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TWISTOR SPINORS AND SOLUTIONS OF THE EQUATION (E) ON RIEMANNIAN MANIFOLDS

by

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1. Introduction

Let M^n be a Riemannian spin manifold and denote by R its scalar curvature. The conformally invariant twistor operator $\mathcal{D}$ acting on sections ψ of the spinor bundle S is defined by the covariant derivative and the projection onto the kernel of the Clifford multiplication. The kernel $Ker(\mathcal{D})$ of this operator is the space of all spinor fields ψ satisfying the first order differential equation

$$\nabla_X \psi + \frac{1}{n} X \cdot D \psi = 0 \qquad \langle 1 \rangle$$

where D denotes the Dirac operator (see [3],[4]).
Killing spinors, i.e. spinor fields satisfying for some complex number $\lambda \in \mathbb{C}$ the equation

$$\nabla_X \psi = \lambda X \cdot \psi \qquad \langle 2 \rangle$$

are special solutions of the twistor equation. A. Lichnerowicz (see [5]) studied the so-called equation (E) for spinor fields:

$$\nabla_X (D\psi) + \frac{R}{4(n-1)} X \cdot \psi = 0. \qquad \langle 3 \rangle$$

In particular, A. Lichnerowicz proved that if a connected Riemannian manifold admits a non-trivial solution of the equation (E) then its scalar curvature R is constant. Consequently, in case of a compact Riemannian manifold the space of all twistor spinors coincides with $Ker(E)$ and with the space of all Killing spinors (see [2]). The aim of this note is to compare the kernel of the twistor operator with the space Ker(E) of all solutions of equation (E).

2. A relation between Ker(E) and Ker($\mathcal{D}$).

We consider a Riemannian spin manifold (M^n,g) with a non-trivial solution of equation (E). Then its scalar curvature R is constant.

Proposition 1:

$$\mathrm{Ker}(E) = \mathrm{Ker}\left(D^2 - \frac{n\,R}{4(n-1)}\right) \cap D^{-1}(\mathrm{Ker}(\mathcal{D}))$$

Proof: If $\psi \in \mathrm{Ker}(E)$ we obtain from equation <3>

$$D^2\psi = \sum_{i=1}^{n} e_i \cdot \nabla_{e_i}(D\psi) = -\sum_{i=1}^{n} e_i \cdot \frac{R}{4(n-1)}\, e_i \cdot \psi = \frac{n\,R}{4(n-1)}\psi\,,$$

i.e. ψ belongs to the kernel of the operator $D^2 - \frac{n\,R}{4(n-1)}$. Furthermore, we have

$$\nabla_X(D\psi) + \frac{1}{n} X \cdot D(D\psi) = \nabla_X(D\psi) + \frac{1}{n} X \cdot \frac{n\,R}{4(n-1)}\psi =$$
$$= \nabla_X(D\psi) + \frac{R}{4(n-1)} X \cdot \psi = 0$$

and, consequently, $D\psi$ is a twistor spinor. Conversely, if $\psi \in \mathrm{Ker}(D^2 - \frac{n\,R}{4(n-1)}) \cap D^{-1}(\mathrm{Ker}(\mathcal{D}))$ then we have $D^2\psi = \frac{n\,R}{4(n-1)}\psi$ as well as $\nabla_X(D\psi) + \frac{1}{n} X \cdot D(D\psi) = 0$ and ψ belongs to $\mathrm{Ker}(E)$.

Theorem 1: Let (M^n,g) be a Riemannian spin manifold with constant scalar curvature $R \neq 0$. Then the map

$$\mathrm{Ker}(E) \ni \psi \longrightarrow D\psi \in \mathrm{Ker}(\mathcal{D})$$

is an isomorphism.

Proof: By proposition 1 the map is well-defined. If $D\psi = 0$

and $\psi \in \mathrm{Ker}(E)$ the equation <3> implies

$$\frac{R}{4(n-1)} X\cdot\psi = 0.$$

Since $R \neq 0$ we conclude $\psi = 0$, i.e. the map $\mathrm{Ker}(E) \to \mathrm{Ker}(\mathcal{D})$ is injective. On the other hand, given a twistor spinor $\varphi \in \mathrm{Ker}(\mathcal{D})$ we consider

$$\psi = \frac{4(n-1)}{n\,R} D\varphi.$$

Then $D\psi = \frac{4(n-1)}{n\,R} D^2\varphi = \frac{4(n-1)}{n\,R}\ \frac{n\,R}{4(n-1)}\varphi = \varphi$

and $\nabla_X(D\psi) + \frac{R}{4(n-1)} X\cdot\psi =$

$$= \nabla_X\left(\frac{4(n-1)}{n\,R} D^2\varphi\right) + \frac{1}{n} X\cdot D\varphi =$$

$$= \nabla_X(\varphi) + \frac{1}{n} X\cdot D\varphi = 0.$$

This means that ψ belongs to $\mathrm{Ker}(E)$ and is the preimage of φ, i.e. the map $\mathrm{Ker}(E) \to \mathrm{Ker}(\mathcal{D})$ is surjective.

<u>Corollary:</u> Let (M^n,g) be a connected Riemannian manifold with scalar curvature $R \neq 0$. Then

$$\dim \mathrm{Ker}(E) = \dim \mathrm{Ker}(\mathcal{D}) \leq 2^{\left[\frac{n}{2}\right]+1}.$$

Proof: see [3].

<u>Proposition 2:</u> If $\psi \in \mathrm{Ker}(E)$ and $R \neq 0$ then

$$\nabla_X \psi = \frac{2(n-1)}{R(n-2)}\left(\frac{R}{2(n-1)} X - \mathrm{Ric}(X)\right)\cdot D\psi.$$

Proof: If ψ belongs to $\mathrm{Ker}(E)$ then by proposition 1 $D\psi$ is a twistor spinor. The general Lichnerowicz formula

$$\nabla_X(D\varphi) = \frac{n}{2(n-2)}\left(\frac{R}{2(n-1)} X\cdot\varphi - \mathrm{Ric}(X)\cdot\varphi\right) \qquad \langle 4\rangle$$

valid for any twistor spinor φ (see [4]) now yields

$$\nabla_X(D^2\psi) = \frac{n}{2(n-2)}\left(\frac{R}{2(n-1)} X - \mathrm{Ric}(X)\right)\cdot D\psi .$$

Moreover, since $\psi \in \mathrm{Ker}(E)$ it satisfies the equation

$$D^2\psi = \frac{n\,R}{4(n-1)}\,\psi$$

and we obtain

$$\frac{n\,R}{4(n-1)}\nabla_X\psi = \frac{n}{2(n-2)}\left(\frac{R}{2(n-1)}X - Ric(X)\right)\cdot D\psi .$$

Theorem 2:

a) Let (M^n,g) be an Einstein manifold with scalar curvature $R \neq 0$. Then $Ker(E)$ and $Ker(\mathcal{D})$ coincide, i.e.

$$Ker(E) = Ker(\mathcal{D}).$$

b) Let (M^n,g) be a connected Riemannian manifold such that $Ker(E)\cap Ker(\mathcal{D}) \neq \{0\}$. Then (M^n,g) is an Einstein space.

Proof: Suppose first that M^n is an Einstein space, $Ric(X) = \frac{R}{n}X$. If $\psi \in Ker(E)$ we obtain from proposition 2

$$\nabla_X\psi = \frac{2(n-1)}{R(n-2)}\left(\frac{R}{2(n-1)}X - Ric(X)\right)\cdot D\psi = -\frac{1}{n}X\cdot D\psi$$

and ψ is a twistor spinor. Conversely, if $\psi \in Ker(\mathcal{D})$ we use the Lichnerowicz formula $\langle 4\rangle$

$$\nabla_X(D\psi) = \frac{n}{2(n-2)}\left(\frac{R}{2(n-1)}X - Ric(X)\right)\cdot\psi$$

which reduces in an Einstein space to

$$\nabla_X(D\psi) + \frac{R}{4(n-1)}X\cdot\psi = 0.$$

This means that in an Einstein space every twistor spinor is a solution of equation (E). We consider now an arbitrary Riemannian manifold as well as a non-trivial solution $\psi \in Ker(E)\cap Ker(\mathcal{D})$. Using the formulas $\langle 3\rangle$ and $\langle 4\rangle$ we obtain the condition

$$-\frac{R}{4(n-1)}X\cdot\psi = \frac{n}{2(n-2)}\left(\frac{R}{2(n-1)}X - Ric(X)\right)\cdot\psi$$

and, finally,

$$Ric(X)\cdot\psi = \frac{R}{n}X\cdot\psi .$$

Since ψ is a twistor spinor, the zeroes of ψ are isolated points (see [3]) and therefore we conclude

$$Ric(X) = \frac{R}{n}X,$$

i.e. M^n is an Einstein space.

3. An example

Let M^2 be the simply connected Riemannian surface with constant scalar curvature R. Then M^2 admits two Killing spinors φ and ψ. According to the decomposition of the spinor bundle $S = S^+ \oplus S^-$ on M^2 we decompose the Killing spinors into $\psi = \psi^+ + \psi^-$, $\varphi = \varphi^+ + \varphi^-$. We consider the three-dimensional Riemannian manifold $M^3 = M^2 \times R^1$. The general solution of equation (E) on $M^3 = H^2 \times R^1$ (case $R < 0$) is given by

$$\begin{aligned}\psi(x,t) = &\left\{A_0 \cos(\tfrac{1}{2} t) + A_1 \sin(\tfrac{1}{2} t)\right\} \psi^+(x) + \\ &+ \left\{A_0 \sin(\tfrac{1}{2} t) - A_1 \cos(\tfrac{1}{2} t)\right\} \psi^-(x) + \\ &+ \left\{B_0 \cos(\tfrac{1}{2} t) + B_1 \sin(\tfrac{1}{2} t)\right\} \varphi^+(x) + \\ &+ \left\{-B_0 \sin(\tfrac{1}{2} t) + B_1 \cos(\tfrac{1}{2} t)\right\} \varphi^-(x).\end{aligned}$$

A similar formula we can get on $M^3 = S^2 \times R^1$ (case $R > 0$):

$$\begin{aligned}\psi(x,t) = &\left\{A_0 \cosh(\tfrac{1}{2} t) + A_1 \sinh(\tfrac{1}{2} t)\right\} \psi^+(x) - \\ &- i \left\{A_0 \sinh(\tfrac{1}{2} t) + A_1 \cosh(\tfrac{1}{2} t)\right\} \psi^-(x) + \\ &+ \left\{B_0 \cosh(\tfrac{1}{2} t) + B_1 \sinh(\tfrac{1}{2} t)\right\} \varphi^+(x) + \\ &+ i \left\{B_0 \sinh(\tfrac{1}{2} t) + B_1 \cosh(\tfrac{1}{2} t)\right\} \varphi^-(x).\end{aligned}$$

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