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In a recent paper [1], M. De Wilde and B. Tsirulnikov have shown that if E is a dense barrelled subspace of a B-complete locally convex space X, then there is an order-reversing correspondence between the closed subspaces M of X with $M \cap E = \{0\}$ and the weaker barrelled topologies ξ on E. Under this correspondence, the topologies ξ which are assigned to infinite-dimensional subspaces M are nontrivial in the sense that they give rise to duals of E which are of infinite codimension in the original dual of E.

In connection with this, the following question has been asked in [1]:

Suppose E is a dense barrelled subspace of uncountable codimension in a Banach space X. Does there always exist an infinite-dimensional subspace M of X such that $M \cap E = \{0\}$?

The Theorem below answers this question in the negative. Its proof depends heavily upon the results presented at this Winter School in the talk of Z. Lipecki [2].

T H E O R E M. Every infinite-dimensional Banach (or Fréchet) space X has a dense Baire (hence barrelled) subspace E of codimension at least $c = 2^{\aleph_0}$ such that if M is a closed subspace of X with $M \cap E = \{0\}$, then M is of finite dimension.

P r o o f. Let Y be a closed subspace of X such that the dimension of $Z = X/Y$ is c. (E.g., take $Y = \bigcap_n \ker f_n$, where (f_n) is a linearly independent sequence in X^*.) Call a subspace $W \subset Z$ a $\varkappa$-subspace of Z if every linearly independent sequence (z_n) in Z with $\sum_n z_n$ subseries convergent has a subsequence (w_n) such that $0 \neq \sum_n w_n \in W$. Clearly, every $\varkappa$-subspace W in Z has property (K) ([2]), and it is not hard to see that $\dim W \geq c$. Now, using Theorem 1 and Proposition 1 of [2], we can find two dense $\varkappa$-subspaces U and V in Z such that $Z = U \oplus V$ (algebraically). As observed above, each of them has property (K) and is of dimension at least c.

Let $Q: X \longrightarrow Z$ be the quotient map and define $E = Q^{-1}(U)$. Since U is dense in Z, E is dense in X; moreover, $\operatorname{codim} E = \dim V \geq c$. Since U has property (K), so does E; hence, by Theorem 3 in [2], E is Baire.

Finally, suppose that there is an infinite-dimensional closed subspace M in X with $M \cap E = \{0\}$. Choose a linearly independent sequence (x_n) in M for which $\sum_n x_n$ is subseries convergent. Then also the sequence (Qx_n) is linearly independent and the series $\sum_n Qx_n$ is subseries convergent. Since U is a $\varkappa$-subspace in Z, there is a subsequence (y_n) of (x_n) such that $0 \neq \sum_n Qy_n \in U$. It follows that $0 \neq y = \sum_n y_n \in E$. On the other hand, however, we have $y \in M$ because all y_n are in M and M is closed. Hence $M \cap E \neq \{0\}$; a contradiction.

Thus E has all the required properties.

R e m a r k. The above theorem remains valid for all infinite dimensional complete metric linear spaces X which have a Hausdorff quotient X/Y od dimension c. We do not know if this is always the case.

Let us note also that the main results of [1] can be extended to the non-locally convex setting.

R e f e r e n c e s

[1] M. De Wilde and B. Tsirulnikov, Barrelled spaces with a B-complete completion, to appear.

[2] Z. Lipecki, On some nonclosed subspaces of metric linear spaces, this volume, pp. 110 - 113 .