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Component categories have been investigated by several authors (see [2], [4]), for topological functors $G : \underline{X} \to \mathit{Ens}$, where *Ens* is the category of sets. We give a generalization to arbitrary functors $G : \underline{X} \to \mathit{Ens}$, following an idea of Pumplün and Holmann (unpublished) and we are let to a generalization of a Galois correspondence given by Maranda [3]. The results are part of my thesis.

Theorem: Let $G : X \to \mathit{Ens}$ be a functor, and $\underline{A} \subset \underline{X}$ a full subcategory such that $G | \underline{A}$ is pointwise non-void, i.e. $G(A) \neq \emptyset$ for all $A \in Ob(\underline{A})$. Then there is a functor $Q_{G,\underline{A}} : \underline{X} \to \mathit{Ens}$ and a natural transformation $\zeta_{G,\underline{A}} : G \to Q_{G,\underline{A}}$ with the following properties:

(i) For all $A \in Ob(\underline{A})$ the cardinality of $G(A)$ is 1.
(ii) If $\alpha : G \to P$ is a natural transformation, such that $\alpha(A)$ is of cardirality 1 for all $A \in Ob(\underline{A})$, then there is a unique natural transformation $\xi : Q_{G,\underline{A}} \to P$ with $\xi\zeta_{G,\underline{A}} = \alpha$.
(iii) For all $X \in Ob(\underline{X})$, $\zeta_{G,\underline{A}}(X)$ is onto.

Definition: Let $\alpha : G \to P$ be a natural transformation, such that $\alpha(X)$ is onto for all $X \in Ob(\underline{X})$. Then *Connect* (α) denotes the full subcategory of $\underline{X}$ generated by all $\underline{X}$-objects A where $P(A)$ is a singleton.

Now the above theorem can be interpreted as a Galois adjunction between *Connect* and $\zeta_{G,-}$, considered as meta-functors between the meta-category of all full subcategories $\underline{A}$ of $\underline{X}$ with $G|\underline{A}$ pointwise non-void and the meta-category of all pointwise surjective natural transformations with domain G.

Definition: Let $G : \underline{X} \to \mathit{Ens}$ be a functor. A full subcategory $\underline{A} \subset \underline{X}$ is called a G-component category, iff $\underline{A} = \mathit{Connect}\ (\zeta_{G,\underline{A}})$ (or, equivalently, iff there exist a functor P and a natural transformation $\alpha : G \to P$, such that $\underline{A}$ is the full subcategory generated by all objects $A \in \mathit{Ob}(\underline{X})$ where $P(A)$ is a singleton).

Corollary: If $A \subset X$ is a full subcateogry and $G : \underline{X} \to \mathit{Ens}$ is a functor with $G|\underline{A}$ pointwise non-void, then $\mathit{Connect}\ (\zeta_{G,A})$ is the smallest G-component category containing $\underline{X}$.

Theorem: Let $G : X \to \mathit{Ens}$ be a mono-fibration (i.e. any injective map $m : U \to G(X)$ has an initial lifting to an X-morphism $\overline{m} : \overline{U} \to X$). Let M denote the class of all G-initial liftings of injective maps and let $\underline{A} \subset \underline{X}$ be a full and replete subcategory with $G | A$ pointwise non void. Let $\underline{A}$ contain all $A \in \mathit{Ob}(X)$ for which $P(A)$ is a singleton. Then the following statements are equivalent.

(i) $\underline{A}$ is a G-component category.

(ii) $\underline{A}$ is strongly locally M-coreflective in the ense of [1], i.e. for any $X \in \mathit{Ob}(\underline{X})$ there is a family $(u_i : Z_i \to X)_{i \in I}$, all Z_i are in $\underline{A}$, such that for any $f : A \to \underline{X}$ with $A \in \mathit{Ob}(\underline{A})$ there is a unique pair (i,h) with $i \in I$, $h : A \to Z_i$, and $u_i h = f$.

(iii) $\underline{A}$ fulfills the following conditions:

1) If $A \in \mathit{Ob}(\underline{A})$, $f : A \to B$ is an $\underline{X}$-morphism, $G(f)$ is onto, then $B \in \mathit{Ob}(\underline{A})$.

2) Let $X \in \mathit{Ob}(\underline{X})$, $(m_i : A_i \to X)_{i \in I}$, $I \neq \emptyset$ be a family of G-initial morphisms, such that $G(m_i)$ is onto one for all $i \in I$. If now $\cap\{G(m_i)[A_i]\} \neq \emptyset$, $\cup\{G(m_i)[A_i]\} = G(X)$ then $X \in \mathit{Ob}(\underline{A})$.

This characterization leeds to a general investigation of full replete strongly coreflective subcategories of an arbitrary category.

Definition: Let $\underline{X}$ be a category.

(i) If $A \in Ob(\underline{X})$, $(Y, (X_i \xrightarrow{m_i} Y)_{i \in I})$ is a sink, A is called locally uniquely projective with respect to $(Y, (X_i \xrightarrow{m_i} Y)_{i \in I})$, iff for any $\underline{X}$-morphism $f : A \to Y$ there is a unique pair (i,h) with $i \in I$, $h\, m_i = f$. Equivalently, we say $(Y, (X_i \xrightarrow{m_i} Y)_{i \in I})$ is locally uniquely coextendable with respect to A.

(ii) If $\underline{A} \subset \underline{X}$ is a full subcategory let $P_1^{loc}(\underline{A})$ denote the conglomerate of all locally coextendable sinks with respect to all $A \in Ob(\underline{A})$.

(iii) If S is a conglomerate of $\underline{X}$-sinks, let $C_1^{loc}(S)$ denote the full subcategory of $\underline{X}$ generated by all locally uniquely projective objects with respect to all S-sinks.

(iv) If M is a class of $\underline{X}$-morphisms, let $\hat{M}$ denote the conglomerate of all sinks $(Y, (X \xrightarrow{m_i} Y_i))_{i \in I})$ with $m_i \in M$ for all $i \in I$.

(v) Let $\underline{A}$ be a full replete subcategory of $\underline{X}$ and M a class of morphisms. $\underline{A}$ is called strongly locally M-coreflective, if for any $Y \in Ob(\underline{X})$ there is a sink $(Y, (X_i \xrightarrow{m_i} Y)_{i \in I}) \in P_1^{loc}(\underline{A}) \cap \hat{M}$ with $X_i \in Ob(\underline{A})$ for all $i \in I$. $\underline{A}$ is called strongly locally coreflective, iff $\underline{A}$ is strongly locally $\underline{X}$-coreflective.

As P_1^{loc} and C_1^{loc} form a Galois correspondence, we look at the full subcateogires closed under the correspondence. We get the following

<u>Theorem</u>: Let $\underline{X}$ be a category, $\underline{A} \subset \underline{X}$ a full subcategory, M a class of $\underline{X}$-morphisms.

(i) If $\underline{A} = P_1^{loc} C_1^{loc}(\underline{A})$, then $\underline{A}$ is closed under the formation of connected colimits.

(ii) If $\underline{A}$ is strongly locally M-coreflective, then $\underline{A} = P_1^{loc}(C_1^{loc}(\underline{A}) \cap \hat{M})$.

(iii) If $\underline{A}$ has locally coorthogonal (E,M)-factorizations (see [5]), then $\underline{A} = P_1^{loc}(C_1^{loc}(\underline{A}) \cap \hat{M})$ implies that $\underline{A}$ is strongly locally M-coreflective.

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