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Markoff property of generalized random fields

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This note arose from the discussions about the problem of characterization of Markoff generalized Gaussian random fields posed by R. Schrader last year at this School. All the participants of these discussions contributed to the results of this note, especially the contribution of R. Hájek cannot be neglected. We learned that in the literature there is a lot of confusion and mistakes concerning Markoff random fields. In this note we give two possible definitions of the Markoff property, study their relationship and on one example we show which statements claimed in literature are not true.

A generalized random field means here a linear and continuous mapping ϕ from the real Schwartz space $\mathcal{D}(R^n)$ into the space $L_1(\Omega,\Sigma,\mu)$ of the (classes of) integrable real-valued functions on some probability space (Ω,Σ,μ). For each open set $G\subset R^n$ we define $\Sigma(G)$ as the σ-algebra generated by the set $\{\phi_f \mid f\in\mathcal{D}(R^n),\ \mathrm{supp} f\subset G\}$ and containing sets of measure zero. For arbitrary $S\subset R^n$ we then define $\Sigma(S)=\bigcap_{\substack{G\subset S\\ G\text{ open}}}\Sigma(G)$. (1)

<u>Remark:</u> Other versions of the above definitions might be used- the intersection in (1) might be taken only over ε-neighbourghoods $U_\varepsilon(S)$ of S ("sequential definition") and also, if ϕ is canonically represented, the condition that $\Sigma(G)$ contains zero sets might be omitted. Resulting σ-algebras may differ depending on definitions adopted.

As an example we have:

If $G \subset \mathbb{R}^2$ is open, let $\mathcal{A}(G)$ be the σ-algebra on $\mathcal{D}'(\mathbb{R}^2)$ generated by all functions $\langle f, \cdot \rangle$ with $f \in \mathcal{D}(\mathbb{R}^2)$, supp $f \subset G$. Then $\bigcap_{\substack{G \supset L \\ G \text{ open}}} \mathcal{A}(G) \subsetneqq \bigcap_{\varepsilon > 0} \mathcal{A}(U_\varepsilon(L))$ for each line $L \subset \mathbb{R}^2$.

This can be proved by taking a sequence $\{f_i\}$ of functions from $\mathcal{D}(\mathbb{R}^2)$ with disjoint supports such that:

$\int f_i^2 = 1$ for each i,

$\bigcup \text{supp } f_i$ is a closed set disjoint with L,

every compact contains only finite number of supp f_i,

diam(supp f_i) $\to 0$ and dist(supp f_i, L) $\to 0$.

Clearly $A = \{\omega \mid \lim \langle f_i, \omega \rangle = 1\} \in \bigcap_{\varepsilon > 0} \mathcal{A}(U_\varepsilon(L))$ and

$$A \notin \mathcal{A}' = \{B \mid \omega \in B \Rightarrow \omega + \sum f_i \in B\} \supset \mathcal{A}(\mathbb{R}^2 - \bigcup \text{supp } f_i) \supset \bigcap_{\substack{G \supset L \\ G \text{ open}}} \mathcal{A}(G).$$

We do not know any similar example in the case of σ-algebras containing zero sets.

We shall discuss the following two definitions of Markoff property:

<u>M 1:</u> A generalized random field is said to be M1-Markoff if $E(v \mid \Sigma(G)) = E(v \mid \Sigma(G-F))$ for every pair of sets $F \subset G \subset \mathbb{R}^n$, F closed, G open and for each $\Sigma(\mathbb{R}^n - F)$-measurable bounded function v.

<u>M 2:</u> A generalized random field is said to be M2-Markoff if $E(v \mid \Sigma(\overline{G})) = E(v \mid \Sigma(\partial G))$ for each open $G \subset \mathbb{R}^n$ and for each $\Sigma(\mathbb{R}^n - G)$-measurable bounded function v.

Note that the definition M1 uses only σ-algebras $\Sigma(G)$ for open sets.

The relation between M1 and M2 is given by

<u>Theorem</u>: A generalized random field is M1-Markoff if and only if it is M2-Markoff and the following condition holds:

(C) whenever $F \subset G \subset R^n$, F closed, G open, then the smallest σ-algebra containing $\Sigma(F)$ and $\Sigma(G-F)$ equals $\Sigma(G)$.

<u>Proof</u>:

M1 $\Rightarrow$ M2 by a simple use of martingale convergence theorems.

M1 $\Rightarrow$ (C) will be shown by proving that for each $f \in \mathcal{D}$, supp $f \subset G$, the function $\phi_f - E(\phi_f | \Sigma(G-F))$ is $\Sigma(F)$ measurable: For each open $H \supset F$, $\bar{H} \subset G$, there are $f_1, f_2 \in \mathcal{D}$ such that $f = f_1 + f_2$, supp $f_1 \subset G-F$, supp $f_2 \subset H$. Thus $\phi_f - E(\phi_f | \Sigma(G-F)) = \phi_{f_2} - E(\phi_{f_2} | \Sigma(G-F))$ is $\Sigma(H)$-measurable since $E(\phi_{f_2} | \Sigma(G-F)) = E(\phi_{f_2} | \Sigma(H-F))$ by M1.

M2 & (C) $\Rightarrow$ M1 : if $F \subset G \subset R^n$, F closed, G open and $v \in L_1(\Sigma(R^n - F))$, then $E(v | \Sigma(G))$ can be by martingale convergence theorems approximated in L_1 by $E(v | \Sigma(H))$ with H open, $F \subset H \subset \bar{H} \subset G$. Hence it suffices to prove $\Sigma(G-F)$-measurability of $E(v | \Sigma(H))$. By (C) it reduces to proving this for $v = w_1 w_2$ with w_1, w_2 bounded and $\Sigma(G-H)$, $\Sigma(H-F)$-measurable, respectively. But then $E(w_1 w_2 | \Sigma(H)) = w_2 E(w_1 | \Sigma(H)) = w_2 E(w_1 | \Sigma(\partial H))$.

<u>Example</u>: The Gaussian generalized random field ϕ on R^1 with mean zero and covariance $D(f,g) = \int [f(x)g(x) + f'(x)g'(x)] dx$ is M2-Markoff and is not M1-Markoff.

<u>Proof</u>:

As D is exactly the scalar product of the Sobolev space $W^{1,2}$

(of real locally absolutely continuous functions f such that $\int[f^2(x)+f'^2(x)]\,dx < \infty$), Φ can be continuously extended to an isometry (denoted also by Φ) from $W^{1,2}$ into $L_2(\Omega,\Sigma,\mu)$. Hence Φ is the standard Gaussian process on $W^{1,2}$.

For $V\subset W^{1,2}$ denote $\mathcal{A}(V)$ the σ-algebra (containing zero sets) generated by $\{\Phi_f \mid f\in V\}$. A simple consequence of formulas for moments of Gaussian processes is that whenever $V,W\subset W^{1,2}$ are mutually orthogonal subspaces, then $\mathcal{A}(V)$ and $\mathcal{A}(W)$ are independent.

For $S\subset R$ denote $H_S=\{f\in W^{1,2} \mid f=0 \text{ on } R-S\} = \overline{\{f\in W^{1,2} \mid \operatorname{supp} f\subset S\}}$. Obviously $H_S=\bigcap_{\substack{G\supset S\\ G \text{ open}}} H_G$. As $\bigcup_{\substack{G\supset S\\ G \text{ open}}} H_G^{\perp}$ is a linear space, $H_S^{\perp}=\overline{\bigcup_{\substack{G\supset S\\ G \text{ open}}} H_G^{\perp}}$ and thus $H_S+\bigcup_{\substack{G\supset S\\ G \text{ open}}} H_G^{\perp}$ is dense in $W^{1,2}$. We shall use a consequence of this: If $S\subset R$ and v is a bounded $\Sigma(R)$-measurable function with $E(vw_1w_2) = 0$ whenever $G\supset S$ is open and w_1,w_2 are bounded $\mathcal{A}(H_S)$, $\mathcal{A}(H_G)$-measurable functions, respectively, then $v=0$. It holds since according to a martingale convergence theorem $v = E(v\mid\Sigma(R)) = \lim_{\substack{G\supset S\\ G \text{ open}}} E(v\mid\sigma(\mathcal{A}(H_S)\cup\mathcal{A}(H_G)))$.

Now we shall prove $\Sigma(S) = \mathcal{A}(H_S)$.

$\Sigma(S)\supset\mathcal{A}(H_S)$ is obvious.

$\Sigma(S)\subset\mathcal{A}(H_S)$: if v is a bounded $\Sigma(S)$-measurable function with $E(v\mid\mathcal{A}(H_S)) = 0$ and $G\supset S$, G open, w_1,w_2 bounded $\mathcal{A}(H_S)$, $\mathcal{A}(H_G)$-measurable functions, respectively, then $E(vw_1w_2) = E(vw_1)E(w_2)$ as vw_1 is $\mathcal{A}(H_G)$-measurable (for G open, $\mathcal{A}(H_G)= \Sigma(G)$). Further $E(vw_1) = E(E(v\mid\mathcal{A}(H_S))w_1) = 0$, hence $v = 0$.

If $G\subset R$ is open, then $H_{\partial G} = \{0\}$, hence $\Sigma(\partial G)$ is triv l l' Markoff pro, rty of Φ follows by indep n enc of $\Sigma(\bar{G}) = \mathcal{A}(H_{\bar{G}})$ and $\Sigma(\overline{R^n - G}) = \mathcal{A}(H_{\overline{R^n - G}})$.

As $\mathcal{A}((H_{(-\infty,0\rangle} \oplus H_{\langle 0,\infty)})^{\perp})$ is a nontrivial σ-algebra independent of $\mathcal{A}(H_{(-\infty,0\rangle} \oplus H_{\langle 0,\infty)})$, the latter cannot be equal to $\Sigma(R)$. This implies that the condition (C) does not hold.

Let us note some remarkable facts:

An often mistake in articles about generalized Markoff random fields is the statement that the condition (C) holds for all generalized random fields (see Lemma 1 in [N]) or at least for all Gaussian fields (see Lemma 2 in [KM]). As shown by the Example even the latter statement is false.

The Example shows also that the Theorem 1 from [KM] is false. The Gaussian field from the Example is clearly nonlocal, i.e. it does not fulfil the condition (A_1) from [KM]. Using the implication M1 => (C), instead of the Lemma 2 from [KM] the implication M1 =>locality may be proved.

The problem, whether locality of a Gaussian field (such that a dual field exists) implies some kind of Markoff property, is still open. The proof of the implication locality => M2 in [KM] is not correct (the proof of Lemma 8 contains the same mistake as that of Lemma 2). This difficulty may be overcome by taking the conclusion of the Lemma 8 of [KM] as an additional assumption. One thus recovers the results announced in [M] stating that locality implies M1 under such an assumption.

Refe ences:

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