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THE GRADED REPRESENTATIONS OF AN AFFINE LIE ALGEBRAS

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Introduction. Let $A=(a_{ij})_{i,j=1}^{n}$ be a generalized Cartan matrix which satisfies the following conditions:

I) $a_{ii}=2$, $i=\overline{1,n}$.

2) $a_{ij}\in\mathbb{Z}$, $a_{ij}\leq 0$, $i\neq j$.

3) A is symmetrizable i.e. there exists a diagonal matrix $D=(d_1,\dots,d_n)$ with non-zero entries such that DA is symmetric.

4) All proper principal minors of DA are positive and $\det A=0$.

The complex affine Lie algebra $\mathfrak{g}=\mathfrak{g}(A)$ is generated by $e_1,\dots,e_n$, $f_1,\dots,f_n,h_1,\dots,h_n$ with following defining relations [3] : $[h_i,e_j]=a_{ij}e_j$, $[h_i,f_j]=-a_{ij}f_j$, $[e_i,f_j]=\delta_{ij}h_j$, $[h_i,h_j]=0$,

$$(ad\, e_i)^{1-a_{ij}}(e_j)=(ad\, f_i)^{1-a_{ij}}(f_j)=0,\quad i\neq j.$$

Denote by H a Cartan subalgebra generated by $h_1,\dots,h_n$.

Let Δ denotes the set of roots, $\pi=\{\alpha_1,\dots,\alpha_n\}$ be some base of Δ , $Q=\{\sum_{K=1}^{n}K_i\alpha_i \mid K_i\in\mathbb{Z}\}$ the lattice of roots, $\Delta^+(\pi)$ the set of positive roots with respect to π .

We have a root space decomposition of $\mathfrak{g}$ with respect to H : $\mathfrak{g}=H\oplus\sum_{\alpha\in\Delta}\mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha=\{g\in\mathfrak{g}\mid [h,g]=\alpha(h)g$ for all $h\in H\}$.

Let $\Delta^{im}=\{K\delta \mid K\in\mathbb{Z}\setminus\{0\}\}$ the set of imaginary roots, where δ is a minimal positive imaginary root.

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The universal enveloping algebra $U(\mathcal{G})$ is a Q-graded algebra: $U(\mathcal{G}) = \bigoplus_{\alpha\in Q} U_\alpha(\mathcal{G})$. $\mathcal{G}$-module V is called

$$Q\text{-graded if} \quad V = \bigoplus_{\alpha\in Q} V_\alpha, \quad U_\alpha(\mathcal{G}) V_\beta \subset V_{\alpha+\beta}$$

for all $\alpha, \beta \in Q$. Let K denotes the category of all Q-graded $\mathcal{G}$-modules. It is clear that Verma modules and generalized Verma modules [4], for example, belong to K. The objects of K constructing by parabolic induction were studied in [5]. In this paper we construct the new families of irreducible objects of K which aren't quotients of generalized Verma modules but have many analogous properties. The first examples of such modules were built in [I].

I. The graded irreducible $\mathcal{G}$-modules.

A subset $F \subset \Delta$ is called closed if $\alpha+\beta \in F$ for all roots $\alpha, \beta \in F$ such that $\alpha+\beta \in \Delta$. Denote by $\langle F \rangle$ the closed subset generated by F. A closed subset $P \subset \Delta$ such that $P \cup -P = \Delta$ we called a parabolic subset. The classification of all parabolic subsets in affine case is contained in [2].

Let $\emptyset \neq F \subset \Delta$, $\mathcal{G}_F = \langle \mathcal{G}_\alpha, \alpha \in F \rangle$, $H_F = \{ h \in H \mid \alpha(h) = 0 \text{ for all } \alpha \in F\} / Z$, where $Z \subset H$ is a center of $\mathcal{G}$ and $H_\emptyset = H$.

We say that $\mathcal{G}$-module V is graded irreducible if V hasn't Q-graded $\mathcal{G}$-submodules.

Let $\mathfrak{A} \subset \mathcal{G}$, V is graded $\mathcal{G}$-module, $V^{\mathfrak{A}} = \{ v \in V \mid Xv = 0 \text{ for all } X \in \mathfrak{A} \}$. For any parabolic subset P we denote by $K(P)$ the subcategory of K of graded $\mathcal{G}$-modules V such that $V^{\mathcal{G}_{P\setminus(-P)}} \neq 0$.

Definition. Any element $v \in V^{\mathcal{G}_{P\setminus(-P)}}$ is called P-primitive.

If $P = \Delta^+(\pi)$ then we have the well-known definition of primitive element.

Let P be a parabolic subset, $P \neq \Delta$,

$I = U(\mathfrak{g})\mathfrak{g}_{P\setminus(-P)} \cap U_0(\mathfrak{g})$.

Proposition I.I. (I) $P \setminus (-P)$ is a closed subset in Δ.
(2) I is the ideal in $U_0(\mathfrak{g})$.
(3) $U_0(\mathfrak{g})/I \simeq U_0(\mathfrak{g}_{P\cap -P}) \otimes U(H_{P\cap -P})$.

Proposition I.2. (I) If W is an irredusible $U_0(\mathfrak{g}_{P\cap -P})\otimes U(H_{P\cap -P})$-module then there exist the unique graded irreducible $\mathfrak{g}$-module $V\in K(P)$ such that $V_0 \simeq W$.
(2) If $V\in K(P)$ is the graded irreducible $\mathfrak{g}$-module, $\alpha\in Q$, $0\neq v\in V_\alpha \cap V^{\mathfrak{g}_{P\setminus(-P)}}$ then V_α is the irreducible $U_0(\mathfrak{g}_{P\cap -P})\otimes U(H_{P\cap -P})$-module.

Proof. Let W be the irreducible $U_0(\mathfrak{g}_{P\cap -P})\otimes U(H_{P\cap -P})$-module. Then we may consider W as $U_0(\mathfrak{g})$-module defining $I\omega = 0$ for all $\omega\in W$. Let $M(P, W) = U(\mathfrak{g}) \otimes_{U_0(\mathfrak{g})} W$. Then $M(P, W)_0 \simeq W$ as $U_0(\mathfrak{g})$-modules. The module $M(P, W)$ has unique maximal graded submodule and thus we have the graded irreducible quotient $L(P, W)$ such that $L(P, W)_0 \simeq W$. Moreover, $\mathfrak{g}_{P\setminus(-P)} L(P, W)_0 = 0$ and $L(P, W) \in K(P)$.

If L is another irreducible module such that $L_0 \simeq W$ then there exists epimorphism $\rho : M(P, W) \to L$. Thus $L \simeq L(P, W)$. It proves (I). The point (2) follows from proposition I.I.

The universal module $M(P, W)$ is very "big" with complicated structure. More convenient to have the "smaller" universal module. Now we shall construct such $\mathfrak{g}$-modules generated by P-primitive elements. The first construction is analogous to the construction of generalized Verma modules.

Let $Q_{P\cap -P} = \{ \sum K_i \alpha_i \mid \alpha_i \in P\cap -P,\ K_i \in \mathbb{Z} \}$,

V be an irreducible a $Q_{P\cap -P}$ -graded $U(\mathfrak{g}_{P\cap -P})$-module, $\lambda \in H^{*}_{P\cap -P}$, $\Lambda_1 = U(\mathfrak{g}_{P\cap -P}) + U(H_{P\cap -P}) + U(\mathfrak{g}_{P\setminus(-P)})$, $M_1^{\lambda}(P, V) = U(\mathfrak{g}) \otimes_{\Lambda_1} V$, where $\mathfrak{g}_{P\setminus(-P)} v = 0$, $hv = \lambda(h) v$ for all $v \in V$, $h \in H_{P\cap -P}$.

It's easy to prove.

Proposition I.3. (I) $M_1^{\lambda}(P, V)$ is Q-graded $\mathfrak{g}$-module.
(2) The element $1 \otimes v$ is P-primitive and $M_1^{\lambda}(P, V)$ is generated by one for any $v \in V$.
(3) $M_1^{\lambda}(P, V)$ has unique maximal graded submodule $\mathcal{M}_1$ and $L_1^{\lambda}(P, V) = M_1^{\lambda}(P, V)/\mathcal{M}_1$ is graded irreducible quotient.
(4) $L_1^{\lambda}(P, V) \in K(P)$ and $L_1^{\lambda}(P, V)^{\mathfrak{g}_{P\setminus(-P)}} \simeq V$.

Now consider another construction of $\mathfrak{g}$-module generated by P-primitive element.

Let $\lambda \in H^{*}_{P\cap -P}$, $\Lambda_2 = U_0(\mathfrak{g}_{P\cap -P}) + U(H_{P\cap -P}) + U(\mathfrak{g}_{P\setminus(-P)})$, $M_2^{\lambda}(P, W) = U(\mathfrak{g}) \otimes_{\Lambda_2} W$, where W is the irreducible Λ_2-module, such that $\mathfrak{g}_{P\setminus(-P)} \omega = 0$, $h\omega = \lambda(h)\omega$ for all $h \in H_{P\cap -P}$, $\omega \in W$.

Proposition I.4. (I) $M_2^{\lambda}(P, W)$ is Q-graded $\mathfrak{g}$-module.
(2) The element $1 \otimes \omega$ is P-primitive and $M_2^{\lambda}(P, W)$ is generated by one for any $\omega \in W$.
(3) $M_2^{\lambda}(P, W)$ has unique maximal graded submodule $\mathcal{M}_2$, $L_2^{\lambda}(P, W) = M_2^{\lambda}(P, W)/\mathcal{M}_2$ is unique graded irreducible quotient.
(4) $M_2^{\lambda}(P, W)_0 \simeq W$ and $M_2^{\lambda}(P, W)_0 \subset M_2^{\lambda}(P, W)^{\mathfrak{g}_{P\setminus(-P)}}$.

The next result shows the universal nature of modules $M(P, W)$, $M_1^{\lambda}(P, V)$, $M_2^{\lambda}(P, W)$.

Theorem I.5. Let $\lambda \in H^{*}_{P\cap -P}$, $\alpha \in Q_{P\cap -P}$.

(I) If V be the graded irreducible $U(\mathfrak{g}_{P\cap -P})$-module and $V_\alpha \neq 0$ then $L(P, V_\alpha) \simeq L_1^\lambda(P, V) \simeq L_2^\lambda(P, V_\alpha)$, $hv = \lambda(h)v$, $v \in V_\alpha$, $h \in H_{P\cap -P}$.
(2) If U is graded irreducible $\mathfrak{g}$-module and U_α contains a P-primitive element u, such that $hu = \lambda(h)u$ for all $h \in H_{P\cap -P}$ then $U \simeq L_2^\lambda(P, U_\alpha)$.

The proof of the theorem follows from propositions I.2-I.4.

Remarks. (I) There exist epimorphisms $\varphi_1 : M(P, V_\alpha) \to M_1^\lambda(P, V)$, $\varphi_2 : M(P, V_\alpha) \to M_2^\lambda(P, V_\alpha)$, $\varphi_3 : M_2^\lambda(P, V_\alpha) \to M_1^\lambda(P, V)$.
(2) If $\delta \notin P\cap -P$ then $L(P, V_\alpha)$, $L_1^\lambda(P, V)$, $L_2^\lambda(P, V_\alpha)$ are irreducible modules and in "general position" they aren't quotients of Verma modules or generalized Verma modules if $P \neq \Delta^+(\Pi) \cup \langle -\Pi' \rangle$ for any Π, $\Pi' \subset \Pi$.

Like that we have the constructions of irreducible graded $\mathfrak{g}$-modules with P-primitive elements. The next theorem gives the characterization of modules without P-primitive elements in particular case $\mathfrak{g} = A_1^{(1)}$.

Theorem I.6. Let V is the irreducible graded $A_1^{(1)}$-module without P-primitive elements for any parabolic subset P. Then $V_\alpha \neq 0$ for any $\alpha \in Q$.

Hypothesis. The theorem I.6 is correct for all affine Lie algebras.

2. The structure of the subalgebra $\mathfrak{g}_{P\cap -P}$. Let P be a parabolic subset and $P \cap -P \neq \emptyset$.

Theorem 2.I. (I) If $\delta \in P \setminus (-P)$ then $\mathfrak{g}_{P\cap -P}$ is a finite dimensional semisimple Lie subalgebra in $\mathfrak{g}$.
(2) If $\delta \in P\cap -P$ then $\mathfrak{g}_{P\cap -P} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$,

where $\mathfrak{g}_1$ is an affine Lie algebra, $\operatorname{rank} \mathfrak{g}_1 \leq \operatorname{rank} \mathfrak{g}$, $\mathfrak{g}_2 \subset \mathfrak{g}_{\Delta^{im}}$ and $\mathfrak{g}_2 \oplus (\mathfrak{g}_{\Delta^{im}} \cap \mathfrak{g}_1) = \mathfrak{g}_{\Delta^{im}}$.

It's easy to prove

Lemma 2.2. Let $\pi = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$, $\alpha \in \pi$, $X_\varphi \in \mathfrak{g}_\varphi$. Then

(I) $\mathfrak{g}_\delta = \{ [X_{\delta-\alpha_i}, X_{\alpha_i}] \mid i = \overline{1,n} \}$.

(2) If $[X_{\delta-\alpha}, X_\alpha] \neq 0$ then $[[X_{\delta-\alpha}, X_\alpha], X_\alpha] \neq 0$.

The proof of the theorem 2.I is based on computations for every affine Lie algebra and used the lemma 2.2.

The next table containes all kinds of the subalgebra $\mathfrak{g}_1$ for different ρ.

$\mathfrak{g}$	$\mathfrak{g}_1$
$A_{n-1}^{(1)}$	$A_{k-1}^{(1)}$, $2 \leq k \leq n-1$
$B_{n-1}^{(1)}$	$A_{k-1}^{(1)}$, $2 \leq k \leq n-1$, $C_2^{(1)}$, $B_{k-1}^{(1)}$, $4 \leq k \leq n-1$
$C_{n-1}^{(1)}$	$A_{k-1}^{(1)}$, $2 \leq k \leq n-1$, $C_{k-1}^{(1)}$, $3 \leq k \leq n-1$
$D_{n-1}^{(1)}$	$A_{k-1}^{(1)}$, $2 \leq k \leq n-1$, $D_{k-1}^{(1)}$, $5 \leq k \leq n-1$
$G_2^{(1)}$, $D_4^{(3)}$	$A_1^{(1)}$
$F_4^{(1)}$	$A_1^{(1)}$, $A_2^{(1)}$, $C_2^{(1)}$, $C_3^{(1)}$, $B_3^{(1)}$
$E_\ell^{(1)}$, $\ell = 6,7,8$	$A_{k-1}^{(1)}$, $2 \leq k \leq \ell$, $D_{k-1}^{(1)}$, $5 \leq k \leq \ell$, $E_k^{(1)}$, $6 \leq k \leq \ell-1$
$A_{2n-2}^{(2)}$	$A_{k-1}^{(1)}$, $2 \leq k \leq n-1$, $A_{2k-2}^{(2)}$, $2 \leq k \leq n-1$
$D_n^{(2)}$	$A_{k-1}^{(1)}$, $2 \leq k \leq n-1$, $D_k^{(2)}$, $3 \leq k \leq n-1$
$A_{2n-3}^{(2)}$	$A_{k-1}^{(1)}$, $2 \leq k < n-1$, $A_{2k-3}^{(2)}$, $4 \leq k \leq n-1$, $D_3^{(2)}$
$E_6^{(2)}$	$A_1^{(1)}$, $A_2^{(1)}$, $D_3^{(2)}$, $D_4^{(2)}$, $A_5^{(2)}$

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