Javier Lafuente; Beatriz Salvador

From the Fermi-Walker to the Cartan connection

In: Jan Slovák and Martin Čadek (eds.): Proceedings of the 19th Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 2000. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 63. pp. 149--156.

Persistent URL: http://dml.cz/dmlcz/701657

RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO
Serie II, Suppl. 63 (2000) pp. 149-156

# FROM THE FERMI-WALKER TO THE CARTAN CONNECTION

JAVIER LAFUENTE & BEATRIZ SALVADOR

ABSTRACT. Let $M$ be a differentiable manifold and $\mathcal{C}=\{e^{2\sigma}\mathbf{g} \;/\; \sigma : M \to \mathbb{R}\}$ a Riemannian conformal structure on $M$. Given any regular curve in $M$, $\gamma : I \to M$, there is a natural way of defining an operator, $D/dt : \mathfrak{X}(\gamma) \to \mathfrak{X}(\gamma)$, the *Fermi-Walker* connection along $\gamma$, which only depends on the conformal structure $\mathcal{C}$, and such that it coincides with the Fermi-Walker connection along $\gamma$ -in the classical sense- of any $g \in \mathcal{C}$ such that $g(\gamma'(t),\gamma'(t)) = 1 \;\forall t \in I$. This *Fermi-Walker* connection enables us to construct a lift-function $\kappa_b : T_p^2M \to T_bCO(M)$ for every $b \in CO(M)$, and $p = \pi(b)$, $\pi : CO(M) \to M$ being the usual projection. In some sense, $\kappa_b$ combines all the different lift-functions $T_pM \to T_bCO(M)$ given by the Levi-Civita connections of the compatibles metrics $g \in \mathcal{C}$. But over all, $\kappa_b$ determines the conformal structure $\mathcal{C}$ over $M$, so that it may be used to know about the normal Cartan connection and the Weyl conformal curvature tensor.

## 1. THE FERMI-WALKER CONNECTION

Let $M$ be a differentiable n-dimensional manifold and $\mathcal{C}=\{e^{2\sigma}\mathbf{g} \;/\; \sigma : M \to \mathbb{R}$ differentiable$\}$ a Riemannian conformal structure on $M$.

Any curve in $M$ with parametrization $\gamma : I \to M$ being an embedding, admits a non empty family $\mathcal{C}_\gamma$ of compatible metrics $g \in \mathcal{C}$ such that $\gamma$ is $g$-geodesic ([6]). The requirement of $\gamma$ being a curve with velocity field not null assures the existence of several compatible metrics $\overline{g} \in \mathcal{C}$ such that $\overline{g}(\gamma'(t),\gamma'(t)) = 1$, $\forall t \in I$ ; then, a metric $g$ in the family $\mathcal{C}_\gamma$ can be obtained from this $\overline{g}$ by defining $g = e^{2\sigma}\overline{g}$ , where $\sigma$ is a differentiable function verifying the following conditions along $\gamma$ :

a) $(grad_{\overline{g}}\sigma)(\gamma(t)) = -\frac{\overline{\nabla}}{dt}\gamma'(t)$, $\forall t \in I$

b) $\sigma(\gamma(t)) = 0$, $\forall t \in I$

These two conditions are compatible (observe that the first one implies that $\sigma$ is constant along $\gamma$) and is easy to see that such a function can be constructed (locally), further more, there are several of them.

It can be checked that every $g$ in this family $\mathcal{C}_\gamma$ induces the same connection along $\gamma$ through its Levi-Civita connection. Thus, since any regular curve is locally an embedding, along every regular curve in $M$, $\gamma : I \to M$, there is a connection, $D/dt : \mathfrak{X}(\gamma) \to \mathfrak{X}(\gamma)$, which only depends on the conformal structure $\mathcal{C}$, and it will be called *Fermi-Walker connection along $\gamma$ associated to $\mathcal{C}$.*

This paper is in final form and no version of it will be submitted for publication elsewhere.

The justification of this name follows from the fact that for any $g \in \mathcal{C}$ such that $g(\gamma'(t), \gamma'(t)) = 1\ \forall t \in I$, $D/dt$ coincides with the Fermi-Walker connection along $\gamma$ associated with $g$ in the classical sense of the Theory of Relativity, that is used to define a standard of non-rotation for curves that are not geodesics ([4]). So, if $\nabla$ is the Levi-Civita connection of $g$, then $D/dt$ is determined by the conditions:

a) $\frac{D}{dt}\gamma'(t) = 0, \forall t \in I$

b) $\frac{D}{dt}X(t) = pr(\frac{\nabla}{dt}X(t))$
for any $X(t) \in \mathfrak{X}(\gamma)$ orthogonal to the field $\gamma'(t)$, and being $pr : T_{\gamma(t)}M \to \gamma'(t)^{\perp}$ the orthogonal projection over $\gamma'(t)^{\perp}$.

Or equivalently, $\frac{D}{dt}$ is defined by the formula:

$$\frac{D}{dt}X(t) = g(\frac{\nabla}{dt}X(t), \gamma'(t)).\gamma'(t) + pr(\frac{\nabla}{dt}(prX)(t)) \text{ , for any field } X(t) \in \mathfrak{X}(\gamma).$$

## 2. The lift-function $\kappa$ associated to $(M, \mathcal{C})$

It is well known that a linear connection on a $G$-structure $\mathcal{B}$ over $M$ gives rise to a family of lift functions from $M$ to the fiber bundle $\mathcal{B}$. These lift-functions elevate tangent vectors of $M$ to tangent vectors of the fibre bundle, in such a way that, if $b$ is any element of the fiber over $p$, then $T_pM$ is linearly elevated to a horizontal subspace of $T_b\mathcal{B}$ complementary to the vertical subspace ([2], [1], [8]). In this section we are going to do a similar construction. As in the linear case, a family of special lift-functions develops from the *Fermi-Walker* connection associated to a conformal structure. The fiber bundle that appears in this case is the conformal frame bundle $CO(M)$, as it might be expected.

Let $\gamma : I \to M$ be a regular curve on $M$, then a field along $\gamma$, $X(t) \in \mathfrak{X}(\gamma)$, is said to be *Fermi-Walker parallel* if $\frac{D}{dt}X(t) = 0$. The set of *Fermi-Walker* parallel fields $\mathfrak{X}_{|\cdot|}(\gamma) = \{\ X(t) \in \mathfrak{X}(\gamma) : \frac{D}{dt}X(t) = 0\}$ is a real vector subspace of $\mathfrak{X}(\gamma)$ and the following application

$$\begin{array}{ccc} \mathfrak{X}_{|\cdot|}(\gamma) & \to & T_{\gamma(t_0)}M \\ X & \mapsto & X(t_0) \end{array}$$

is a linear isomorphism, for every $t_0 \in I$.

Given any $v \in T_{\gamma(t_0)}M$, $\mathbb{P}_\gamma v \in \mathfrak{X}_{|\cdot|}(\gamma)$ is the unique parallel field with $(\mathbb{P}_\gamma v)(t_0) = v$, and the curve in $TM$ $\mathbb{P}_\gamma v : I \to TM$ is the *Fermi-Walker* horizontal lift of $\gamma$. For any $t \in I$, $(\mathbb{P}_\gamma v)(t) \in T_{\gamma(t)}M$ is called the *Fermi-Walker* parallel transported vector of $v$ to $T_{\gamma(t)}M$.

It is clear that the *Fermi-Walker* connection $D/dt$ transports vectors along $\gamma : I \to M$ preserving the angles; that is why we can use $D/dt$ for transporting along $\gamma$ elements of the conformal frame bundle $\pi : CO(M) \to M$, $CO(M) = \{b = (v_1, ..., v_n) : \exists g \in \mathcal{C}, \exists p \in M \text{ st.}\{\ v_1, ..., v_n\}$ is a $g$-orthonormal frame of $T_pM\} = \{b : \mathbb{R}^n \to T_pM$ isomorphism : $\exists g \in \mathcal{C}$ st. $b^*g_p =$canonical metric of $\mathbb{R}^n\}$.

Given any $b \in CO(M)$ with $\pi(b) = p = \gamma(0)$, we get the horizontal lift in $CO(M)$ of the curve $\gamma$ with initial value $b$, $\mathbb{P}_\gamma b : I \to CO(M)$, by means of the *Fermi-Walker* parallel transport:

$$\mathbb{P}_\gamma b(t) = (\mathbb{P}_\gamma v_1(t), ..., \mathbb{P}_\gamma v_n(t)) \in CO(M), \forall t \in I, \ \textit{being}\ b = (v_1, ..., v_n).$$

When working with linear connections, the 1-jet of the lift-curve in the bundle is determined by the 1-jet of $\gamma$ in $M$ (this fact allows to construct a lift-function between tangent spaces). This not being the case now, it can still be verified that the 1-jet $j_0^1(\mathbb{P}_\gamma b) \in T_bCO(M)$ is completely determined by the 2-jet defined by $\gamma$, $j_0^2(\gamma) \in T^2_{\pi(b)}M$. This fact is easily proved taking a chart of $M$ near $\pi(b)$ and studying the expressions of these jets in the induced charts of $T^2M$ and $TCO(M)$.

This shows that, for each $b \in CO(M)$, $D/dt$ defines a lift-function $\kappa_b$ from the open subset of $T^2_{\pi(b)}M$ of 2-jets given by regular curves in $M$ emanating from $\pi(b)$, $A^2_{\pi(b)} \subset T^2_{\pi(b)}M$, to the set of 1-jets of curves in $CO(M)$ emanating from $b$, that is $T_bCO(M)$.

$$\kappa_b : A^2_{\pi(b)} \subset T^2_{\pi(b)}M \to T_bCO(M)$$
$$j_0^2(\gamma) \mapsto j_0^1(\mathbb{P}_\gamma b)$$

It is clear from its definition, that $\kappa_b$ verifies the formula:

(1) $$d\pi(b) \circ \kappa_b(j_0^2(\gamma)) = j_0^1(\gamma) \in T_{\pi(b)}M$$

Remember that the lift-functions given by linear connections are linear monomorphisms from $T_pM$ to $T_bCO(M)$, but there is no sense in talking about the linearity of $\kappa_b$ since $A^2_{\pi(b)} \subset T^2_{\pi(b)}M$ has not structure of vectorial space. However, we are going to see that the mapping $\kappa_b$ is an injective differentiable function that combines the linear lift-functions $T_{\pi(b)}M \to T_bCO(M)$ associated to the different compatible metrics of $\mathcal{C}$, in the following sense.

Every $g \in \mathcal{C}$ defines, in a natural way, a global section $\sigma^g$ of the fiber bundle $T^2M \to TM$, such that if $v = j_0^1(\gamma) \in T_pM$ is a tangent vector, then its image through $\sigma^g$ is the 2-jet of the $g$-geodesic with initial velocity vector $v$.

Now, if we compose the section $\sigma^g$ with $\kappa_b$, it is clear that the mapping thus obtained is the lift-function of the Levi-Civita connection of $g$ (it follows from the fact that for any $g$-geodesic in $M$, $\gamma : I \to M$, the *Fermi-Walker* connection $D/dt$ coincides with the Levi-Civita connection of $g$ along $\gamma$).

Moreover, every 2-jet in $A^2_{\pi(b)}$ can be reached by this construction for some $g \in \mathcal{C}$; we only have to take $\gamma : I \to M$, a regular curve defining the 2-jet, and any $g \in \mathcal{C}_\gamma$, this means that $\gamma$ is $g$-geodesic, and then $\sigma^g(j_0^1(\gamma)) = j_0^2(\gamma)$.

This fact also motivates extending the definition of $\kappa$ all over the set of 2-jets of constant curves in $M$, $\gamma_p(t) = p, \forall t \in I$. So we define $\kappa_b(j_0^2(\gamma_p)) = (\kappa_b \circ \sigma^g)(\gamma_p'(0)) = 0$ (this last equality comes from the fact that $\gamma_p'(0) = 0$ and $\kappa_b \circ \sigma^g$ is a linear application).

Therefore, we have the following diagram

$$T_pM \xrightarrow{\sigma^g} A^2_p \cup \{j_0^2(\gamma_p)\} \subset T^2_pM \xrightarrow{\kappa_b} T_bCO(M)$$
$$\underrightarrow{\kappa_b^g \ g\ \textit{lift-function (linear application)}}$$

**Lemma 1.** *If $g, \overline{g} \in \mathcal{C}$ are related by $\overline{g} = e^{2f}g$ , then, a computation with the expressions of the geodesics with initial velocity $v$ for $g$ and for $\overline{g}$ in a $g$-normal chart of $M$*

*around p, gives the relationship*

$$\kappa_b(\sigma^{\overline{g}}(v)) - \kappa_b(\sigma^g(v)) = (H(\eta, V))^{\#}(b) \tag{2}$$

*being:*

$V = b^{-1}(v) \in \mathbb{R}^n$, *a column vector,*
$\eta = df(p) \circ b \in \mathbb{R}^{n*}$, *a row vector,*
$H : \mathbb{R}^{n*} \otimes \mathbb{R}^n \to \mathfrak{co}(\mathfrak{n})$ *homomorphism,* $H(\eta, V) = \eta.V.I_n + (\eta^T.V^T - V.\eta)$
$(H(\eta, V))^{\#}$ *the fundamental field given by* $H(\eta, V) \in \mathfrak{co}(\mathfrak{n})$.

**Remark 1.** *It is well known that the first prolongation of the algebra* $\mathfrak{co}(\mathfrak{n})$, *this is* $\mathfrak{co}(\mathfrak{n})_1 = \{\alpha \in L(\mathbb{R}^n, \mathfrak{co}(\mathfrak{n})) \subset \mathbb{R}^{n*} \otimes \mathbb{R}^{n*} \otimes \mathbb{R}^n \ / \ \alpha(V)(W) = \alpha(W)(V) \in \mathbb{R}^n\}$, *is isomorphic to* $\mathbb{R}^{n*}$ *([7], [9]). The homomorphism above,* $H : \mathbb{R}^{n*} \otimes \mathbb{R}^n \to \mathfrak{co}(\mathfrak{n})$, *can be seen as*

$$\begin{array}{cccc} H: & \mathbb{R}^{n*} & \to & \mathfrak{co}(\mathfrak{n})_1 \subset L(\mathbb{R}^n, \mathfrak{co}(\mathfrak{n})) \\ & \eta & \mapsto & H(\eta) \end{array}$$

*being* $H(\eta)(V) = H(\eta, V) = \eta.V.I_n + (\eta^T.V^T - V.\eta) \in \mathfrak{co}(\mathfrak{n})$, $\forall V \in \mathbb{R}^n$, *then, H is an isomorphism that gives the identification* $\mathfrak{co}(\mathfrak{n})_1 \overset{\Phi}{\simeq} \mathbb{R}^{n*}$.

**Proposition 2.** *The lift-function* $\kappa_b : A^2_{\pi(b)} \subset T^2_{\pi(b)}M \to T_bCO(M)$ *is an injective differentiable function and* $\kappa_b(j^2_0(\gamma)) \neq 0 \in T_bCO(M), \forall j^2_0(\gamma) \in A^2_{\pi(b)}$.

**Proof.** Any element of $A^2_{\pi(b)}$ is given by the 2-jet of a regular curve $\gamma$, so using 1 we have $\pi_* \circ \kappa_b(j^2_0(\gamma)) = j^1_0(\gamma) = v \in T_pM\backslash\{0\}$. Since $\pi_*$ is a linear application, this implies that $\kappa_b(j^2_0(\gamma)) \neq 0$.

Now we are going to study the injectivity of $\kappa_b$.

Let $j^2_0(\gamma), j^2_0(\overline{\gamma}) \in A^2_{\pi(b)}$ such that $\kappa_b(j^2_0(\gamma)) = \kappa_b(j^2_0(\overline{\gamma})) \in T_bCO(M)$. We can repeat the construction above, and by 1 we get

$$j^1_0(\gamma) = \pi_* \circ \kappa_b(j^2_0(\gamma)) = \pi_* \circ \kappa_b(j^2_0(\overline{\gamma})) = j^1_0(\overline{\gamma}) = v \in T_pM\backslash\{0\}$$

Let $g, \overline{g} = e^{2f}g \in \mathcal{C}$ be compatible metrics such that $\sigma^g(v) = j^2_0(\gamma)$, $\sigma^{\overline{g}}(v) = j^2_0(\overline{\gamma})$. Then, using the formula 2,

$$\begin{gathered} 0 = \kappa_b(\sigma^{\overline{g}}(v)) - \kappa_b(\sigma^g(v)) = (H(\eta, V))^{\#}(b) \implies \\ 0 = H(\eta, V) = \eta.V.I_n + (\eta^T.V^T - V.\eta), V = b^{-1}(v) \neq 0 \implies \\ 0 = \eta = df(p) \circ b \in \mathbb{R}^{n*} \implies 0 = df(p). \end{gathered}$$

This means that $g$ and $\overline{g}$ have the same Levi-Civita connection at $p$, therefore $\sigma^g(v) = \sigma^{\overline{g}}(v) \iff j^2_0(\gamma) = j^2_0(\overline{\gamma}) \in A^2_{\pi(b)}$, and the injectivity has been proved.

The differentiability of $\kappa_b$ is easier to check if we previously fix a compatible metric $g \in \mathcal{C}$. We know that the lift function $\kappa^g_b : T_pM \to T_bCO(M)$ of the Levi-Civita connection $\nabla$ of $g$ is a differentiable function. Using again the formula 2, we have

$$\kappa_b(\sigma^{\overline{g}}(v)) = \kappa_b(\sigma^g(v)) + (H(\eta, V))^{\#}(b) = \kappa^g_b(b(V)) + (H(\eta, V))^{\#}(b) \tag{3}$$

It is clear that the last expression of this equality is a differentiable function over $V \in \mathbb{R}^n$, $\eta \in \mathbb{R}^{n*}$. Now consider the following diffeomorphism :

$$\begin{array}{ccc} A^2_{\pi(b)} & \to & \mathbb{R}^n\backslash\{0\} \times \mathbb{R}^n \\ j^2_0(\gamma) & \mapsto & (b^{-1}(\gamma'(0)), b^{-1}(\frac{\nabla}{dt}\gamma'(0))) \end{array}$$

Through an easy calculation, it can be proved that $\sigma^{\overline{g}}(v) \in A^2_{\pi(b)}$ is sent to $(V, -2(\eta.V).V + (V^T.V)\eta^T) \in \mathbb{R}^n\backslash\{0\} \times \mathbb{R}^n$. And observe that the application $\mathbb{R}^n\backslash\{0\} \times \mathbb{R}^{n*} (V,\eta) \mapsto (V, -2(\eta.V).V + (V^T.V)\eta^T) \in \mathbb{R}^n\backslash\{0\} \times \mathbb{R}^n$ is a diffeomorphism. So, we have constructed a diffeomorphism from $A^2_{\pi(b)}$ to $\mathbb{R}^n\backslash\{0\} \times \mathbb{R}^{n*}$ such that $\sigma^{\overline{g}}(v)$ is identified with $(V,\eta)$. Using the formula 3, we have that, with this identification, the action of $\kappa_b$ is given by $\kappa_b((V,\eta)) = \kappa_b^g(b(V)) + (H(\eta, V))^{\#}(b)$, which is differentiable. □

In conclusion, we can give to $\kappa_b$ the following meaning. For any $g \in \mathcal{C}$ we have the pair $\{\sigma^g, \kappa_b^g\}$ of functions over $T_pM$, where $\sigma^g : T_pM \to T_p^2M$ is the $g$-section of the bundle $T_p^2M$ over $T_pM$, and $\kappa_b^g : T_pM \to T_bCO(M)$ is the linear lift-function associated to the Levi-Civita connection of $g$. We have seen that $\bigcup_{g\in\mathcal{C}} \sigma^g(T_pM) = A_p^2 \cup \{j_0^2(\gamma_p)\} \subset T_p^2M$ and that this family of pairs of functions verifies the following condition:

$$\forall g, \overline{g} \in \mathcal{C}, v, w \in T_pM,\ \sigma^g(v) = \sigma^{\overline{g}}(w) \in T_p^2M \Rightarrow \begin{array}{c} v = w \in T_pM \\ \kappa_b^g(v) = \kappa_b^{\overline{g}}(w) \in T_bCO(M) \end{array}$$

It is clear that this condition of compatibility allows us to define a function from $A_p^2 \cup \{j_0^2(\gamma_p)\}$ to $T_bCO(M)$ that glues the family of $\kappa_b^g : T_pM \to T_bCO(M)$, $g \in \mathcal{C}$, by means of the sections $\sigma^g : T_pM \to A_p^2 \cup \{j_0^2(\gamma_p)\} \subset T_p^2M$; the function thus obtained is just the lift-function $\kappa_b : A_p^2 \cup \{j_0^2(\gamma_p)\} \to T_bCO(M)$ associated to the Fermi-Walker connection.

## 3. Getting the Cartan connection

Fixing a compatible torsion free connection $\nabla$ in $M$, the lift-function $\kappa_b$ enables us to get the first prolongation of the conformal frame bundle of $M$, $CO(M)_1 \subset L(CO(M))$. This special subbundle of $L(CO(M))$ has the property that it can be used to obtain the normal conformal connection of $(M, \mathcal{C})$ in a manner as natural as restricting the canonical fundamental form to it ([7], [9]). The normal conformal connection is an interesting Cartan connection that gives information about the conformal structure; in particular, the well known Weyl curvature tensor, that can be used to check the locally conformal flatness of $(M, \mathcal{C})$, can be obtained from it ([3], [11], [5]).

First of all, there is a natural trivialization given by the linear connection $\nabla$ of the fiber bundle $T^2M \simeq^{\nabla} TM \times TM$ just by identifying $j_0^2(\gamma) \in T_p^2M$ with $(\gamma'(0), \frac{\nabla}{dt}\gamma'(0)) \in T_pM \times T_pM$. Observe that under this identification the section $\sigma^g$ given by any $g \in \mathcal{C}$ such that $\nabla$ is its Levi-Civita connection corresponds to the zero-section,

$$\begin{array}{ccc} T_pM & \to & T_pM \times T_pM \\ v & \mapsto & (v, 0) \end{array}$$

To prepare for the following construction, for any $\eta \in \mathbb{R}^{n*}$, $b \in CO(M)$, $p = \pi(b)$, ( $b$ is seen here as the isomorphism $b : \mathbb{R}^n \to T_pM$ that sends the canonical frame of $\mathbb{R}^n$ to the frame $b$ of $T_pM$ ), we shall define the mapping

$$\begin{array}{cccc} \Phi^{\nabla}_{(b,\eta)} : & T_pM & \to & T_pM \times T_pM \simeq^{\nabla} T_p^2M \\ & v & \mapsto & (v, -2\eta(b^{-1}(v)).v + g_p(v,v).(\mathrm{d}_b^g)^{-1}(\eta)) \end{array}$$

being $g \in \mathcal{C}$ and $\mathbf{d}^g_b: \begin{array}{ccl} T_pM & \to & \mathbb{R}^{n*} \\ v & \mapsto & \mathbf{d}^g_b(v) = g_p(v, b(.)) \end{array}$

Observe that the definition of $\Phi^{\nabla}_{(b,\eta)}$ is independent of the choice of the $g \in \mathcal{C}$ because the term $g_p(v,v).(\mathbf{d}^g_b)^{-1}(\eta)$ only depends on $\mathcal{C}$ ( it keeps invariant by conformal changes of the metric $g$).

It is clear that $\Phi^{\nabla}_{(b,\eta)}$ is an injective differentiable function, furthermore, is an embedding from $T_pM$ to $T^2_pM$.

**Remark 2.** $\Phi^{\nabla}_{(b,\eta)}$ *coincides with the section* $\sigma^{\overline{g}} : T_pM \to T^2_pM$, *for any* $\overline{g} \in \mathcal{C}$ *which Levi-Civita connection* $\overline{\nabla}$ *differs from* $\nabla$ *at* $p$ *in* $\eta \in \mathbb{R}^{n*}$, *in the following sense:*

$$(\overline{\nabla} - \nabla)_p(v,w) = \lambda(v).w + \lambda(w).v - g_p(v,w).(\mathbf{d}^g)^{-1}(\lambda_p), \forall v, w \in T_pM$$

*being* $\mathbf{d}^g : T_pM \to T_pM^*$ *the isomorphism* $\mathbf{d}^g(u) = g(u,.)$

*This affirmation follows from the formula we obtained for* $\sigma^{\overline{g}}(v)$ *in the identification of the proof of the proposition in the previous section.*

We are ready to define the application that will give the key for getting the first prolongation of $CO(M)$. For any $\eta \in \mathbb{R}^{n*}$, $b \in CO(M)$,

$$Z^{\nabla}_{(b,\eta)} : \begin{array}{ccl} \mathbb{R}^n + \mathfrak{co}(n) & \to & T_bCO(M) \\ (V, \alpha) & \mapsto & Z^{\nabla}_{(b,\eta)}(V,\alpha) = \kappa_b \circ \Phi^{\nabla}_{(b,\eta)}(b(V)) + \alpha^{\#}(b) \end{array}$$

where $\alpha^{\#}$ is the fundamental vertical field in $CO(M)$ given by $\alpha \in \mathfrak{co}(n)$.

$Z^{\nabla}_{(b,\eta)}$ is a linear application, moreover, it is a linear isomorphism from $\mathbb{R}^n + \mathfrak{co}(n)$ to $T_bCO(M)$. That means that $Z^{\nabla}_{(b,\eta)}$ is an element of the bundle $L(CO(M))$.

So we have obtained the application

$$Z^{\nabla}_{(.,.)} : \begin{array}{ccl} CO(M) \times \mathbb{R}^{n*} & \to & L(CO(M)) \\ (b, \eta) & \mapsto & Z^{\nabla}_{(b,\eta)} : \mathbb{R}^n + \mathfrak{co}(n) \to T_bCO(M) \end{array}$$

**Remark 3.** *If we change the fixed compatible torsion free connection* $\nabla$ *for another* $\overline{\nabla}$ *such that the form* $\lambda \in \Lambda^1(M, \mathbb{R})$ *measures the difference between them (i.e.* $(\overline{\nabla} - \nabla)(v,w) = \lambda(v).w + \lambda(w).v - g_p(v,w).(\mathbf{d}^g)^{-1}(\lambda_p)$, $\forall v, w \in T_pM$, *being* $\mathbf{d}^g : T_pM \to T_pM^*$ *the isomorphism* $\mathbf{d}^g(u) = g(u,.)$*), then, the relation between* $Z^{\nabla}_{(.,.)}$ *and* $Z^{\overline{\nabla}}_{(.,.)}$ *is given by*

$$(4) \qquad Z^{\nabla}_{(.,.)}(b,\eta) = (Z^{\overline{\nabla}}_{(.,.)})^{-1}(b, \eta - \lambda_p \circ b) \qquad \forall (b,\eta) \in CO(M) \times \mathbb{R}^{n*}.$$

Now, recall that $CO(M)_1$, the first prolongation of $CO(M)$, is the subbundle of $L(CO(M))$ of the frames $Z \in L(CO(M))$ such that, if we think of $Z$ as an isomorphism $Z : \mathbb{R}^n + \mathfrak{co}(n) \to T_bCO(M)$, then it verifies the following conditions:

a) $Z(0,\alpha) = \alpha^{\#}(b)$, $\forall \alpha \in \mathfrak{co}(n)$.

b) $\theta^L(Z(V,\alpha)) = V$, $\forall V \in \mathbb{R}^n$, $\forall \alpha \in \mathfrak{co}(n)$, with $\theta^L \in \Lambda^1(LM, \mathbb{R}^n)$ the canonical form.

c) The torsion of $Z$, $\tau_Z = d\,\theta^L \circ (Z(\,.\,,0), Z(\,.\,,0)) \in A^2(n)$, is identically zero $\tau_Z = 0$.

And $CO(M)_1$ is a $\mathfrak{co}(n)_1$-principal bundle over $CO(M)$ with projection function $\Pi : CO(M)_1 \to CO(M)$, being $\Pi$ the restriction to $CO(M)_1$ of the projection of the bundle $L(CO(M))$ over $CO(M)$ ( [7], [9]).

**Remark 4.** *It is clear from the definition of* $Z^{\nabla}_{(.,.)}$ *that* $Z^{\nabla}_{(b,0)} : \mathbb{R}^n + \mathfrak{co}(n) \to T_bCO(M)$ *is the isomorphism given by the connection* $\nabla$ *and therefore* $Z^{\nabla}_{(b,0)} \in CO(M)_1$, *since it verifies conditions a), b), c). Using the formula 4, for any* $\eta \in \mathbb{R}^{n*}$, *we have that* $Z^{\nabla}_{(b,\eta)}$ *coincides with* $Z^{\overline{\nabla}}_{(b,0)} : \mathbb{R}^n + \mathfrak{co}(n) \to T_bCO(M)$, *which is the isomorphism given by the connection* $\overline{\nabla}$ *(also a torsion free connection) related to* $\nabla$ *by means of* $\eta \circ b^{-1}$ *at* $p = \pi(b)$ *in the sense of the Remark 3, and so we conclude that* $Z^{\nabla}_{(b,\eta)} \in CO(M)_1, \forall (b,\eta) \in CO(M) \times \mathbb{R}^{n*}$.

**Theorem 3.** *The image of* $CO(M) \times \mathbb{R}^{n*}$ *under* $Z^{\nabla}_{(.,.)}$ *is the first prolongation of the conformal frame bundle of* $M$, *and* $Z^{\nabla}_{(.,.)} : CO(M) \times \mathbb{R}^{n*} \to CO(M)_1$ *is a trivialization of the fibre bundle* $CO(M)_1$ *over* $CO(M)$.

$$\begin{array}{ccccc} CO(M) \times \mathbb{R}^{n*} & \xrightarrow{Z^{\nabla}_{(.,.)}} & CO(M)_1 & \\ p_1 \downarrow & & \downarrow & \Pi \\ CO(M) & \xrightarrow[id]{} & CO(M) & \end{array}$$

As announced at the beginning of this section, we are now in a position to get the normal Cartan connection and the Weyl curvature tensor of the conformal structure $(M,\mathcal{C})$. The specific description about the way of getting them from $CO(M)_1$ falls out of the concern of this paper and it can be found in several works, however we will sketch the basic steps to follow and will supply the reader with a list of references.

Let $(\omega^i;\omega^i_j)$ be the restriction to the subbundle $CO(M)_1$ of $L(CO(M))$ of the canonical fundamental form $\theta = (\theta^i;\theta^i_j) \in \Lambda^1(L(CO(M)),\mathbb{R}^n \times \mathfrak{co}(n))$, then it determines completely a $\omega_j \in \Lambda^1(CO(M)_1,\mathbb{R}^{n*})$ such that the 1-form given by $\omega = (\omega^i;\omega^i_j;\omega_j) \in \Lambda^1(CO(M)_1,\mathbb{R}^n \times \mathfrak{co}(n) \times \mathbb{R}^{n*})$ is a Cartan connection over $CO(M)_1$ and verifies another condition, the condition of normalization; $\omega = (\omega^i;\omega^i_j;\omega_j)$ is the well known normal conformal connection of $(M,\mathcal{C})$ ([7], [9]).

The Cartan connection $\omega$ has the property that, given its curvature form $\Omega = (0;\Omega^i_j;\Omega_j)$, then $\Omega^i_j \in \Lambda^2(CO(M)_1,\mathfrak{co}(n))$ restricted to $CO(M)$ gives rise to the Weyl conformal curvature tensor of $(M,\mathcal{C})$ ([7], [2], [10]).

The interpretation of these forms by means of the Fermi-Walker connection shall be the concern of our next research work.

## References

[1] B. O'Neill, *Semi-Riemannian Geometry with applicatins to relativity.* Academic Press Inc, 1983.
[2] Walter A. Poor, *Differential Geometric structures.* McGraw-Hill Book Company, 1981.
[3] S.-T. Yau R. Schoen, *Lectures on Differential Geometry.* International Press, 1994.
[4] H.Wu R.K.Sachs, *General Relativity for Mathematicians.* Springer-Verlag, 1977.
[5] J Lafontaine S. Gallot, D. Hulin, *Riemannian Geometry.* Springer-Verlag, 1993.
[6] J. A. Schouten, *Ricci-Calculus.* Springer-Verlag, 1954.
[7] S.Kobayashi, *Transformation Groups in Differential Geometry.* Springer-Verlag, 1972.
[8] K.Nomizu S.Kobayashi, *Foundations of Differential Geometry.* Interscience Publisher, 1963.
[9] S. Sternberg, *Lectures on Differential Geometry.* Prentice Hall Inc., 1964.
[10] S. Haperlin W. Greub, *Connections, Curvature and Cohomology.* Academic Press, 1973.
[11] Kentaro Yano, *Integral formulas in Riemannian Geometry.* Marcel Dekker Inc., 1970.

Dep. de Geometria y Topologia
Facultad de Mathematicas
Universidad Complutence de Madrid
Avda. Complutence s/n
28.040 Madrid, SPAIN
*E-mail:* beatriz_salvador@mat.ucm.es