [10] describing nice spaces X with exactly X^n paracompact and X^m normal for reasonable n and m is a meld of techniques. These are not necessarily the latest or most significant papers; they are papers containing useful ideas.

In spite of all the activity, the situation with a number of the most basic old questions is the most frustrating imaginable. We know that, say <u>yes</u>, is consistent with the usual axioms for set theory but we have been unable to find out whether <u>no</u> is consistent ot not:

(1) Is there a p-point in $\beta N-N$? (A point such that the intersection of every countable family of neighborhoods of the point contains a neighborhood.) Such problems involving ultrafilters are set theory in the raw; one hesitates to mention topological formulations. Under a large number of special set theoretic assumptions there are p-points in $\beta N-N$; this is a problem which has had serious attack by

both topologists and logicians. It is probably consistent that there have been no p-points in $\beta N-N$, but no such models have been found. There has been no movement on this problem for several years and in a way the subject has moved on leaving it as an irrelevant island. But if there were a real p-point, the construction would almost surely have many other applications for it is exactly such countable versus uncountable properties which are vital in set theoretic topology.

(2) Is every normal Moore space metrizable?

More generally we have comparison problem: does normal imply collectionwise normal in 1st countable T_2 spaces? Basically nothing is known except that under Martin's axiom plus the continuum hypothesis the answer is no. In translating this problem into set theoretic terms one is given a cardinal λ and one needs to know certain intersection properties for the subsets of subsets of λ. The intractability of this problem is based on the fact that the set theoretic questions are 3rd order. The related question: whether normal imples collectionwise normal for 1st countable T_2 spaces, is independent of the usual axioms for set theory; here we only need to know about intersection properties for subsets of λ. We have effective tools for dealing with 2^{λ}, but not with $2^{2^{\lambda}}$.

(3) Can a T_3 (1st countable) space have only one of hereditary separability and hereditary Lindelöfness? I suspect this problem is much more set theoretically basic and widely applicable that the previous two problems although people outside of general topology consider the question ridiculously esoteric. Assuming the continuum for hypothesis or the existence of a Souslin line, the answer is yes as a rich variety of examples now show; but the basic question remains. This seems to be a partition calculus problem. $\omega_1 \to (\omega_1; \omega_1)^2$ would imply that the answer is no; and Galvin has shown that this is a consequence of Martin's axiom plus the negation of the continuum hypothesis plus $\omega_1 \to [\omega_1]^2_5$. But $\omega_1 \to [\omega_1]^2_5$ may be false for all we know.

(4) Is there a Dowker space with small cardinal functions?

(Is there a 1st countable or separable or cardinality $\aleph_1$ T_2 space which is normal but not countably paracompact?) There is a normal T_2

space which is not countably paracompact, but to be useful we really need examples which are 1st countable or separable. Consistency examples of such spaces exist . The situation is similar to that of problem (1): we know there are normal T_2 spaces which are not collectionwise normal but we only know that the existence of 1st countable ones is consistent with the usual axioms for set theory. Similarly the existence of a regular hereditarily separable non Lindelöf or 1st countable space in answer to problem (3) would only raise the question of the existence of a 1st countable one. It is typical of set theoretic topology as opposed to set theory that the basic problem is which pathologies are eliminated by which countability conditions.

This year my principal concern has been a problem whose motivation came from Banach spaces. An Eberlein compact is a topological space which is homeomorphic to a weakly compact subset of a Banach spase. (Eberlein proved [11] that a closed subset of a Banach space is compact in the weak topology if and only if it is sequentially compact.) If Γ is a set, let $c_0(\Gamma) = \{f \in I^\Gamma | \{\alpha \in \Gamma | f(\alpha) > \varepsilon\}$ is finite for all $\varepsilon > 0\}$. Amir and Lindenstrauss proved [12] that (for some Γ) every Eberlein compact is homeomorphic to a compact subset of $c_0(\Gamma)$ where the topology on $c_0(\Gamma)$ is just the subspace topology inherited from the product space I^Γ . They conjectured that every (T_2) continuous image of an Eberlein compact is an Eberlein compact. It is.

Rosenthal proved [13] that a compact T_2 space is an Eberlein compact if and only if it has a σ-point-finite point-separating-in-the-T_0 -sense family of open F_σ s. (Recall that a compact T_2 space with a point-countable point-separating-in-the-T_1 -sense family of open sets is metric). Our solution [14] of the continuous image problem shows, using a good deal of (often finite) combinatories and one straight forward topological lemma, that any T_2 continuous image of a compact subset of $c_0(\Gamma)$ satisfies Rosenthal's characterization of an Eberlein compact.

However, my interest in this problem (which is obviously a topological problem) stemmed from the fact that I thought that the problem was set theoretic. People working on the problem had been asking me questions like: Is every compact T_2 space of cardinality $\leq c$ sequentially compact? Is every compact, T_2 , ccc space with a point-countable point-separating-in-the-T_0 -sense family of open F_σ s metrizable? The answer to either of these questions is independent of the usual axioms for set theory. Even those of us who consider ourselves "experts" in set theoretic topology often cannot tell in advance when a problem is indeed set theoretic. The clue to the non set theoretic nature of problems concerning Eberlein compacts is that the weight and cellularity are the same for these spaces and are preserved by irreducible continuous functions; there is little room for cardinal pathology.

A much deeper problem from Banach space theory is: is every injective Banach space isomorphic to $C(S)$ for some extremally disconnected S ? M. Zippin has recently proved by a topological argument that the answer is yes for separable Banach spaces. Perhaps the general problem is purely topological; perhaps set theoretic; it is tempting to try such problems in hopes of finding broader applications for one's set theoretic topological techniques.

I close with a lemma which was the clue to the solution of the Eberlein compact problem mentioned earlier. Neither the lemma nor its proof will come as a surprise to anyone who has proved that metric spaces are paracompact, that locally finite closed refinements yield paracompactness, ...

If λ is an ordinal, we say that a sequence $\{S_\alpha\}_{\alpha<\lambda}$ of subsets of a topological space is left separated provided $S_\alpha \cap \overline{\bigcup_{\beta>\alpha} S_\beta} = \emptyset$ for all $\alpha < \lambda$.

Lemma. If U is an open subset of a T_4 space, then the union of countably many left separated sequences of closed subsets of U covers U.

Proof. Let $\{p_\alpha\}_{\alpha>\lambda}$ be an indexing of the points of U. Fix α. We define $T_{\alpha n}$ for all $n<\omega$ for induction; our induction hypothesis is that $T_{\alpha n}$ is a closed subset of U. Let $T_{\alpha 0} = \{p_\alpha\}$ and, if $T_{\alpha n}$ has been defined, choose a closed set $T_{\alpha,n+1}$ such that $T_{\alpha n} \subset$ interior $T_{\alpha,n+1} \subset U$. Observe that $T_\alpha = \bigcup_{n<\omega} T_{\alpha n}$ is open.

For each $\alpha<\lambda$ and $n \in \omega$, let $S_{\alpha n} = T_{\alpha n} - \bigcup_{n<\omega} T_\beta$. It is trivial that $\{S_{\alpha n}\}_{\alpha<\lambda}$ is a left separated sequence of closed subsets of U. If $p_\alpha \in U$ there is a smallest $\beta \leq \alpha$ such that $p_\alpha \in S_{\beta n}$. So $U = \bigcup_{n<\omega} \bigcup_{\beta<\lambda} \{S_{\beta n}\}$ and the lemma is proved.

BIBLIOGRAPHY

[1] M.Starbird: The Alexander linear isotopy theorem for dimension 3, (to appear).

[2] R.Shori,J.West: 2^I is homeomorphic to the Hilbert cube,Bull.Amer. Math.Soc.78,no.3(1972), pp.402-406.

[3] M.E.Rudin: Lectures on set theoretic topology, CBMS regional conference series (1974), no.23.

[4] A.J.Ostaszewski: On countably compact, perfectly normal spaces, J.London Math.Soc.

[5] I.Juhász: Cardinal functions in topology, Math.Centre Tracts (1975) Amsterdam, no.34.

[6] B.É.Šapirovskii: Discrete subspaces of topological spaces.Weight, tightness and Souslin number, Dokl.Akad.Nauk SSSR 202 (1972), pp. 779-782 = Soviet Math.Dokl. 13 (1972) no. 1, pp.215-219.

[7] R.Hodel: New proof of a theorem of Hajnal and Juhász on the cardinality of topological spaces, Dept. of Mathematics, Durham,Nort Carolina.

[8] B.A.Efimov: Extermally disconnected compact spaces and absolutes, Trudy Maskov.Mat.Obsc., 23 (1970), pp. 243-285.

[9] J.Chaber:Conditions implying compactness in countably compact spaces. Bull.Acad.Polon.Sci.Sér.Sci.Math.Astronom.Phys.(to appear).

[10] T.Przymusinski: Normality and paracompactness in finite and countable cartesian products,(to appear).

[11] W.Eberlein: Weak compactness in Banach spaces,I.Proc.Acad.Sci. U.S.A., 33 (1947), pp.51-53.

[12] D.Amir,J.Lindenstrauss: The structure of weakly compact sets in Banach spaces, Ann.of Math. 88(1968), pp. 33-46.

[13] H.Rosenthal: The hereditary problem for weakly compactly generated Banach spaces, Comp.Math. 28(1974), pp. 83-111.

[14] J.Benjamini, M.E.Rudin;W.Wage: (to appear).

[15] R.Pol: Short proofs of theorems on cardinality of topological spaces, Pol.Acad.Sci., vol. 22 (1974), no. 2, p. 1245-1249.