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In: Jan Slovák (ed.): Proceedings of the 16th Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 1997. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 46. pp. [107]--116.

Persistent URL: http://dml.cz/dmlcz/701600

RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO
Serie II, Suppl. 46 (1997), pp. 107-116

BRS-Transformations in a finite dimensional setting*

Margarita Kraus

Abstract

In order to get a mathematical understanding of the BRS-transformation and the Slavnov-Taylor identities, we will treat them in a finite dimensional setting. We will show that in this setting the BRS-transformation is a vector field on a certain supermanifold. The connection to the BRS-complex will be established. Finally we will treat the generating functional and the Slavnov-Taylor identity in this setting.

1 Introduction

In classical gauge theories one starts with a gauge invariant action $S[\Phi] = \int d^4x L[\Phi]$, where Φ denotes the fields and $L[\Phi]$ the Lagrangian, which is integrated over the Minkowski space $\mathbb{R}^{1,3}$. For example the Lagrangian of QCD is given by

$$L[A, \Psi, \tilde{\Psi}] = \tilde{\Psi}(i\gamma^\mu D_\mu - m)\Psi - \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a,$$

where A denotes the gauge potential, Ψ the Dirac spinor and $\tilde{\Psi}$ its conjugate. The covariant derivative D_μ is given by $D_\mu = \partial_\mu + igA^a_\mu t_a$ and the field strength tensor $F^{\mu\nu}_a$ by

$$F^{\mu\nu}_a = \partial^\mu A^\nu_a - \partial^\nu A^\mu_a - gf_{abc}A^\mu_b A^\nu_c,$$

where f_{abc} are the structure constants of the Lie algebra $\mathfrak{g}$ of the gauge group G given by a basis $(t_1, \dots, t_n)$ of $\mathfrak{g}$.

The infinitesimal form of the gauge transformation is

$$\begin{aligned} \delta_\theta \Psi(x) &= -it_a\theta^a(x)\Psi(x) \\ \delta_\theta A^a_\mu(x) &= -\frac{1}{g}D^{ab}_\mu \theta^b(x) := -(\frac{1}{g}\partial_\mu\theta^a(x) - f_{abc}\theta^b(x)A^c_\mu(x)), \end{aligned} \tag{1}$$

which has a mathematical interpretation as the fundamental vector field given by the action of the gauge transformation group $\mathcal{G} = C^\infty(\mathbb{R}^{1,3}, G)$ on the space $\mathcal{C}$ of gauge fields and $\mathcal{E}$ of Dirac fields [15, p. 25].

*The paper is in final form and no version of it will be submitted elsewhere.

The quantization of such a theory by standard methods is not possible. One has to modify the theory to make it accessible to these methods. For that purpose new fields – the anticommuting ghost and antighost fields C and $\bar{C}$ and the auxiliary field B – are introduced. A gauge fixing map $\mathcal{F}$, e.g. the Lorentz gauge $\mathcal{F}(A) = \partial^\mu A_\mu$, is chosen and the gauge invariant action is replaced by the effective action with effective Lagrangian (cf. e.g. [18, (2.3.179)]):

$$L_{\text{eff}}[A, \Psi, \tilde{\Psi}, B, C, \bar{C}] = L[A, \Psi, \tilde{\Psi}] + B^a \mathcal{F}^a(A) + \frac{\alpha}{2} B^a B_a - \bar{C}^b \mathcal{M}_{\mathcal{F}}^{ba}(A) C^a, \quad (2)$$

with the Fadeev-Popov operator $\mathcal{M}_{\mathcal{F}}(A)$, which in the Lorentz gauge is given by

$$\mathcal{M}_{\mathcal{F}}(A)(x,y)^{ab} = -\partial^\mu (\frac{1}{g} \partial_\mu \delta^{ab} - f^{abc} A_\mu^c) \delta^4(x-y).$$

L_{eff} is no longer gauge invariant, but in 1974 C. Becchi, A. Rouet and R. Stora [2], [3] found a new symmetry of it: the invariance under BRS-transformation, which has important consequences. This transformation usually is given by its action on the fields (cf. e.g. [18, (2.3.180)], [1, (2.11)]):

$$\begin{aligned} sA_\mu^a &= -\frac{1}{g}(D_\mu C)^a \\ s\Psi &= -\mathrm{i} t_a C^a \Psi \\ sC^a &= -\frac{1}{2} f_{abc} C^b C^c \\ s\bar{C}^a &= B^a \\ sB^a &= 0 \end{aligned} \quad (3)$$

On functionals in the fields it acts as a derivation, which is given in [4, (26)] by

$$sS[A, C, \bar{C}] = \frac{\partial S}{\partial A} sA + \frac{\partial S}{\partial C} sC + \frac{\partial S}{\partial \bar{C}} s\bar{C} \quad (4)$$

and fulfils [1, (2.12)]

$$s(XY) = (sX)Y + (-1)^{|X|} X(sY),$$

where $|X| = 0$ for bosonic and $|X| = -1$ for fermionic fields X.

On gauge fields and Dirac fields the BRS-transformation acts as an infinitesimal gauge transformation (1) with θ replaced by the anticommuting ghost field C.

But while the mathematical meaning of the infinitesimal gauge transformation as an infinitesimal action is clear, the mathematical meaning of the formulas (3) is not obvious, it is not an infinitesimal action in the ordinary sense.

There are several mathematical interpretations of these transformations in the literature. Eg. [4], [6],[20] treat the BRS-transformation on A and C, [7], [11], [23] the BRS-transformation on A, C and $\bar{C}$ and [17] the BRS-transformation of Yang-Mills theory as a dynamical system.

If one would try to assume $C \in C^\infty(\mathbb{R}^{1,3}, \mathfrak{g}) =: L\mathcal{G}$, then $sC = 0$ because of the antisymmetry of the structure constants. Therefore, in order to give (3) a mathematical meaning, the first step is to give an interpretation of the anticommutativity of the fields C and $\bar{C}$. The attempt to model C as $C^a \in \Lambda^1(C^\infty(\mathbb{R}^{1,3}))$ would yield $C^a(x) \wedge C^b(x) = 0$, so this is not the right interpretation either.

But if one interprets C as the identity on $L\mathcal{G}$, then it is possible to read $sC = -\frac{1}{2}[C \wedge C] \in \mathrm{Alt}^2(L\mathcal{G}, L\mathcal{G})$, where $\mathrm{Alt}^k(V, W)$ denotes the space of the k-linear alternating maps $V \times \ldots \times V \to W$. In the same way we will interpret the other fields as identities.

For the sake of simplicity we will not include the Dirac spinors in our discussion but just consider a finite dimensional version of Yang-Mills theory. We will see that supergeometry is the right tool to treat theories with anticommuting fields. The BRS-transformation will then appear as a vector field on a supermanifold and its relation to the BRS-complex becomes clear. In a last section we will treat a consequence of the BRS-transformation, the first Slavnov-Taylor identity.

2 The BRS-Transformation

We introduce our finite dimensional setting of Yang-Mills theory by choosing finite dimensional versions of "physical" data as follows.

"physics":	"finite dimensional setting":
group $\mathcal{G}$ of gauge transformations	finite dimensional Lie group G
space $L\mathcal{G}$ of infinitesimal gauge transformations	Lie algebra $\mathfrak{g}$ with scalar product $\langle \cdot, \cdot \rangle$
space $\mathcal{C}$ of gauge fields	finite dimensional vector space V with (non-linear) G-action $\Phi : G \times V \to V$
infinitesimal gauge transformation given by $\delta_\theta A^a_\mu(x) = -\frac{1}{g} D^{ab}_\mu \theta^b(x)$	infinitesimal action $\rho \in C^\infty(V, \mathrm{Hom}(\mathfrak{g}, V))$, given by the G-action Φ as $\rho(v) := (d\Phi(\cdot, v))(1)$
gauge invariant action S	G-invariant function $\ell \in C^\infty(V)$
gauge fixing map $\mathcal{F}$	$f \in \mathrm{Hom}(V, \mathfrak{g})$
Fadeev-Popov operator $\mathcal{M}_\mathcal{F}$	$m_f := f \circ \rho \in C^\infty(V, \mathrm{End}(\mathfrak{g}))$.

As we have have argued already in the introduction we will interpret the fields as identities. Thus we replace them by coordinate maps:

"physics":	"finite dimensional setting":
gauge field A	$a := \mathrm{id} \in C^\infty(V, V)$
auxiliary field B	$b := \mathrm{id} \in C^\infty(\mathfrak{g}_0, \mathfrak{g}_0)$
ghost field C	$c := \mathrm{id} \in \mathrm{Alt}^1(\mathfrak{g}, \mathfrak{g})$
antighost field $\bar{C}$	$\bar{c} := \mathrm{id} \in \mathrm{Alt}^1(\bar{\mathfrak{g}}, \bar{\mathfrak{g}})$

where $\mathfrak{g} =: \mathfrak{g}_0 =: \bar{\mathfrak{g}}$.

Therefore, as a finite dimensional model of the effective action S_{eff} of Yang-Mills theory, given by the effective Lagrangian (2):

" physics":
$L_{\text{eff}}[A,B,C,\bar{C}] = L[A] + B^a\mathcal{F}^a(A) + \frac{\alpha}{2}B^aB_a - \bar{C}^b\mathcal{M}^{ba}_{\mathcal{F}}(A)C^a$,

we will use the welldefined mathematical object

"finite dimensional setting":
$\ell_{\text{eff}} := \ell + \langle b, f\rangle + \frac{1}{2}\langle b, b\rangle - \langle \bar{c}\wedge m_f \circ c\rangle \in C^\infty(V\oplus \mathfrak{g}_0)\otimes\Lambda(\mathfrak{g}\oplus\bar{\mathfrak{g}})^*$.

Before we proceed, let us recall some results of the theory of supermanifolds, which is a well elaborated theory (cf. eg. [12], [16], [8]). By a superspace we mean a pair $(V, C^\infty(V)\otimes\Lambda W^*)$ with vector spaces V and W. We will consider ℓ_{eff} as superfunction on the superspace $(V\oplus\mathfrak{g}_0, C^\infty(V\oplus\mathfrak{g}_0)\otimes\Lambda(\mathfrak{g}\oplus\bar{\mathfrak{g}})^*)$.

For vector fields, which means in our special case nothing else than graded derivations of $C^\infty(V\oplus\mathfrak{g}_0)\otimes\Lambda(\mathfrak{g}\oplus\bar{\mathfrak{g}})^*$, we use the following result (cf. (cf. [12, p. 197])).

Theorem 1 *Let V and W be vector spaces of dimensions m and n. Then the vector space of graded derivations* $\text{Der}(C^\infty(V)\otimes\Lambda W^*)$ *as a $C^\infty(V)\otimes\Lambda W^*$-modul is freely generated by the even partial derivatives $\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_m}$ and the odd partial derivatives $\frac{\partial}{\partial \xi_1},\ldots,\frac{\partial}{\partial \xi_n}$, where $(x_1,\ldots,x_m)$ are coordinates for V and $(\xi_1,\ldots,\xi_n)$ is a basis of W^*. The action of $\frac{\partial}{\partial x_i}$ on superfunctions is given by*

$$\frac{\partial}{\partial x_i}(g\xi_1^{\mu_1}\ldots\xi_n^{\mu_n}) = (\frac{\partial}{\partial x_i}g)\xi_1^{\mu_1}\ldots\xi_n^{\mu_n},\ \textit{for}\ \mu_i\in Z_2$$

and of $\frac{\partial}{\partial \xi_l}$ by

$$\frac{\partial}{\partial \xi_l}g\xi_{i_1}\ldots\xi_{i_k} = g\sum_{j=1}^{k}\delta_{li_j}(-1)^{j-1}\xi_{i_1}\ldots\hat{\xi}_{i_j}\ldots\xi_{i_k}.$$

We now define the even and odd derivatives of superfunctions $f\in C^\infty(V)\otimes\Lambda W^*$ by

$$d_V f := \sum_{i=1}^{m}\frac{\partial}{\partial x_i}f dx_i \in C^\infty(V)\otimes\Lambda W^*\otimes V^*$$

$$d_W f := \sum_{i=1}^{n}\frac{\partial}{\partial \xi_i}f\otimes\xi_i \in C^\infty(V)\otimes\Lambda W^*\otimes W^*$$

and in a similar way for subspaces of V and W.

Then for our superspace $(V\oplus\mathfrak{g}_0, C^\infty(V\oplus\mathfrak{g}_0)\otimes\Lambda(\mathfrak{g}\oplus\bar{\mathfrak{g}})^*)$ we get as a consequence of Theorem 1

Corollary 1 *Let δ be a Derivation of $C^\infty(V\oplus\mathfrak{g}_0)\otimes\Lambda(\mathfrak{g}\oplus\bar{\mathfrak{g}})^*$ and $f\in C^\infty(V\oplus\mathfrak{g}_0)\otimes\Lambda(\mathfrak{g}\oplus\bar{\mathfrak{g}})^*$. Then*

$$\delta f = \delta(a)\cdot d_V f + \delta(b)\cdot d_{\mathfrak{g}_0}f + \delta(c)\cdot d_{\mathfrak{g}}f + \delta(\bar{c})\cdot d_{\bar{\mathfrak{g}}}f,$$

where we have abbreviated $(\delta\otimes\text{id})(x) =: \delta(x)$ for $x = a, b, c, \bar{c}$ and " $\cdot$ " denotes multiplication in $C^\infty(V\otimes\mathfrak{g}_0)\otimes\Lambda(\mathfrak{g}\oplus\mathfrak{g})^*$ followed by evaluation on $V\otimes V^*$, resp. $\mathfrak{g}\otimes\mathfrak{g}^*$.

We replace the functional derivatives in (4) in the finite dimensional setting by even and odd derivatives

$\frac{\partial}{\partial A}$, $\frac{\partial}{\partial C}$, $\frac{\partial}{\partial \bar{C}}$	d_V, $d_{\mathfrak{g}}$, $d_{\bar{\mathfrak{g}}}$

Now it is obvious that in the finite dimensional setting the BRS-transformation (3) has to be replaced by the vector field on the superspace $(V \oplus \mathfrak{g}_0, C^\infty(V \oplus \mathfrak{g}_0) \otimes \Lambda(\mathfrak{g} \oplus \bar{\mathfrak{g}})^*)$, which is defined on the coordinate maps by the righthand side in the following table:

BRS-transformation:	BRS-vector field:
$sA^a_\mu = -\frac{1}{g}(D_\mu C)^a$	$Da = \rho(a) \circ c$
$sB^a = 0$	$Db = 0$
$sC^a = -\frac{1}{2} f^{abc} C^b C^c$	$Dc = -\frac{1}{2}[c \wedge c]$
$s\bar{C}^a = B^a$	$D\bar{c} = b$

We will use the remainder of this section to illuminate the mathematical meaning of this special vector field D by explaining its relation to the BRS-complex, which has been worked out in another context by several authors (e.g. [10], [13], [21], [9], for a survey cf. [22], [24]).

Let ∂ be the Koszul differential on $C^\infty(V \oplus \mathfrak{g}_0) \otimes \Lambda\bar{\mathfrak{g}}^*$ given by

$$\begin{aligned} \partial : C^\infty(V \oplus \mathfrak{g}_0) \otimes \Lambda^{-i}\bar{\mathfrak{g}}^* &\to C^\infty(V \oplus \mathfrak{g}_0) \otimes \Lambda^{-(i+1)}\bar{\mathfrak{g}}^*, \\ (f \otimes \alpha) &\mapsto ((v, x) \mapsto f(v, x) \otimes (x \lrcorner \alpha)), \end{aligned}$$

where $(v, x) \in V \oplus \mathfrak{g}_0$. For the cohomology of this complex there is the following result (cf. [9, p. 241])

$$H^0(C^\infty(V \oplus \mathfrak{g}_0) \otimes \Lambda\bar{\mathfrak{g}}^*, \partial) = C^\infty(V)$$

and

$$H^i(C^\infty(V \oplus \mathfrak{g}_0) \otimes \Lambda\bar{\mathfrak{g}}^*, \partial) = 0 \quad \text{for} \quad i \neq 0.$$

Next we consider the Lie algebra differential $\delta_{\rho,\mathfrak{g}} = \delta_\rho + \mathrm{id} \otimes \delta_{\mathfrak{g}}$ with coefficients in $C^\infty(V \oplus \mathfrak{g}_0) \otimes \Lambda\bar{\mathfrak{g}}^*$, where $\delta_\rho : C^\infty(V \oplus \mathfrak{g}_0) \otimes \Lambda\bar{\mathfrak{g}}^* \otimes \Lambda^i \mathfrak{g}^* \to C^\infty(V \oplus \mathfrak{g}_0) \otimes \Lambda\bar{\mathfrak{g}}^* \otimes \Lambda^{i+1}\mathfrak{g}^*$ is given by

$$\delta_\rho(f \otimes \alpha \otimes \beta)(v_0, v_1, \ldots, v_i) = \sum_{k=0}^{i} (-1)^k v_{k\rho} f \otimes \alpha \otimes \beta(v_0, \ldots, \hat{v}_k, \ldots, v_i),$$

with $v_k \in \mathfrak{g}$, and $v_{k\rho}$ denoting the fundamental vector field given by the infinitesimal action ρ, and $\delta_{\mathfrak{g}}$ by

$$(\delta_{\mathfrak{g}}\beta)(v_0, \ldots, v_i) = \sum_{k<l} (-1)^{k+l} \beta([v_k, v_l], v_0, \ldots, \hat{v}_k, \ldots, \hat{v}_l, \ldots v_i).$$

Then $(C^\infty(V \oplus \mathfrak{g}_0) \otimes \Lambda\bar{\mathfrak{g}}^* \otimes \Lambda\mathfrak{g}^*, \partial \otimes \mathrm{id}, \delta_{\rho,\mathfrak{g}})$ is a double complex, which is called the BRS-double complex. By [9, p. 239] we have

Theorem 2 *If $(\mathcal{K}, \mathcal{D})$ denotes the complex corresponding to the BRS-double complex, with the BRS-operator $\mathcal{D} = \partial \otimes \mathrm{id} + \delta_{\rho,\mathfrak{g}}$, then $H^0(\mathcal{K}, \mathcal{D}) = C^\infty(V)^G$, where $C^\infty(V)^G$ denotes the G-invariant functions on V .*

This BRS-operator $\mathcal{D}$ is a graded derivation on $C^\infty(V \oplus \mathfrak{g}_0) \otimes \Lambda(\mathfrak{g} \oplus \bar{\mathfrak{g}})^*$, therefore a vector field.

Theorem 3 *The BRS-operator is the BRS-vector field, $\mathcal{D} = D$.*

Proof By Theorem 1 $\mathcal{D}$ is determined by its action on the coordinate maps. By a straightforward calculation we get $\mathcal{D}x = Dx$ for $x = a, b, c, \bar{c}$, which shows $\mathcal{D} = D$. □

The vector space V is a model for the space $\mathcal{C}$ of gauge fields A^a_μ, hence $H^0(\mathcal{K}, \mathcal{D})$ models the space of gauge invariant functionals on $\mathcal{C}$. Especially for the gauge invariant action this yields the following

Theorem 4 *For $\ell \in \mathcal{K}^0$ and $\ell_{\mathrm{eff}} \in \mathcal{K}^0$ we have $[\ell]_\mathcal{D} = [\ell_{\mathrm{eff}}]_\mathcal{D}$*

Proof It is $\mathcal{D}\ell = \delta_{\rho,\mathfrak{g}}\ell = 0$, because of the G-invariance of ℓ, and a straightforward calculation shows

$$\ell_{\mathrm{eff}} = \ell + \mathcal{D}(\frac{1}{2}\langle \bar{c} \wedge b\rangle + \langle \bar{c} \wedge f\rangle).$$

□

3 The Slavnov-Taylor Identities

In the physical literature the BRS-transformations are used to obtain the first Slavnov-Taylor identity by manipulating the mathematically ill-defined path integral (cf. [5, p. 152]). In the finite dimensional setting of the last section we get a corresponding identity by strictly mathematical methods.

The Slavnov-Taylor identity is an identity for the generating functional using the effective action $\tilde{S}_{\mathrm{eff}}$ without auxiliary field given by

"physics":
$\tilde{L}_{\mathrm{eff}}[A, C, \bar{C}] = L[A] - \frac{1}{2}\mathcal{F}(A)^a\mathcal{F}(A)_a - \bar{C}^b\mathcal{M}^{ba}_{\mathcal{F}}(A)C^a$,

which is modelled by

"finite dimensional setting":
$\tilde{\ell}_{\mathrm{eff}} = \ell - \frac{1}{2}\langle f \cdot f\rangle - \langle \bar{c} \wedge m_f \circ c\rangle \in C^\infty(V) \otimes \Lambda(\mathfrak{g} \oplus \bar{\mathfrak{g}})^*$.

This action $\tilde{S}_{\mathrm{eff}}$ is invariant under a slightly modified BRS-transformation $\tilde{s}$ modelled by a vector field $\tilde{D} =: \delta_{\rho,\mathfrak{g}} + \tilde{\partial}$ as shown in the following table:

"physics":	"finite dimensional setting":
$\tilde{s}A = sA, \quad \tilde{s}C = sC$	$\tilde{D}a := Da, \quad \tilde{D}c := Dc$
$\tilde{s}\bar{C}^a = -\mathcal{F}(A)^a$	$\tilde{D}\bar{c} := -f$

To model the generating functional we replace the path integral by an integral over finite dimensional spaces.

Recall that the fermionic integral of superfunctions $g \in C^\infty(V) \otimes \Lambda(\mathfrak{g} \oplus \bar{\mathfrak{g}})^*$ is defined by

$$\int^F (\sum_{\mu,\nu} g_{(\mu,\nu)} \xi_1^{\mu_1} \bar{\xi}_1^{\nu_1} \ldots \xi_n^{\mu_n} \bar{\xi}_n^{\nu_n}) \mathrm{d}\xi \mathrm{d}\bar{\xi} := g_{(1,\ldots,1)}$$

(cf. [19]), where $(\xi_1, \ldots, \xi_n)$ is a basis of $\mathfrak{g}^*$, with ξ_i interpreted as the element $\xi_i \otimes 1$ in $\Lambda\mathfrak{g}^* \otimes \Lambda\bar{\mathfrak{g}}^*$ and $\bar{\xi}_i := 1 \otimes \xi_i$. This integral has the same properties as the fermionic integral defined in the physical literature, cf. [19, 10.5].

Using this integration we model the generating functional

$$T\{J, \omega, \bar{\omega}\} = \frac{1}{N} \int \mathrm{d}[A, C, \bar{C}] e^{\mathrm{i}(\tilde{S}_{\text{eff}} + \int (J_a^\mu(x) A_\mu^a(x) + \bar{\omega}^a(x) C^a(x) + \bar{C}^a(x) \omega^a(x)) \mathrm{d}^4 x)},$$

where J, ω and $\bar{\omega}$ are sources of the fields, by the Fourier transform of the superfunction $e^{\tilde{l}_{\text{eff}}}$, that is by

$$t := \int_V \mathrm{d}x \int^F \mathrm{d}\xi \mathrm{d}\bar{\xi} e^{\tilde{l}_{\text{eff}} + \mathrm{i}(\langle j_a \cdot a \rangle + \langle j_c \wedge c \rangle + \langle j_{\bar{c}} \wedge \bar{c} \rangle)} \in C^\infty(V^*) \otimes \Lambda(\mathfrak{g} \oplus \bar{\mathfrak{g}}) \otimes \mathbb{C},$$

where we have replaced the sources of the fields by coordinate maps in $C^\infty(V^*) \otimes \Lambda(\mathfrak{g} \oplus \bar{\mathfrak{g}})$:

$J, \quad \omega, \quad \bar{\omega}$	$j_a := \mathrm{id}_{V^*}$, $j_c := \mathrm{id}_{\mathfrak{g}^*}$, $j_{\bar{c}} := \mathrm{id}_{\bar{\mathfrak{g}}^*}$

(For the fermionic Fourier transform cf. [19, p. 229].)

The derivation of the first Slavnov-Taylor identity in the physical literature starts from the assumption that the 'measure in the path integral' is BRS-invariant. In the finite dimensional setting we have the following corresponding theorem:

Theorem 5 *Let G be a unimodular Lie group, V a finite dimensional vector space with measure $\mathrm{d}x$ and Φ an action of G on V, such that the measure $\mathrm{d}x$ is G-invariant. Let $(T_1, \ldots, T_n)$ be a basis in $\mathfrak{g}$. If the fundamental vector fields $(T_i)_\Phi$ are polynomial and $g \in \mathcal{S}(V) \otimes \Lambda(\mathfrak{g} \oplus \bar{\mathfrak{g}})^*$, where $\mathcal{S}(V)$ denotes the rapidly decreasing functions on V in the generalized sense (cf [14, p. 117]) then*

$$\int_V (\int^F \mathcal{D} g \mathrm{d}\xi \mathrm{d}\bar{\xi}) \mathrm{d}x = 0.$$

Proof Obviously we have $\int^F (\tilde{\partial} g) \mathrm{d}\xi \mathrm{d}\bar{\xi} = 0$, where $\tilde{\partial}$ denotes the vector field defined by $\tilde{\partial}\bar{c} = -f$. But $\mathfrak{g}$ is unimodular, so $\delta_{\mathfrak{g}} \alpha = 0$ for $\alpha \in \Lambda^{n-1}\mathfrak{g}^*$, therefore $\int^F (\mathrm{id} \otimes \delta_{\mathfrak{g}}) g \mathrm{d}\xi \mathrm{d}\bar{\xi} = 0$. Finally $\int_V \int^F \delta_\rho g \mathrm{d}\xi \mathrm{d}\bar{\xi} \mathrm{d}x = 0$ is a consequence of the invariance of the measure on V. For details cf. [14, p. 203]. □

The first Slavnov-Taylor identity is given by (cf. [5, (2.3.136'), (2.3.139')]) "physics":

$$\frac{\mathrm{i}}{\alpha}\mathcal{F}^a\left(\frac{\delta}{\mathrm{i}\delta J(y)}\right)T\{J\} = \int \mathrm{d}^4x J_b^\mu(x) D_\mu^{bc}\left(\frac{\delta}{\mathrm{i}\delta J(x)}\right)T^{ca}\{J,x,y)$$

with

$$T^{ca}\{J,x,y) := \frac{\delta}{\mathrm{i}\delta\bar{\omega}_c(x)}\mathrm{i}\frac{\delta}{\delta\omega_a(y)}\Big|_{\omega=\bar{\omega}=0} T\{J,\omega,\bar{\omega}\}.$$

In the finite dimensional setting as a consequence of Theorem 5 and some properties of the Fourier transform we get

Theorem 6 *Let G be a unimodular Lie group, V a finite dimensional vector space with measure $\mathrm{d}x$ and Φ an action of G in V, such that the infinitesimal action ρ is polynomial and $\mathrm{d}x$ is G-invariant. If $e^{\tilde{l}_{\mathrm{eff}}} \in \mathcal{S}(V) \otimes \Lambda(\mathfrak{g} \oplus \bar{\mathfrak{g}})^*$, then*

$$\mathrm{i}f\left(\frac{\partial}{\mathrm{i}\partial j_a}\right)\tilde{t} = \langle j_a \wedge \rho\left(\frac{\partial}{\partial j_a}\right)t_{\bar{\mathfrak{g}},\mathfrak{g}}\rangle \quad \textit{with} \quad t_{\bar{\mathfrak{g}},\mathfrak{g}} := (\mathrm{d}_{\bar{\mathfrak{g}}^*}\mathrm{d}_{\mathfrak{g}^*}t)^{\tilde{}}$$

and $\tilde{}$ denoting the projection $C^\infty(V^*) \otimes \Lambda(\mathfrak{g} \oplus \bar{\mathfrak{g}}) \to C^\infty(V^*)$. In coordinates this is

"finite dimensional setting":

$$\mathrm{i}f^\alpha\left(\tfrac{\partial}{\mathrm{i}\partial j_a}\right)\tilde{t} = j_a^\beta\left(\rho\left(\tfrac{\partial}{\partial j_a}\right)\right)^{\beta\gamma}(t_{\bar{\mathfrak{g}},\mathfrak{g}})^{\gamma\alpha}$$

with

$$(t_{\bar{\mathfrak{g}},\mathfrak{g}})^{\gamma\alpha} := \left(\tfrac{\partial}{\mathrm{i}\partial j_c^\gamma}\tfrac{\partial}{\mathrm{i}\partial j_{\bar{c}}^\alpha}t\right)^{\tilde{}}.$$

Proof cf. [14, p. 216].

The last formula corresponds in the finite dimensional setting exactly to the first Slavnov-Taylor identity.

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