Wojciech Dorabiala

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FINITE GROUP ACTIONS ON 2-DIMENSIONAL CW-COMPLEXES.

Wojciech Dorabiala[1]

The main purpose of this paper is an investigation of a finite group action on 2-dimensional CW-complexes.
More exactly let

$$X = (S^1 \vee S^1 \vee \ldots \vee S^1) \cup_f e^2$$

be a 2-dimensional CW-complex obtained by attaching 2-cell to the finite bouquet of circles. We shall call a CW-complex of this form a μ-complex. For example any compact connected 2-dimensional manifold is a μ-complex.
The conditions under which such a μ-complex is the Eilenberg-MacLane space of type K(G,1) are well known, see [4]. The question whether the finite covering of a μ-complex is itself a μ-complex has also been investigated, see [6].

In this work we try to answer the following questions:
(1) Does a finite group G can act freely on a μ-complex,
(2) Does the orbit space of a given action on the μ-complex is itself a μ-complex.

It turns out that a finite group G can act freely on the μ-complex X in the following three cases:

(a) X is a closed surface.

(b) X is a surface with boundary and the boundary has one component

(c) when the 2-cell e^2 is attached to one of the circles of the bouquet more than twice.

In these cases the answer to the second question (2) is affirmative.

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[1] this paper is in final form and no version of it will be submitted for publication elsewhere.

1. Actions on the 2-dimensional sphere and disk

In this section we recall some known facts about effective actions of a finite group G on the sphere S^2 and the disk D^2. Of course S^2 and D^2 are μ-complexes.

Definition 1 Let the group

$$G = \mathbb{Z}_2 \oplus D_{2n} = \{a,b,c \ : a^2, b^{2n}, c^2, aca^{-1}c^{-1}, bcb^{-1}c^{-1}, abab\}$$

act effectively on $S^1 \times D^1$ by

$$T_a(t,y) = (1-t,y),\ T_b(t,y) = (t+\frac{1}{n},y),\ T_c(t,y) = (t,-y).$$

We quote from [5] the

THEOREM 1.1 Any effective action of a finite group on $S^1 \times D^1$ is conjugated with (a subgroup of) the transformation group G given above.

In the same paper [5] one can find the following.

THEOREM 1.2 Any effective action of a finite abelian group G on S^2 is an extension of some action on $S^1 \times D^1$.

From these theorems we infer

Corollary. The orbit space of the effective action of a finite group G on S^2 is homeomorphic to one of the fllowing spaces:

(1) the projective space RP^2

(2) the 2-dimensional disk D^2

(3) the 2-dimensional sphere S^2

thus the orbit space in all above cases is (up to homeomorphism) a μ-complex.

THEOREM 1.3 [3] Any periodic homeomorphism of the 2-dimensional disk is (conjugated to) a rotation or a reflection.

Corollary. (of theorem 1.3)

The orbit space of an effective action of a finite abelian group on 2-dimensional disk is homeomorphic to the 2-disk.

2. The free action of a finite group

In this section we shall present some necessary conditions under which a finite group can act freely on a μ-complex.

Comparing the Euler characteristics of the space X and the orbit space $X/_G$ one gets the following.

LEMMA 2.1 Let X be the complex obtained by attaching a 2-dimensional cell to the circle, $X = S^1 \cup_f e^2$. Then there does not exist the free action of a non-trivial finite group G on the complex X.

Now we consider the case when the 2-cell of the complex X is attached to all circles of the bouquet. Then a finite group G can act freely on the complex X.

LEMMA 2.2 Let X be a μ-complex of the form $X = X' \vee \bigvee_{i=1}^{k} S_i^1$ where $X' \subset X$ is the subcomplex of X such that the 2-cell e^2 is contained in X'. Then there does not exist the free action of any finite group G on the complex X.

Proof: Let $g: X \longrightarrow X$ be a homeomorphism. Considering a neighborhood of points $x_1 \in X$ and $x_2 \in \bigvee_{i=1}^{k} S_i^1$ it is not difficult to prove that $g(X') = X'$ and $g(\bigvee_{i=1}^{k} S_i^1) = \bigvee_{i=1}^{k} S_i^1$. Hence $\bigvee_{i=1}^{k} S_i^1$ is G-equivariant subcomplex of X for any action of group G. The free action of G on X gives us the free action on $\bigvee_{i=1}^{k} S_i^1$.

Now let $f: \bigvee_{i=1}^{k} S_i^1 \longrightarrow \bigvee_{i=1}^{k} S_i^1$ be the homeomorphism and x_o be the vertex of the bouquet.
Thus

$$f: \bigvee_{i=1}^{k} S_i^1 \setminus \{x_o\} \longrightarrow \bigvee_{i=1}^{k} S_i^1 \setminus \{f(x_o)\}$$

is the homeomorphism and induces the isomorphism on fundamental groups.

$$\begin{array}{ccc}
\pi_1\left[\bigvee_{i=1}^{k} S_i^1 \setminus \{x_0\}\right] & \xrightarrow{f_*} & \pi_1\left[\bigvee_{i=1}^{k} S_i^1 \setminus \{f(x_0)\}\right] \\
\| & & \| \\
0 & \xrightarrow{\cong} & \pi_1\left[\bigvee_{i=1}^{k} S_i^1 \setminus \{f(x_0)\}\right] \neq 0
\end{array}$$

This is valid for $k > 1$ when $f(x_0) = x_0$. We infer that any homeomorphism of a bouquet of circles fixes the vertex of this bouquet. This implies that there does not exist a free action on $X = X' \vee \bigvee_{i=1}^{k} S_i^1$ for $k > 1$. In order to finish the proof of the lemma it is enough to consider the case $X = X' \vee S^1$. Using the same type of argument as above we can show that the vertex of the bouquet $X' \vee S^1$ is fixed by any homeomorphism.

Corollary (of lemma 2.2)

The group G can act freely on the complex X when X is obtained by attaching 2-cell to all circles of the bouquet.

In order to formulate further results we need the following.

Definition

Let X be the μ-complex with the fundamental group $\pi_1(X) = \langle x_1, x_2, x_3, \ldots x_n : r \rangle$ where $r = f_*(1)$. We denote by $\lambda_{x_i}(r)$ the sum of absolute values of exponents occurring at x_i in the relation r.

Example

(a) $r = x_1^1 x_3^{-6} x_1^4 x_2^{-4} x_3^2$ then $\lambda_{x_3}(r) = 8$, $\lambda_{x_1}(r) = 5$, $\lambda_{x_2}(r) = 4$

(b) if X is a closed surface then $\lambda_{x_i}(r) = 2$ for any i.

THEOREM 2.3 let X be a μ-complex such that $\lambda_{x_i}(r) > 2$ for at least two generators x_i. Then any homeomorphism $f: X \longrightarrow X$ has a fixed point.

Proof: Here, for convenience, we will consider the complex X as the quotient space of a polygon where the relation is given by $r = f_*(1)$ on the boundary of the polygon. The assumption $\lambda_{x_l}(r) > 2$ means that the side x_l occurs in the boundary of the polygon more than twice. Let $f: X \longrightarrow X$ be a homeomorphism and let x_k be another side of the boundary for which $\lambda_{x_k}(r) > 2$.

Our aim is to prove that any point of the side x_k via the homeomorphism f is mapped into the side x_l or x_k for which $\lambda_{x_k}(r) > 2$. In order to prove this we consider the point $b \in x_k$ different from the point a, the vertex of the bouquet. Let us assume that $f(b) \in \overset{\circ}{e}{}^2$. Then there exists an open neighborhood U of f(b) homeomorphic to $\overset{\circ}{D}{}^2$. Since $f^{-1}: U \longrightarrow f(U)$ is a homeomorphism we see that f(U) is the neighborhood of the point b. This is a contradiction with the fact that b has a neighborhood homeomorphic to $\overset{\circ}{D}{}^2$. To finish the proof we have to consider the vertex of the bouquet. In this case it is enough to assume that x_k occurs at least tree times in the boundary of a polygon. Then a neighborhood of the point a will contain the identification of three parts of a disc with respect to the common edge.

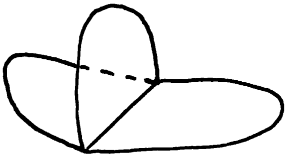


Fig.1

Hence the vertex a does not have a neighborhood homeomorphic to $\overset{\circ}{D}{}^2$. What we have proved above imply that any homeomorphism $f: X \longrightarrow X$ preserves the bouquet of the circles obtained from x_k and x_l. Now because any homeomorphism f restricted to a bouquet of circles has a fixed point (this has been proven in lemma 2.2) one can see that the homeomorphism f on X has a fixed point too.

THEOREM 2.4 Let X be the μ-complex such that $\lambda_{x_i}(r) = 1$ for at least two generators x_i whereas $\lambda_{x_k}(r) = 2$ for others. Then any homeomorphism $f: X \longrightarrow X$ has a fixed point.

Proof: The proof of this theorem is similar to the previous one.

Corollary (of theorem 2.3 and 2.4)

The group G can act freely on the complex X only in the following cases :

(1) if $\lambda_{x_i}(r) = 2$ for $i = 1,2,3...,n$,

(2) if $\lambda_{x_k}(r) = m > 2$ for exactly one k and $\lambda_{x_i}(r) = 2$ otherwise,

(3) if $\lambda_{x_k}(r) = 1$ for exactly one k and $\lambda_{x_i}(r) = 2$ otherwise.

LEMMA 2.5 Let $\lambda_{x_i}(r) = 2$ for $i = 1,...,n$. Assume that there exists a homeomorphism $f:X \longrightarrow X$ without fixed points.Then X is a surface and

$$r = x_1^2 x_2^2 x_3^2 ... x_n^2 \quad \text{or} \quad r = [x_1,x_2][x_3,x_4]...[x_{n-1},x_n].$$

Proof: It is not difficult to verify that for any point b different from the vertex a of the bouquet there exists a neighborhood of b homeomorphic to $\mathring{D}^2$. This follows easily from the fact that $\lambda_{x_i}(r) = 2$ for any $i = 1,2...,n$.

Let us assume that there does not exist a neighborhood of a homeomorphic to $\mathring{D}^2$. Let $b = f(a)$ and $b \neq a$. We know that there exists the neighborhood U of b homeomorphic to $\mathring{D}^2$. Since f is a homeomorphism, $f: f^{-1}(U) \longrightarrow U$ is also a homeomorphism and $f^{-1}(U)$ is the neighborhood of the point b. This is a contradiction. Thus we have $f(a) = a$. But this contradicts the assumption that f does not have a fixed point.This completes the proof of the lemma.

LEMMA 2.6 Let $\lambda_{x_k}(r) = 1$ for exactly one k and $\lambda_{x_i}(r) = 2$ otherwise. Assume further that there exist a homeomorphism $f: X \longrightarrow X$ without fixed points. Then the complex X is a surface with one-component boundary.

Proof: The proof of this lemma is similar to the preceding one.

3.The orbit space of complex X.

In the previous chapter we gave the profound answer to the question when a group G can act freely on the μ-complex X

We now turn our attention to the question when the orbit space $X/_G$ of any action on the μ-complex is also the μ-complex.

First we prove the following theorem.

THEOREM 3.1 If a finite abelian group G acts on the complex $X = X' \vee \bigvee_{i=1}^{k} S^1_i$ such that $\lambda_{x_i}(r) \neq 2$ for i=1,2,3,...,n then the orbit space of this action is (up to homotopy type) a μ-complex.

<u>Proof</u>: To get a clearer view of the complex X, as in chapter 2 we consider the complex X as a quotient space of the polygon. The relation on the boundary of polygon is given by r. According to what was said in the first chapter the orbit space of any action of a group G on a 2-disk is homeomorphic to 2-disk. Moreover, the orbit space of any action of a group on the bouquet of the circles $\bigvee_{i=1}^{k} S^1_i$ is itself homotopy equivalent to the bouquet of circles. The assumption $\lambda_{x_i}(r) \neq 2$ for any i = 1,2,3,...,n guarantees that the boundary of the polygon is G-invariant. Let us consider the action of a group G on the circle. An importent action on the circle is the reflection.

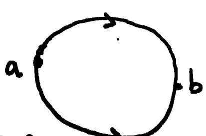


Fig.2

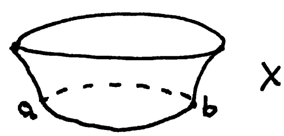


Fig.3

Consider the complex X in this case . Assume that G acts on one circle as it is shown on the picture 2, then we get the situation depicted in picture 3.

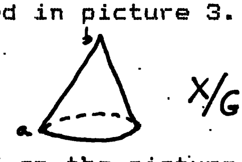


Fig.4

But the complex X_1 on the picture 4 is homotopy equivalent to X_1' in which the side ab is contracted to a point. Now the proof goes by series of contractions which reduce the complex X_1 to the complex X_1' with one 2-cell. In view of the previous reduction we considered the case when the complex X has no free circles. But when X has " free " circles then $X/_G = X'/_G \vee \bigvee_{i=1}^{k} S^1_i/G$ so that we may indeed take into account X without " free " circles.

THEOREM 3.2 If the finite abelian group G acts effectively on the surface then the orbit space of this action is also a surface.

This theorem is the classical fact already known, see [7]

THEOREM 3.3 If a finite abelian group G acts freely on the complex X such that $\lambda_{x_i}(r)$ satisfy conditions (1) and (3) from page 6 then the orbit space of this action is homeomorphic to a μ-complex.

Proof: The proof of this theorem is a consequence of the theorems proved in this chapter.

4.The free action which induce the trivial action on cohomology.

In the previous parts of the paper all the results were obtained by geometric methods. In what follows we shall use algebraic methods and consider the free action of a group G on the complex,inducing the trivial action on cohomology.
The assumption that the group G acts trivially on the cohomology allows us to compute cohomology groups of the orbit space $X/_G$.
For the finite group G we have the isomorphism
$H^*(X,\mathbb{Q})^G \cong H^*(X/_G,\mathbb{Q})$ see[1].
Because the action on cohomology is trivial we have
$H^*(X/_G,\mathbb{Q}) \cong H^*(X,\mathbb{Q})$.

LEMMA 4.1 If n = 2 then there does not exist a free action of a finite nontrivial group G on the complex $X = X' \vee \bigvee_{=1}^{k} S^1_i$ which induces a trivial action on cohomology.

Proof: Previous remarks give us the equalities for Euler characteristics:

$$\chi(X) = \chi(X/_G) \text{ and } \chi(X) = 2 - n.$$

As far as the action on X is free and $\|G\| < \infty$ one obtains $\chi(X) = \|G\| \chi(X/_G) = \|G\| \chi(X)$. Dividing both sides of the equality by $\chi(X) \neq 0$ we obtain $\|G\| = 1$ which means that G is trivial.

The next theorem gives the answer to the question when the trivial action on X induces the trivial action on cohomology.
Before we formulate the theorem we will need some more information about the cohomology of a μ-complex.

Definition Let r be an element of the free group generated by $x_1, x_2, \ldots, x_n$. We denote by $\delta_{x_i}(r)$ the sum of exponents at x_i occurring in the element r. For the μ-complex X we define the number $d = d(X) = \text{g.c.d.}\ (\delta_{x_1}(r), \delta_{x_2}(r), \ldots, \delta_{x_n}(r))$. Then we have

$$H^i(X,\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } i=0 \\ \mathbb{Z}^n & \text{for } i=1 \\ \mathbb{Z} & \text{for } i=2 \end{cases} \quad \text{if } \delta_{x_i}(r)=0 \text{ for every } i$$

$$H^i(X,\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } i=0 \\ \mathbb{Z}^{n-1} & \text{for } i=1 \\ \mathbb{Z}_d & \text{for } i=2 \end{cases} , \quad \text{if } \delta_{x_i}(r)\neq 0 \text{ for every } i$$

THEOREM 4.2 Assume that a finite group G of order m acts on the complex $X = (S^1 \vee S^1) \cup_f e^2$. Let m be an integer satisfying the conditions g.c.d(m,2)=g.c.d(m,3)=g.c.d(m,d)=1. Then the induced action on cohomology is trivial.

Proof: We have two possibilities for the cohomology group of the complex X.

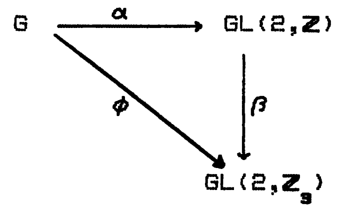
$$(1)\quad H^i(X,\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } i = 0 \\ \mathbb{Z} & \text{for } i = 1 \\ \mathbb{Z}_d & \text{for } i = 2 \end{cases}$$

and

$$(2)\quad H^i(X,\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } i = 0 \\ \mathbb{Z}\oplus\mathbb{Z} & \text{for } i = 1 \\ \mathbb{Z} & \text{for } i = 2 \end{cases} .$$

When the cohomology group is a cyclic group then the action on cohomology is given by multiplication by an integer a such that $a^m=1$. The assumption g.c.d(m,2) = g.c.d(m,3) = = g.c.d(m,d) = 1. implies a = 1.

In order to show the triviality of the action on the first cohomology group in the second case it is enough to consider the commutative diagram.



where β is induced by the canonical homomorphism $\mathbb{Z} \longrightarrow \mathbb{Z}_3$ and α represents the action of a group G on $H^1(X,\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$. Now because g.c.d.($\|G\|$, $\|GL(2,\mathbb{Z}_3\|$),we have that any homomorphism. $\phi: G \longrightarrow GL(2,\mathbb{Z}_3)$ is trivial. Applying the Minkowski theorem which says that $\alpha(G) \cap \ker\beta = \{1\}$ and the fact that ϕ is trivial we obtain the triviality of α.

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WOJCIECH DORABIALA
INSTYTUT MATEMATYKI
UNIWERSYTET MATEMATYKI
UL.WIELKOPOLSKA 15
70-451 SZCZECIN
POLAND