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COHERENCE CONSTRAINTS FOR OPERADS, CATEGORIES AND ALGEBRAS

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ABSTRACT. Coherence phenomena appear in two different situations. In the context of category theory the term 'coherence constraints' refers to a set of diagrams whose commutativity implies the commutativity of a larger class of diagrams. In the context of algebra coherence constrains are a minimal set of generators for the second syzygy, that is, a set of equations which generate the full set of identities among the defining relations of an algebraic theory.

A typical example of the first type is Mac Lane's coherence theorem for monoidal categories [9, Theorem 3.1], an example of the second type is the result of [2] saying that pentagon identity for the 'associator' Φ of a quasi-Hopf algebra implies the validity of a set of identities with higher instances of Φ.

We show that both types of coherence are governed by a homological invariant of the operad for the underlying algebraic structure. We call this invariant the (space of) coherence constraints. In many cases these constraints can be explicitly described, thus giving rise to various coherence results, both classical and new.

1. INTRODUCTION

We remind the reader of some definitions and results of [9]. A *category with a multiplication* is a category $\mathcal{C}$ together with a covariant bifunctor $\square : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. An *associativity isomorphism* for $(\mathcal{C}, \square)$ is then a natural transformation

$$(1) \qquad a : \square(\mathbb{1} \times \square) \longrightarrow \square(\square \times \mathbb{1})$$

($\mathbb{1}$ denotes the identity functor) such that each $a(A, B, C) : A\square(B\square C) \to (A\square B)\square C$ has a two-sided inverse in $\mathcal{C}$, for $A, B, C \in \mathcal{C}$; here we denote, as usual, $\square(\mathbb{1} \times \square)(A, B, C)$ by $A\square(B\square C)$, etc. Having such an associativity isomorphism, we can consider diagrams whose vertices are iterates of $\square$ and edges expansions of instances of a. The category $\mathcal{C}$ is called *coherent* if all these diagrams commute. The easiest of these diagrams is the pentagon (see Figure 1). There is no a priori reason for the commutativity of any of these diagrams, but the celebrated Mac Lane's coherence

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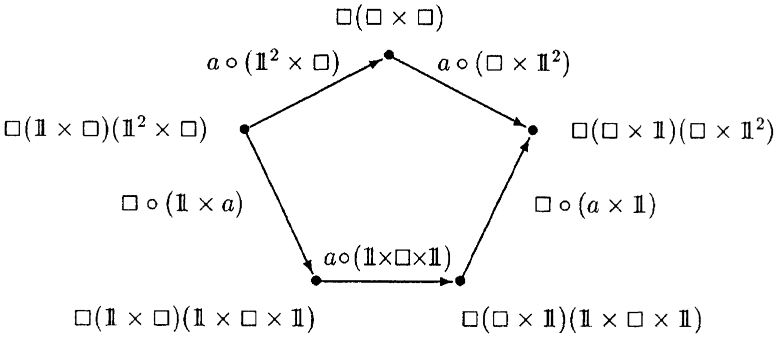

FIGURE 1. The Pentagon

theorem [9, Theorem 3.1] says that the commutativity of one diagram, the pentagon, implies the commutativity of all these diagrams.

Consider the case of a **k**-vector space U which is a module over a unital associative, not necessary coassociative bialgebra $V = (V, \cdot, \Delta, 1)$. We adopt the Drinfel'd convention and consider the associativity a^{-1} represented by the action of an invertible element $\Phi = \sum_i \Phi_{1,i} \otimes \Phi_{2,i} \otimes \Phi_{3,i} \in V^{\otimes 3}$. Let $* : U \otimes U \to U$ be a bilinear product which is 'Φ-associative,'

$$\Phi(a * (b * c)) = ((a * b) * c), \text{ for } a, b, c \in U, \tag{2}$$

where $\Phi(a * (b * c))$ is an abbreviation for $\sum(\Phi_1 a * (\Phi_2 b * \Phi_3 c))$. Assuming $(U, *)$ is a V-module algebra, [10, Chapter 10], that is

$$v \cdot (a * b) = v_{(1)} \cdot a * v_{(2)} \cdot b, \text{ for } v \in V,\ a, b, c \in U,$$

(we will follow usual conventions and delete the summation sign and summation indices use the Sweedler abbreviated notation with $v_{(1)} \otimes v_{(2)}$ standing for $\Delta(v) = \sum_i v_{(1)i} \otimes v_{(2)i}$). One derives easily from (2) that

$$(((a * b) * c) * d) = P(\Phi)(((a * b) * c) * d)\,, \tag{3}$$

where $P(\Phi) := (\Phi \otimes 1)^{-1}(1\!\!1 \otimes \Delta \otimes 1\!\!1)(\Phi)^{-1}(1 \otimes \Phi)^{-1}(1\!\!1 \otimes \Delta)(\Phi)(\Delta \otimes 1\!\!1)(\Phi)$.

If we do not assume that $P(\Phi) = 1$, then (3) is a new relation imposed on the space of bracketed 4-fold products other than the association relation to other bracketings. The condition $P(\Phi) = 1$ is, of course, the same as

$$(1 \otimes \Phi)(1\!\!1 \otimes \Delta \otimes 1\!\!1)(\Phi)(\Phi \otimes 1) = (1\!\!1^2 \otimes \Delta)(\Phi)(\Delta \otimes 1\!\!1^2)(\Phi)\,, \tag{4}$$

the famous pentagon condition on Φ introduced by Drinfel'd [2], although not from this point of view. It is called so because its five factors correspond to the five sides of the pentagon for the natural monoidal structure on the category of modules over the algebra $(V, \cdot)$.

We may consider $*$-products of order 5 and higher and look for similar equations in Φ. Because of Mac Lane's theorem, all these equations follow from the pentagon condition (4).

We presented two situations where the coherence appears – one in category theory, where it was formulated in terms of commutative diagrams, and another in algebra, where it was expressed in the language of algebraic equations. Both examples above were related to a certain associativity – of the transformation $\square$ in (1) and of the product $*$ in (2). Also the description of the 'coherence constraint' pointed in both cases to the shape of the pentagon. In this case the algebraic result was derived from the category theory. We also want to show how to derive categorial results from algebra.

For an operad $\mathcal{P}$, we introduce, in Definition 1, the space $\mathsf{C}_{\mathcal{P}}$ of coherence constraints of $\mathcal{P}$. It is a certain homological invariant of $\mathcal{P}$ that can be informally described as a second syzygy, spanned by 'the relations among the defining relations' where the 'defining relations' generate the ideal defining an operad as a quotient of a free operad. These coherence constraints can be read off from the bigraded model of $\mathcal{P}$ (Corollary 5) and can be easily described for so called Koszul operads (Theorem 7).

We show that both types of coherence boil down to coherence constraints of the governing operad. For the categorial coherence, it is Theorem 15 which says, roughly speaking, that the commutativity of diagrams corresponding to $\mathsf{C}_{\mathcal{P}}$ implies the commutativity of all diagrams.

The algebraic situation is more subtle. The coherence of a 'quantization' is given in terms of linear equations and it intuitively means that any solution of a linear equation in the original system deforms to a solution of the quantized one.

This is formalized by introducing the 'V-relative' version $\mathcal{P}_V$ of the operad $\mathcal{P}$, the coherence then means that $\mathcal{P}_V$ is a 'flat extension' of $\mathcal{P}$ (Definition 6), that is, the defining relations of the quantized structure have the same rank as the original structure. The characterization of this kind of coherence in terms of coherence constraints of $\mathcal{P}$ is given in Theorem 21.

We illustrate our methods by giving a new proof of Mac Lane's coherence ($\mathcal{P} = \mathcal{A}ss$, the operad for associative algebras), which is just two lines once we know that $\mathcal{A}ss$ is Koszul (see Example 16).

Example 17 is related to the linear logic and Example 18 presents a categorial version of Loday's bigebras.

On the quantum side we discuss Drinfel'd's quasi-Hopf algebras (Example 24, $\mathcal{P} = \mathcal{A}ss$), generalized Lie algebras analogous to the construction given by Gurevich (Example 25, $\mathcal{P} = \mathcal{L}ie$) and a certain form of strictly homotopy associative algebras (Example 27).

As an useful tool, we introduce in Section 4 a series of bipartite graphs, encoding the coherence relations. In many cases these relations can be described by a simple graph which we call the Tel-A-graph (Tel-A= Tel Aviv, the place where the discovery was made). While, for $\mathcal{P} = \mathcal{A}ss$ this Tel-A-graph is the pentagon, for $\mathcal{P} = \mathcal{L}ie$ it is the Peterson graph (Figure 3), and for the operad governing algebras of Example 27 a kind of the Möbius strip!

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2. Operads and coherence constraints

The notion of an *operad* and of an *algebra* over an operad is classical and well known (see [15], or more recent sources [4, 3, 13]). We thus recall only briefly the definitions and notation. We will need also some results on homology and presentations of operads; this part of the paper relies on [13].

Operads make sense in any strict symmetric monoidal category $\mathcal{M} = (\mathcal{M}, \square)$. The most important example for the purposes of this paper is the category $\mathtt{gr\text{-}Vect}_{\mathbf{k}}$ of graded $\mathbf{k}$-vector spaces, where $\mathbf{k}$ is a field of characteristic zero, with monoidal structure given by the standard graded tensor product over $\mathbf{k}$. In Sections 4 and 5 we consider also the category $\mathtt{Sets}$ of sets, with monoidal structure given by the cartesian product, and the category $\mathtt{Ab\text{-}Grp}$ of abelian groups with tensor product over $\mathbb{Z}$.

More precisely, an operad is a sequence $\mathcal{P} = \{\mathcal{P}(n); n \geq 1\}$ of objects of $\mathcal{M}$ such that:

(i) Each $\mathcal{P}(n)$ is equipped with a (right) action of the symmetric group Σ_n on n elements, $n \geq 1$.

(ii) For any $m_1, \dots, m_l \geq 1$ we have the *composition maps*

$$\gamma = \gamma_{m_1,\dots,m_l} : \mathcal{P}(l)\square\mathcal{P}(m_1)\square\cdots\square\mathcal{P}(m_l) \longrightarrow \mathcal{P}(m_1 + \cdots + m_l).$$

These data have to satisfy the usual axioms including the existence of a unit $1 \in \mathcal{P}(1)$, for which we refer to [15]. We sometimes write $\mu(\nu_1, \cdots, \nu_l)$, $\mu(\nu_1\square\cdots\square\nu_l)$ or $\gamma(\mu; \nu_1, \cdots, \nu_l)$ instead of $\gamma(\mu\square\nu_1\square\cdots\square\nu_l)$ (notice that in all three above examples $(\mathtt{gr\text{-}Vect}_{\mathbf{k}}, \otimes)$, $(\mathtt{Ab\text{-}Grp}, \otimes)$ and $(\mathtt{Sets}, \times)$ of the monoidal category $(\mathcal{M}, \square)$ the 'product of elements' $\mu\square\nu_1\square\cdots\square\nu_l$ makes sense).

A *collection* is a sequence $E = \{E(n);\ n \geq 2\}$ of elements of $\mathcal{M}$ such that each $E(n)$ is equipped with an action of the symmetric group Σ_n. The obvious forgetful functor $F_{\mathrm{or}} : \mathtt{Oper} \to \mathtt{Coll}$ from the category of operads to the category of collections has a left adjoint $\mathcal{F} : \mathtt{Coll} \to \mathtt{Oper}$ and we call $\mathcal{F}(E)$ the *free operad* on the collection E.

Remark 1. The notions above, as well as all the results which follow, have obvious *non-*Σ (also called *nonsymmetric*) analogs which we obtain by forgetting everything related to the symmetric group action. We thus have non-Σ operads, non-Σ collections, etc. The reason for considering these objects is that the non-Σ versions are much simpler and there are many examples which live in a non-Σ world, the most prominent being the associative algebra case. We will move freely between Σ and non-Σ worlds clarifying when necessary the context in which we are working.

In the rest of this section, an operad is an operad in the category $\mathtt{gr\text{-}Vect}_{\mathbf{k}}$ of graded $\mathbf{k}$-vector spaces.

A *module* over an operad $\mathcal{P}$ is an abelian group object in the slice category $\mathtt{Oper}/\mathcal{P}$. The axioms were explicitly given for modules over a so-called *pseudo-operad* in [13], in the standard case the axioms are quite analogous. Namely, a module over $\mathcal{P}$ is a collection $M = \{M(n);\ n \geq 1\}$ together with a map

$$m : \bigoplus_{1\leq i\leq l} \{\mathcal{P}(l)\otimes\mathcal{P}(m_1)\otimes\cdots\otimes M(m_i)\otimes\mathcal{P}(m_l)\}\oplus\{M(l)\otimes\mathcal{P}(m_1)\otimes\cdots\otimes\mathcal{P}(m_l)\} \longrightarrow$$
$$\longrightarrow M(m_1 + \cdots + m_l)$$

given for any $m_1, \dots, m_l \geq 1$. This map is supposed to satisfy obvious axioms given by the linearization of the axioms of operads. Just as for the operadic composition map, we sometimes write $a(b_1, \dots, b_l)$, $a(b_1 \otimes \cdots \otimes b_l)$ instead of $m(a \otimes b_1 \otimes \cdots \otimes b_l)$.

We will give some examples of $\mathcal{P}$-modules which we will need in the sequel. The operad $\mathcal{P}$ itself is a $\mathcal{P}$-module. If $\alpha : \mathcal{P} \to \mathcal{S}$ is an operad map, then α induces a

$\mathcal{P}$-module structure on $\mathcal{S}$. Finally, if $\mathcal{I} \subset \mathcal{P}$ is an ideal in $\mathcal{P}$ (see [13]), then $\mathcal{I}$ is naturally a $\mathcal{P}$-module.

The forgetful functor $F_{\mathrm{or}} : \mathcal{P}\text{-}\mathtt{Mod} \to \mathtt{Coll}$ from the category of $\mathcal{P}$-modules to the category of collections has a left adjoint $\mathcal{P}\langle - \rangle : \mathtt{Coll} \to \mathcal{P}\text{-}\mathtt{Mod}$, and we call the $\mathcal{P}$-module $\mathcal{P}\langle E\rangle$ the *free $\mathcal{P}$-module* on the collection E.

For the time being, we suppose that our operads always have $\mathcal{P}(1) = \mathbf{k}$. Let $\mathcal{P}^+ \subset \mathcal{P}$ be the ideal defined by $\mathcal{P}^+(1) := 0$ and $\mathcal{P}^+(n) := \mathcal{P}(n)$ for $n \geq 2$.

For a $\mathcal{P}$-module (M, m) we define the *decomposables* of M to be the collection $D(M) = D_{\mathcal{P}}(M)$ generated by elements either of the form $m(r; p_1, \dots, p_l)$, where $r \in M$, $p_1, \dots, p_l \in P$ and at least one of $p_1, \dots, p_l \in \mathcal{P}$ belongs to $\mathcal{P}^+$, or of the form $m(p; p_1, \dots, r, \dots, p_l)$, where again $r \in M$, $p, p_1, \dots, p_l \in \mathcal{P}$ and at least one of $p, p_1, \dots, p_l \in \mathcal{P}$ belongs to $\mathcal{P}^+$. Define the *indecomposables* of M as the collection $Q(M) = Q_{\mathcal{P}}(M) := M/D_{\mathcal{P}}(M)$.

Each operad can be represented as $\mathcal{P} = \mathcal{F}(E)/(R)$, where E and R are collections and (R) is the operadic ideal generated by R; we write $\mathcal{P} = \langle E; R\rangle$. Because $\mathcal{P}(1) = \mathbf{k}$, we can always suppose that the presentation is minimal [13]. This means, by definition, that $E \cong Q_{\mathcal{P}}(\mathcal{P}^+)$ and that the collection R is isomorphic to the indecomposables of the kernel of the canonical map $\mathcal{F}(E) \to \mathcal{P}$, $R \cong Q_{\mathcal{F}(E)}\{\mathrm{Ker}(\mathcal{F}(E) \to \mathcal{P})\}$.

Following Quillen's paradigm, we consider the higher derived functors of the functor of indecomposables, see [16].

Let $\mathcal{P} = \langle E; R\rangle$. Let $J := \mathcal{F}(E)\langle R\rangle$ be the free $\mathcal{F}(E)$-module on R and let $\pi : J \to (R)$ be the obvious natural epimorphism of $\mathcal{F}(E)$-modules. For $x \in \mathcal{F}(E)\langle R\rangle(l)$, $y \in \mathcal{F}(E)\langle R\rangle$ and $a_1, \dots, a_l \in \mathcal{F}(E)$ the element

$$o' := x(a_1, \dots, a_{s-1}, \pi(y), a_{s+1}, \dots, a_l) - \pi(x)(a_1, \dots, a_{s-1}, y, a_{s+1}, \dots, a_l) \in J$$

belongs to $\mathrm{Ker}(\pi)$, for any $1 \leq s \leq l$. Similarly, for $b \in \mathcal{F}(E)(l)$, $a_1, \dots, a_l \in \mathcal{F}(E)$ and $x, y \in \mathcal{F}(E)\langle R\rangle$, the element

$$\begin{aligned} o'' := b(a_1, \dots, a_{s-1}, \pi(x), a_{s+1}, \dots, a_{t-1}, y, a_{t+1}, \dots, a_l) - \\ - b(a_1, \dots, a_{s-1}, x, a_{s+1}, \dots, a_{t-1}, \pi(y), a_{t+1}, \dots, a_l) \in J \end{aligned}$$

belongs to $\mathrm{Ker}(\pi)$, for any $1 \leq s < t \leq l$. In the spirit of the definition of the cotangent cohomology we call the $\mathcal{F}(E)$-module generated by elements of the above two types the *module of obvious relations* and denote it by $\mathcal{O} = \mathcal{O}_{\mathcal{P}}$. To understand better the meaning of this module we recommend looking at the definition of the module U_0 on page 44 of [7] in the classical commutative algebra situation or to the definition in 2.2 of [11].

Definition 1. The collection of *coherence relations* of the operad $\mathcal{P}$ is the $\mathcal{F}(E)$-module $\mathcal{D} = \mathcal{D}_{\mathcal{P}} := \mathrm{Ker}(\pi : J \to (R))/\mathcal{O}$. The collection of *coherence constraints* of the operad $\mathcal{P}$ is the collection of indecomposables of the $\mathcal{F}(E)$-module $\mathcal{D}$, $\mathsf{C} = \mathsf{C}_{\mathcal{P}} := Q_{\mathcal{F}(E)}(\mathcal{D})$.

Example 2. Let ξ be an independent symbol and let E be the non-Σ collection defined by

$$E(n) := \begin{cases} \mathrm{Span}(\xi), & \text{for } n = 2, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

Let $r := \xi(1,\xi) - \xi(\xi,1) \in \mathcal{F}(E)(3)$ and let R be the (non-Σ) collection generated by r, i.e.

$$R(n) := \begin{cases} \mathrm{Span}(r), & \text{for } n = 3, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

Associative algebras are then the algebras over the non-Σ operad $Ass := \langle E; R\rangle$.

There are five rooted planar binary trees having four leaves so $\dim_{\mathbf{k}}(\mathcal{F}(E)(4)) = 5$. Moreover, there are five rooted planar trees with one bivalent and one trivalent vertex and four leaves so $\dim_{\mathbf{k}}(\mathcal{F}(E)\langle R\rangle)(4) = 5$. Choose as basis elements for $\mathcal{F}(E)(4)$ the trees represented in the standard way by the following bracketings

$$\mathbf{a} := ((12)3)4,\ \mathbf{b} := (12)(34),\ \mathbf{c} := 1(2(34)),$$
$$\mathbf{d} := 1((23)4),\ \mathbf{e} := (1(23))4 \in \mathcal{F}(E)(4).$$

Choose also the basis

$$\mathbf{1} := r(\xi,1,1),\ \mathbf{2} := r(1,1,\xi),\ \mathbf{3} := \xi(1,r),\ \mathbf{4} := r(1,\xi,1),\ \mathbf{5} := \xi(r,1) \in \mathcal{F}(E)\langle R\rangle,$$

for $\mathcal{F}(E)\langle R\rangle(4)$. The map $\pi : \mathcal{F}(E)\langle R\rangle(4) \to (R)(4)$ has the following matrix description:

	a	b	c	d	e
$\pi(\mathbf{1})$	-1	1	0	0	0
$\pi(\mathbf{2})$	0	-1	1	0	0
$\pi(\mathbf{3})$	0	0	1	-1	0
$\pi(\mathbf{4})$	0	0	0	1	-1
$\pi(\mathbf{5})$	-1	0	0	0	$+1$

One sees immediately that $\dim(\mathrm{Ker}(\pi))(4) = 1$, and that the kernel is spanned by the element $p \in J(4) = \mathcal{F}(E)\langle R\rangle(4)$ defined by

$$p := \xi(r,1) - r(\xi,1,1) + r(1,\xi,1) - r(1,1,\xi) + \xi(1,r). \tag{5}$$

The five terms in p correspond to the edges of the pentagon considered as the Stasheff associahedron K_4 with five vertices labeled by the five binary trees with four leaves or, equivalently, by the five possible bracketings of four elements. We may prove, by a step-by-step repeating the arguments of the proof of [9, Theorem 3.1], that p generates the collection of coherence constraints C, as was in fact done in the last section of [11], but we derive this statement using a more sophisticated approach, which we now describe.

3. Coherence and the homology of operads

In this section we work with operads in the monoidal category $\mathtt{gr\text{-}Vect}_{\mathbf{k}}$ of graded vector spaces. Let us recall some notions and results of [13, Section 3].

Let $\mathcal{S}$ be an operad. By a *differential* on $\mathcal{S}$ we mean a degree -1 map $d : \mathcal{S} \to \mathcal{S}$ of collections having the expected Leibniz property with respect to the operadic composition and satisfying $d^2 = 0$. A differential on the free operad $\mathcal{F}(E)$ is uniquely determined by its restriction to the space of generators E.

Suppose we have a collection Z which decomposes as $Z = Z^0 \oplus Z^1 \oplus \cdots$ (meaning, of course, that for each $n \geq 2$ we have a Σ_n-invariant decomposition $Z(n) = Z^0(n) \oplus Z^1(n) \oplus \cdots$ of the graded vector space $Z(n) = \bigoplus Z_j(n)$ into the direct sum of graded vector spaces $Z^k(n) = \bigoplus Z^k_j(n)$, $k \geq 0$). This induces on $\mathcal{F}(Z)$ a grading, $\mathcal{F}(Z) = \bigoplus_{k \geq 0} \mathcal{F}(Z)^k$. We call this grading the *TJ-grading* (from Tate-Jozefiak) here; the reason will became obvious below.

In the situation above, the free operad $\mathcal{F}(Z)$ is bigraded, $\mathcal{F}(Z) = \bigoplus \mathcal{F}(Z)^k_j$, where k refers to the TJ-grading introduced above and j indicates the 'inner' grading given by the grading of $Z = \bigoplus Z_j$.

Suppose that d is an (inner) degree -1 differential on $\mathcal{F}(Z)$ such that

$$d(Z^k) \subset \mathcal{F}(Z)^{k-1}, \text{ for all } k \geq 1 \tag{6}$$

(meaning, of course, that $d(Z^k)(n) \subset \mathcal{F}(Z)^{k-1}(n)$ for all $n \geq 2$), i.e. that d is homogeneous degree -1 with respect to the TJ-grading. Then the homology operad $\mathcal{H}(\mathcal{F}(Z), d)$ is bigraded,

$$\mathcal{H}(\mathcal{F}(Z), d) = \bigoplus \mathcal{H}^k_j(\mathcal{F}(Z), d),$$

the upper grading being induced by the TJ-grading and the lower one by the inner grading. In [13] the first author proved the following theorem.

Theorem 3. *Let $\mathcal{P}$ be an operad (Σ or non-Σ, with trivial differential). Then there exists a collection $Z = Z^0 \oplus Z^1 \oplus \cdots$, a differential d on $\mathcal{F}(Z)$ satisfying (6) and a map $\rho : (\mathcal{F}(Z), d) \to (\mathcal{P}, 0)$ of differential operads such that the following conditions are satisfied:*

(i) *d is minimal in the sense that $d(Z)$ consists of decomposable elements of the operad $\mathcal{F}(Z)$,*
(ii) *$\rho|_{Z^{\geq 1}} = 0$ and ρ induces an isomorphism $\mathcal{H}^0(\rho) : \mathcal{H}^0(\mathcal{F}(Z), d) \cong \mathcal{P}$, and*
(iii) *$\mathcal{H}^{\geq 1}(\mathcal{F}(Z), d) = 0$.*

We call the object $\rho : (\mathcal{F}(Z), d) \to (\mathcal{P}, 0)$ the *bigraded model* of the operad $\mathcal{P}$. This object is an analog of the bigraded model of a commutative graded algebra constructed in [6, Section 3].

Recall that the *suspension* of a graded vector space $V = \bigoplus V_i$ is the graded vector space $\uparrow V$ defined by $(\uparrow V)_i = V_{i-1}$. The following proposition is obvious from the construction of the bigraded model described in [13].

Proposition 4. *Let $\mathcal{P}$ be an operad and let $\mathcal{P} = \langle E; R \rangle$ be a minimal presentation. Let $(\mathcal{F}(Z), d)$ be a bigraded model of $\mathcal{P}$. Then there are the following isomorphisms of collections:*

$$Z^0 \cong E, \ Z^1 \cong \uparrow R, \ Z^2 \cong \uparrow^2 \mathsf{C} \tag{7}$$

The most important for our purpose is the third equation of (7). We formulate the following corollary.

Corollary 5. *The collection $\mathsf{C}_{\mathcal{P}}$ of coherence constraints of an operad $\mathcal{P}$ is isomorphic to the double desuspension $\downarrow^2 Z^2$ of the collection of TJ-degree two indecomposables of the bigraded model of $\mathcal{P}$.*

We say that an operad $\mathcal{P}$ is *quadratic* if it has a presentation $\mathcal{P} = \langle E; R \rangle$ such that $E(n) = 0$ for $n \neq 2$ and $R(n) = 0$ for $n \neq 3$. Each quadratic operad has its *quadratic dual* $\mathcal{P}^!$ [4], which is another operad constructed very explicitly from the presentation of $\mathcal{P}$. V. Ginzburg and M.M. Kapranov introduced in [4] an extremely important notion of the Koszulness of an operad. It is a certain homological property analogous to the Koszulness of commutative algebras; operads sharing this property are called Koszul operads. The following proposition appeared in [13] as Proposition 2.6.

Proposition 6. *Let $\mathcal{P}$ be a quadratic operad and suppose that $\mathcal{P}$ is Koszul. Denote by $\mathcal{P}^!$ its quadratic dual. Then there is, for each $n \geq 1$, the following isomorphism of Σ_n-modules:*

$$Z^i(n) \cong \begin{cases} 0, & n \neq i+2 \\ \mathrm{sgn}\otimes \uparrow^{(n-2)}\mathcal{P}^!(n), & n = i+2, \end{cases}$$

where sgn *denotes the one-dimensional signum representation. For non-Σ operads, $Z^i(n)$ is given by the same formula but without the signum representation.*

Combining Proposition 6 with Corollary 5 we get the following proposition.

Theorem 7. *For a Koszul quadratic operad $\mathcal{P}$, there a Σ_4-equivariant isomorphism*

$$\mathsf{C}_{\mathcal{P}} = \mathsf{C}_{\mathcal{P}}(4) \cong \mathrm{sgn} \otimes \mathcal{P}^!(4).$$

In particular, the only coherence constraints are in degree 4.

Let us formulate a proposition counting the dimension of the space of coherence constraints. We need the generating function $g_{\mathcal{P}}(x)$ of an operad $\mathcal{P}$, which is the formal power series

$$g_{\mathcal{P}}(x) := \sum_{n\geq 1} \frac{\dim(\mathcal{P}(n))}{n!} x^n. \tag{8}$$

As it follows from [4], if $\mathcal{P}$ is Koszul, then

$$g_{\mathcal{P}}(-g_{\mathcal{P}^!}(-x)) = x.$$

The same formula holds also for a non-Σ $\mathcal{P}$ if we drop the $n!$ (= the order of Σ_n) from (8). The formula above enables one to express, for a Koszul operad $\mathcal{P}$, the dimension $\dim(\mathcal{P}^!(4))$ via $\dim(\mathcal{P}(2))$, $\dim(\mathcal{P}(3))$ and $\dim(\mathcal{P}(4))$:

Proposition 8. *Suppose that $\mathcal{P}$ is a non-Σ Koszul operad. Then*

$$\dim(\mathsf{C}_{\mathcal{P}}) = \dim(\mathcal{P}(4)) + 5\dim(\mathcal{P}(2))[\dim(\mathcal{P}(2))^2 - \dim(\mathcal{P}(3))].$$

If $\mathcal{P}$ is symmetric, then

$$\dim(\mathsf{C}_{\mathcal{P}}) = \dim(\mathcal{P}(4)) + 5\dim(\mathcal{P}(2))[3\dim(\mathcal{P}(2))^2 - 2\dim(\mathcal{P}(3))].$$

Example 9. Let ζ be an independent variable and E the symmetric collection defined by

$$E(n) := \begin{cases} \mathrm{Span}(\zeta), & \text{for } n = 2, \text{ and} \\ 0, & \text{otherwise,} \end{cases}$$

with the sign representation of Σ_2 on $E(2)$. Let B_n be the free nonassociative anticommutative algebra on the set $\{1, \ldots, n\}$ and let B'_n denote the subset of B_n spanned by monomials in which each element of $\{1, \ldots, n\}$ appears exactly once. Then $\mathcal{F}(E)(n) \cong B'_n$ for any $n \geq 1$, where $\mathcal{F}(E)$ is now the free *symmetric* operad on the collection E.

Let $\iota := [1[23]] + [2[31]] + [3[12]] \in \mathcal{F}(E)(3)$ and R be the symmetric collection generated by ι. Then *Lie* $:= \langle E; R\rangle$ is the operad governing Lie algebras. This operad is quadratic Koszul [4, page 229] and $Lie^! = Comm$, therefore, by Theorem 7, $\mathsf{C} = Comm(4)$, which is the one dimensional trivial representation of Σ_4. Thus there is only one coherence constraint, as in the associative algebra case. Let us describe this constraint explicitly.

A description of the map $\pi : \mathcal{F}(E)\langle R\rangle(4) \to (R)(4)$ is given by the matrix of Figure 2.

	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o
$\pi(1)$	+1	−1	0	0	0	0	0	0	0	0	0	+1	0	0	0
$\pi(2)$	0	+1	−1	0	0	0	+1	0	0	0	0	0	0	0	0
$\pi(3)$	0	0	−1	+1	0	+1	0	0	0	0	0	0	0	0	0
$\pi(4)$	0	0	0	+1	−1	0	0	0	0	0	0	0	0	0	+1
$\pi(5)$	−1	0	0	0	+1	0	0	0	0	0	0	0	+1	0	0
$\pi(6)$	0	0	0	0	0	0	0	0	−1	0	0	0	+1	−1	0
$\pi(7)$	0	0	0	0	0	−1	0	0	0	+1	0	0	0	+1	0
$\pi(8)$	0	0	0	0	0	0	−1	+1	−1	0	0	0	0	0	0
$\pi(9)$	0	0	0	0	0	0	0	0	0	−1	+1	+1	0	0	0
$\pi(10)$	0	0	0	0	0	0	0	−1	0	0	+1	0	0	0	−1

The basis elements are chosen as:

a $:= [[[12]3]4]$ **b** $:= [[12][34]]$ **c** $:= [1[2[34]]]$ **d** $:= [1[[23]4]]$ **e** $:= [[1[23]]4]$
f $:= [1[3[24]]]$ **g** $:= [2[1[34]]]$ **h** $:= [2[3[14]]]$ **i** $:= [2[4[13]]]$ **j** $:= [3[1[24]]]$
k $:= [3[2[14]]]$ **l** $:= [3[[12]4]]$ **m** $:= [4[2[13]]]$ **n** $:= [[13][24]]$ **o** $:= [[14][23]]$

and:

$\mathbf{1} := -\iota(\zeta,1,1)$ $\mathbf{2} := -\iota(1,1,\zeta)$ $\mathbf{3} := -\zeta(1,\iota)$
$\mathbf{4} := -\iota(1,1,\zeta)\cdot T_{1342}$ $\mathbf{5} := -\zeta(1,\iota)\cdot T_{2341}$ $\mathbf{6} := -\iota(\zeta,1,1)\cdot T_{1324}$
$\mathbf{7} := -\iota(1,1,\zeta)\cdot T_{1324}$ $\mathbf{8} := -\zeta(1,\iota)\cdot T_{2134}$ $\mathbf{9} := -\zeta(1,\iota)\cdot T_{2314}$
$\mathbf{10} := -\iota(\zeta,1,1)\cdot T_{1342}$

FIGURE 2. The matrix of the map $\pi : \mathcal{F}(E)\langle R\rangle(4) \to (R)(4)$. The symbol $T_{i_1i_2i_3i_4}$ denotes the permutation which sends (1234) to $(i_1i_2i_3i_4)$. The upper left 5×5-submatrix coincides, up to sign, with the corresponding matrix for the associative algebra operad.

Let $\ell \in \mathcal{F}(E)\langle R\rangle(4)$ be the element defined as

$$\begin{aligned}\ell \;:=\;& \iota(\zeta,1,1)+\iota(1,1,\zeta)-\zeta(1,\iota)+\iota(1,1,\zeta)\cdot T_{1342}+\zeta(1,\iota)\cdot T_{2341}\\ &-\iota(\zeta,1,1)\cdot T_{1324}-\iota(1,1,\zeta)\cdot T_{1324}+\zeta(1,\iota)\cdot T_{2134}\\ &-\zeta(1,\iota)\cdot T_{2314}+\iota(\zeta,1,1)\cdot T_{1342}\,,\end{aligned}$$

or, in the notation of Figure 2,

$$\ell = -\mathbf{1}-\mathbf{2}+\mathbf{3}-\mathbf{4}-\mathbf{5}+\mathbf{6}+\mathbf{7}-\mathbf{8}+\mathbf{9}-\mathbf{10}\,.$$

It is easy to check that $\ell \in \mathrm{Ker}(\pi)$. By Theorem 7, ℓ is the only coherence constraint for Lie algebras.

4. A GRAPHIC DESCRIPTION OF COHERENCE RELATIONS

A nice feature of the associative operad is that the coherence relations can be represented by commutative diagrams, the closed edge-paths in the one skeleton of the Stasheff associahedra [17]. In this section we show how to describe the coherence relations for any quadratic operad by bipartite graphs, and in certain cases by ordinary graphs.

Associated to any presentation of a quadratic operad $\mathcal{P} = \langle E; R\rangle$ there is a series of bipartite graphs, $\mathcal{T} = \{\mathcal{T}(n)\}_{n\geq 4}$. Recall that J denoted the free $\mathcal{F}(E)$-module on R, $\pi : J \to (R)$ was the canonical epimorphism and $\mathcal{D} := \mathrm{Ker}(\pi)/\mathcal{O}$, see Section 2. Let $n \geq 4$. Fix a basis $(v_1, \dots, v_b)$ (resp. $(r_1, \dots, r_s)$) of $\mathcal{F}(E)(n)$ (resp. of $J(n) = \mathcal{F}(E)\langle R\rangle(n)$). Then define a bipartite graph with vertices partitioned into the two sets $\mathcal{V}(n) := \{v_1, \dots, v_b\}$ and $\mathcal{V}'(n) := \{r_1, \dots, r_s\}$. The vertices $v_i \in \mathcal{V}(n)$ and $r_j \in \mathcal{V}'(n)$ are joined by an edge if and only if v_i occurs with a non-zero coefficient in $\pi(r_j)$.

There is a simple graph encoding the same data whenever for one of the sets $\mathcal{V}(n)$ or $\mathcal{V}'(n)$ there are precisely two edges incident to each of the vertices in that set. For example, if $\mathcal{V}'(n)$ has this property, we create a new graph by concatenating the two edges incident to a vertex of $\mathcal{V}'(n)$, delete the vertex and create a single edge connecting two vertices in $\mathcal{V}(n)$ and give the new edge the same label as the deleted vertex. The new graph has edges labeled by the elements of $\mathcal{V}'(n)$ and vertices labeled by the elements of $\mathcal{V}(n)$. In this case each edge is labeled by a basis element in $\mathcal{F}(E)\langle R\rangle(n)$ relating the two elements of the basis of $\mathcal{F}(E)(n)$ which label the endpoints. In this case we say that the defining relations are *graphlike* and call the corresponding graph $\mathsf{G}(\mathcal{T}(n))$ the *Tel-A-graph* (the idea arose from discussions in Tel Aviv).

Note that the bipartite graph $\mathcal{T}(n)$ does not encode all the data from the presentation of $\mathcal{P} = \langle E; R\rangle$ of an operad since it only shows which terms v_i appear in $\pi(r_j)$ with nonzero coefficient, not the actual values of the coefficients. On the other hand if we can choose the bases $\{v_i\}$ for all the $\mathcal{F}(E)(n)$ such that the coefficients are all ± 1 then all the data of the presentation can be encoded into the graph $\mathcal{T}(n)$ by orienting the edges. The edge connecting the vertex $v_i \in \mathcal{V}(n)$ and the relation $r_j \in \mathcal{V}'(n)$ will be oriented in the direction of v_i if the coefficient of v_i in the relation r_j is $+1$ and away from v_i if the coefficient in -1. If $\mathcal{T}(n)$ is graphlike and oriented, that does not guarantee that the Tel-A-graph has a consistent orientation. What is required is that there exist an orientation such that each vertex with two incident edges in the bipartite graph has one incoming edge and one outgoing edge. In this case whenever such a vertex is deleted, the new edge created has a natural orientation.

We will see that the existence of an oriented Tel-A-graph formalizes the property of an operad being defined by axioms expressed by commutative diagrams. For instance, the operad *Ass* has an oriented Tel-A-graph $\mathsf{G}(\mathcal{T}(n))$ given by the one skeleton of the Stasheff associahedron K_n, with orientation as shown, for $n = 4$, in Figure 1.

A dual situation arises when there are exactly two edges incident to any vertex in $\mathcal{V}(n)$, that is, the basis elements v_i appear in exactly two defining relations. In this case the same procedure of concatenating edges, deleting vertices, and giving the new edge the label of the deleted vertex creates a graph with edges labeled by $\mathcal{V}(n)$ and vertices labeled by $\mathcal{V}'(n)$. In this case we say that the defining relations are *dual graphlike* and call the associated graph $\mathsf{G}'(\mathcal{T}(n))$ the *dual Tel-A-graph.* The defining relations for the operad *Lie* and $n = 4$ are dual graphlike, as shown in the next example. The existence of an orientation for the dual Tel-A-graph means that it is possible to find a basis for $\mathcal{F}(E)(n)$ and for $J(n) = \mathcal{F}(E)\langle R\rangle(n)$ such that any basis element for $\mathcal{F}(E)(n)$ which appears in one of the relations $J(n)$ appears in two relations once with a coefficient $+1$ and once with a coefficient -1.

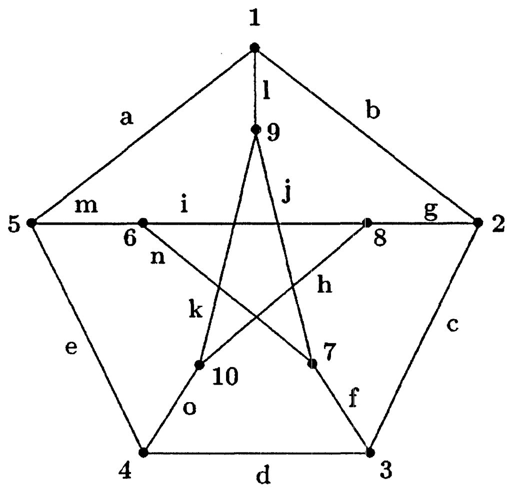


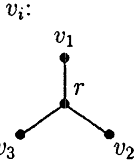
FIGURE 3. The Lie-hedron.

Example 10. For the Lie algebra operad *Lie* introduced in Example 9 there is a dual Tel-A-graph constructed from the bipartite graph $\mathcal{T}(4)$ with vertices $(\mathcal{V}(4), \mathcal{V}'(4))$ as given in Figure 2. The resulting graph is the famous Peterson graph, see Figure 3.

Let us describe $\mathcal{T}(3)$. Let

$$v_1 := [1[23]]\,,\ v_2 := [2[31]]\,,\ v_3 := [3[12]]\,,\ \text{and } r = \iota := v_1 + v_2 + v_2\,.$$

Then $\mathcal{V}(3) := \{v_1, v_2, v_3\}$ and $\mathcal{V}'(3) := \{r\}$. The bipartite graph $\mathcal{T}(3)$ has one trivalent vertex r and three univalent vertices v_i:



so it is neither graphlike, nor dual graphlike. In fact, $n = 4$ is the only value for which $\mathcal{T}(n)$ is (dual) graphlike.

There is a particular class of operads whose defining relations are always graph-like. Such operads $\mathcal{P} \in \mathbf{gr\text{-}Vect_k}$ have a presentation $\mathcal{P} = \langle E; R\rangle$ in which we have chosen a basis $B(n)$ for $E(n)$ and a basis $K(n)$ for the relations $R(n)$ such that $\pi(r_i) = p_i - q_i$ for each $r_i \in K(n)$. For this class we can establish the relation between operad theory and categorical coherence mentioned in the introduction. The following formulation arose from discussions between the first author and Tom Fox.

If $\mathcal{L}$ is an operad in the monoidal category Sets of sets, we can form the collection

$$\mathcal{P} = \mathrm{Span}_{\mathbf{k}}(\mathcal{L}) = \{\mathrm{Span}_{\mathbf{k}}(\mathcal{L})(n)\}_{n\geq 1},$$

where $\mathrm{Span}_{\mathbf{k}}(\mathcal{L})(n) := \mathrm{Span}_{\mathbf{k}}(\mathcal{L}(n))$ is the $\mathbf{k}$-vector space spanned by $\mathcal{L}(n)$. This collection has an obvious $\mathbf{k}$-Vect-operad structure induced by the Sets-operad structure of $\mathcal{L}$. If in addition each $\mathcal{L}(n)$ is graded, then the $\mathbf{k}$-vector space spanned by $\mathcal{L}(n)$ has a gr-$\mathtt{Vect}_{\mathbf{k}}$-operad structure. The preferred basis for each $\mathcal{P}(n)$ is given by the elements of $\mathcal{L}(n)$. In short, an operad has a presentation with defining relations of the type $\pi(r_i) = p_i - q_i$ if and only if it is a $\mathbf{k}$-linearization of an operad defined over the category of sets.

The operad $\mathcal{L}$ can presented as $\mathcal{L} = \mathcal{F}_{\mathbf{S}}/\sim$, for $\mathcal{F}_{\mathbf{S}}$ a free Sets-operad and $\sim$ an equivalence relation given by a list of couples $p_i \sim q_i$ for $p_i, q_i \in \mathcal{F}_{\mathbf{S}}(B)(n)$. Each identification corresponds to a commutative diagram for the maps describing the corresponding $\mathcal{P}$-algebra.

To make the exposition easier, in speaking about operads defined by commutative diagrams, we shall restrict to non-Σ operads. The formal definition is

Definition 2. Let $\mathcal{P}$ be a gr-$\mathtt{Vect}_{\mathbf{k}}$ operad. We say that $\mathcal{P}$ is *defined by commutative diagrams* if there exists a Sets-operad $\mathcal{L}$ and an isomorphism

$$\mathrm{Span}_{\mathbf{k}}(\mathcal{L}) \cong \mathcal{P} \tag{9}$$

of gr-$\mathtt{Vect}_{\mathbf{k}}$-operads.

Categorical coherence means the commutativity of a family of diagrams. Being defined by commutative diagrams is a crucial property of an operad which allows us to establish a relation between the theory of operads and the theory of categorical structures. The following proposition follows immediately from the previous discussion.

Proposition 11. *If $\mathcal{P}$ is an operad defined by commutative diagrams, then for any $n \geq 3$, the bipartite graphs $\mathcal{T}_{\mathcal{P}}(n)$ are graphlike, and the corresponding Tel-A-graphs are oriented.*

The next proposition relates the Tel-A-graph G and the space $\mathcal{D}$ of coherence relations.

Proposition 12. *Let $\mathcal{P} \cong \mathrm{Span}_{\mathbf{k}}(\mathcal{L})$ be an operad defined by commutative diagrams, where $\mathcal{L}$ is the corresponding Sets-operad presented by a collection of generators B and a collection of relations $K \in \mathcal{F}_{\mathbf{S}}(B)$. Let $\mathsf{G}(n) = \mathsf{G}(\mathcal{T}(n))$ for $n \geq 3$ be the corresponding oriented Tel-A-graphs. We interpret the graph $\mathsf{G}(n)$ as a 1-dimensional simplicial complex. To each closed oriented path $\gamma \in \mathsf{G}(n)$ we can associate the a coherence relation. This correspondence defines a natural isomorphism of $\mathbf{k}$-vector spaces:*

$$H_1(\mathsf{G}(n), \mathbf{k}) \cong \mathcal{D}(n). \tag{10}$$

Proof. Let $\mathcal{V}(n) = \{v_1, \ldots, v_b\} = \mathcal{F}_{\mathbf{S}}(B)(n)$ and $\mathcal{V}'(n) = \{r_1, \ldots, r_s\}$. For each $l \leq j \leq s$, we have $\pi(r_j) = v_{a_j} - v_{b_j}$ for some $1 \leq a_j, b_j \leq b$. We can orient the graph $\mathsf{G}(n)$ so that $\partial(r_j) = v_{a_j} - v_{b_j}$, where ∂ is the boundary operator for the one dimensional oriented simplicial complex $\mathsf{G}(n)$ and then $\sum a_i r_i \in \mathrm{Ker}(\partial) = \mathrm{Ker}(\pi)$ if and only if $\sum a_i r_i$ expresses a coherence relation. Moreover $\mathrm{Ker}(\partial) \cong H_1(\mathsf{G}(n), \mathbf{k})$ since there are no boundaries in dimension 1. □

The diagrams whose commutativity is required for coherence correspond to elements of the of the edge-path fundamental group of $\mathsf{G}(n)$, and therefore are more closely related to $H_1(\mathsf{G}(n), \mathbb{Z})$ than to $H_1(\mathsf{G}(n), \mathbf{k})$. Therefore it is natural to consider integral homology. The fact that there are no 1-cycles which are boundaries in $\mathsf{G}(n)$ implies that there is no torsion in $H_1(\mathsf{G}(n), \mathbb{Z})$, so

$$H_1(\mathsf{G}(n), \mathbf{k}) \cong H_1(\mathsf{G}(n), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbf{k}.$$

We also define an integral form for the operad $\mathcal{L}$.

Definition 3. Given the `Sets` operad $\mathcal{L} = \mathcal{F}_{\mathsf{S}}(B)/\sim$ with $\sim$ consisting of a list of couples $p_{n,i} \sim p'_{n,i}$ for $p_{n,i}, p'_{n,i} \in \mathcal{F}_{\mathsf{S}}(B)(n)$, we define an operad $\mathcal{L}_{\mathbb{Z}} := \langle E_{\mathbb{Z}}; R_{\mathbb{Z}} \rangle$ in the symmetric monoidal category `Ab-Grp` of abelian groups. This operad is generated by the collection of free abelian groups $E_{\mathbb{Z}}(n) := \mathbb{Z}^{B(n)}$ and the collection of relations $R_{\mathbb{Z}}(n) := \bigoplus_i \mathbb{Z} \cdot (p_{n,i} - p'_{n,i})$.

The collection of coherence relations $\mathcal{D}_{\mathcal{L}_{\mathbb{Z}}}$ of the operad $\mathcal{L}_{\mathbb{Z}}$ is the $\mathcal{F}(E_{\mathbb{Z}})$-module defined completely analogously to $\mathcal{D}_{\mathcal{P}}$ in Definition 1 and the coherence constraints $\mathsf{C}_{\mathcal{L}_{\mathbb{Z}}}$ is the corresponding collection of indecomposables. The following proposition describes the obvious relation between coherence data for $\mathcal{P} = \mathrm{Span}_{\mathbf{k}}(\mathcal{L})$ and $\mathcal{L}_{\mathbb{Z}}$.

Proposition 13. *Let $\mathcal{P}$ be an operad defined by commutative diagrams, with associated sets operad $\mathcal{L} = \langle B; K \rangle$ such that there is an isomorphism (9). Then*

$$\mathcal{D}_{\mathcal{P}} \cong \mathcal{D}_{\mathcal{L}_{\mathbb{Z}}} \otimes_{\mathbb{Z}} \mathbf{k}, \quad \mathsf{C}_{\mathcal{P}} \cong \mathsf{C}_{\mathcal{L}_{\mathbb{Z}}} \otimes_{\mathbb{Z}} \mathbf{k},$$

and

$$H_1(\mathsf{G}_{\mathcal{P}}(n), \mathbb{Z}) \cong \mathcal{D}_{\mathcal{L}_{\mathbb{Z}}}. \tag{11}$$

Before ending this section we make a final remark about operads with defining relations which are dual graphlike as in the Lie-hedron. Recall that in the dual Tel-A-graph the vertices correspond to relations and the edges correspond to basis elements appearing in the relation. In this case the coherence relations do not correspond to the space of 1-cycles, but to certain 0-chains. If it is possible to orient the dual Tel-A-graph $\mathsf{G}'(\mathcal{T}(n))$, then the sum of all the vertices in a connected component defines a coherence constraint since it represents a linear combination of relations such that each basis element (edge) appears once with a plus sign and once with a negative sign. It is easy to see from the matrix in Figure 2 that by changing the sign of four of the relations, defining $\mathbf{i}' = -\mathbf{i}$ for $\mathbf{i} \in \{\mathbf{3}, \mathbf{6}, \mathbf{7}, \mathbf{9}\}$, we get a matrix with each column having two nonzero entries, one $+1$ one -1, which allows us to orient the Lie-hedron. Then the coherence relation

$$-\ell := \mathbf{1} + \mathbf{2} + \mathbf{3}' + \mathbf{4} + \mathbf{5} + \mathbf{6}' + \mathbf{7}' + \mathbf{8} + \mathbf{9}' + \mathbf{10}$$

is a just a sum over the vertices. The fact that the Lie-hedron is connected implies that the space of coherence constraints in $n = 4$ is one dimensional. We shall discuss this issue further in Example 25.

The following two useful formulas (one for the non-Σ, one for the symmetric case) compute the size of the sets $\mathcal{V}(4)$ and $\mathcal{V}'(4)$ for a quadratic operad $\mathcal{P} = \langle E; R \rangle$. The formulas are immediate consequences of the description of the free operad and free operad module in terms of trees.

(12) $\dim(\mathcal{V}(4)) = 5\dim(E)^3$, $\dim(\mathcal{V}'(4)) = 5\dim(E)\dim(R)$ (the non-Σ case)

(13) $\dim(\mathcal{V}(4)) = 15\dim(E)^3$, $\dim(\mathcal{V}'(4)) = 10\dim(E)\dim(R)$ (the symmetric case)

Summing up the above results, we see for a Koszul operad $\mathcal{P}$ the following nice formula which relates the topological properties of the graph $\mathsf{G}(4)$ and the algebraic properties of the operad $\mathcal{P}$:

$$\dim(\mathsf{C}_{\mathcal{P}}) = \dim(\mathcal{P}^!(4)) = \dim(H_1(\mathsf{G}(4)))\,.$$

5. Relation to Mac Lane coherence

In this section we show how our theory gives coherence theorems à la Mac Lane. We will deal with operads given by commutative diagrams, in the sense of Definition 2. For these operads, the defining relations are always graphlike (Proposition 11) with $\mathsf{G}(n)$ the corresponding Tel-A-graph in degree n.

Let $\mathcal{P}$ be an operad defined by commutative diagrams, and assume that the isomorphism (9) has been fixed once and for all, as well as the presentation $\mathcal{L} = \langle B; K\rangle$. Assume we have oriented the Tel-A-graphs $\mathsf{G}(n)$ as in the proof of Proposition 12 so that $\pi(r) = \partial(r)$. Define the functions $\eta, \xi : \mathcal{V}'(n) \to \mathcal{V}(n)$ by $\pi(r) = \eta(r) - \xi(r)$ for any edge r of $\mathsf{G}(n)$.

Definition 4. A (B, K)-structure on a category $\mathcal{C}$ consists of:

(i) An assignment of an n-polyfunctor $\Phi(b)$ for each element $b \in B(n)$ and thus, by functorial extension, an n-multifunctor $\Phi(p)$ for every element $p \in \mathcal{F}_{\mathsf{S}}(B)(n)$.
(ii) An assignment of a natural isomorphism $a(r)$ between $\Phi(\xi(r))$ and $\Phi(\eta(r))$ for each $r \in K$ and thus, by functorial extension, for every element $r \in \mathcal{F}_{\mathsf{S}}(B)\langle K\rangle$.

The (B, K)-structure is *coherent* if and only if whenever there is a composition of the natural transformations $a(r)$ and their inverses connecting a given pair of multi-functors $\Phi(p)$ and $\Phi(q)$, then all possible compositions connecting these two multi-functors give the same natural transformation. This is equivalent to saying that whenever there is a composition of the $a(r)$'s and their inverses which connects the multi-functor $\Phi(p)$ to itself, the composition is the identity transformation.

Instead of a (B, K)-structure we will often speak simply about a $\mathcal{P}$-structure, $\mathcal{P} = \mathrm{Span}_{\mathsf{k}}(\mathcal{L})$. The (B, K)-notation, on the other hand, underlines the fact that the structure of Definition 4 is very explicitly related to the prezentation $\langle B; K\rangle$ of $\mathcal{L}$.

Each closed path $\gamma \in \mathsf{G}(n)$ determines a diagram $D(\gamma)$ with arrows corresponding to natural isomorphisms of functors. By definition, the (B, K)-structure on the category $\mathcal{C}$ is coherent if and only if all these diagrams are commutative.

Proposition 14. *Let $n \geq 3$ and let $\{\gamma_1, \ldots, \gamma_d\}$ be a sequence of closed paths of the graph $\mathsf{G}(n)$ forming a basis of $H_1(\mathsf{G}(n), \mathbb{Z})$. Then the diagram of natural isomorphisms $D(\gamma)$ is commutative for any closed path γ of $\mathsf{G}(n)$ if and only if it is commutative for any $\gamma \in \{\gamma_1, \ldots, \gamma_d\}$.*

Proof. Since the graph is an one-dimensional simplicial complex, the fundamental group is isomorphic to the free group on a set of closed paths. The set of generators also forms a homology basis. Conversely, each set of closed paths that forms a basis for the cohomology generates the fundamental group.

Thus in order to guarantee commutativity of all diagrams $D(\gamma)$ it is enough to check that the diagrams $D(\gamma_i)$, $1 \leq i \leq d$, are commutative. □

As a corollary we formulate the following 'classical' coherence theorem which involves the coherence constraints $\mathsf{C}_{\mathcal{L}_{\mathbb{Z}}}$ of the operad $\mathcal{L}_{\mathbb{Z}}$ in the category Ab-Grp.

Theorem 15. *A necessary and sufficient condition for the coherence of a* (B, K)*-structure on the category* $\mathcal{C}$ *is that the diagrams* $D(\gamma)$ *are commutative for a set* $\mathcal{B}$ *of closed paths* γ *such that the images* $\{p(\pi(\gamma));\ \gamma \in \mathcal{B}\}$ *form a basis for* $\mathsf{C}_{\mathcal{L}_{\mathbb{Z}}}$.

Thus, in the case of a Koszul operad there are $\dim(\mathcal{P}^!(4))$*-diagrams whose commutativity is necessary and sufficient for coherence.*

Proof. The commutativity of all diagrams $D(\gamma)$, $\gamma \in \mathcal{B}$, is clearly necessary for the coherence. We need to prove that it is sufficient.

By Proposition 14, the condition for coherence is the commutativity of the diagrams $D(\gamma)$ for a basis of $H_1(\mathsf{G}(n), \mathbb{Z})$. By assumption the diagrams with $p(\pi(\gamma))$ forming a basis for $\mathsf{C}_{\mathcal{L}_{\mathbb{Z}}}$ are commutative. By (11) of Proposition 13, these together with the 'obvious relations' give a $\mathbb{Z}$-basis for $H_1(\mathsf{G}(n), \mathbb{Z})$.

Thus, to finish the proof of the first part of the theorem, it is enough to show that each 'obvious relation' corresponds to a commutative diagram. The first type of the 'obvious relation'

$$x(b_1, \dots, b_{s-1}, \pi(y), b_{s+1}, \dots, b_l) - \pi(x)(b_1, \dots, b_{s-1}, y, b_{s+1}, \dots, b_l) = 0$$

is, for $b_1, \dots, b_{s-1}, b_{s+1}, \dots, b_l \in \mathcal{F}_{\mathsf{S}}(B)$, $x, y \in \mathcal{F}_{\mathsf{S}}(B)\langle K\rangle$, represented by the simple closed path ω (recall $\pi(x) = \eta(x) - \xi(x)$, $\pi(y) = \eta(y) - \xi(y)$):

$$\begin{tikzcd}
\xi(x)(b_1,\dots,b_{s-1},\xi(y),b_{s+1},\dots,b_l) \arrow[r, "{x(b_1,\dots,b_{s-1},\xi(y),b_{s+1},\dots,b_l)}"] \arrow[d, "{\xi(x)(b_1,\dots,b_{s-1},y,b_{s+1},\dots,b_l)}"'] & \eta(x)(b_1,\dots,b_{s-1},\xi(y),b_{s+1},\dots,b_l) \arrow[d, "{\eta(x)(b_1,\dots,b_{s-1},y,b_{s+1},\dots,b_l)}"] \\
\xi(x)(b_1,\dots,b_{s-1},\eta(y),b_{s+1},\dots,b_l) \arrow[r, "{x(b_1,\dots,b_{s-1},\eta(y),b_{s+1},\dots,b_l)}"'] & \eta(x)(b_1,\dots,b_{s-1},\eta(y),b_{s+1},\dots,b_l)
\end{tikzcd}$$

for which $D(\omega)$ is commutative by naturality. Also the second 'obvious' relation

$$\begin{aligned} &b(b_1, \dots, b_{s-1}, \pi(x), b_{s+1}, \dots, b_{t-1}, y, b_{t+1}, \dots, b_l) - \\ &\quad - b(b_1, \dots, b_{s-1}, x, b_{s+1}, \dots, b_{t-1}, \pi(y), b_{t+1}, \dots, b_l) = 0 \end{aligned}$$

can be represented by a simple closed path ω such that $D(\omega)$ is commutative in an obvious similar way, again by the naturality.

The second part of the theorem follows from the description of the coherence constraints $\mathsf{C}_{\mathcal{P}}$ given in Theorem 7. □

Example 16. (continuation of Example 2) The non-Σ operad $\mathcal{A}ss$ for associative algebras is quadratic Koszul and $\mathcal{A}ss = \mathcal{A}ss^!$, therefore $\mathsf{C}_{\mathcal{A}ss} = \mathcal{A}ss(4) = \mathrm{Span}(p)$, where p *must* be the element of (5) – there is no other choice! We get from (12) that $\dim(\mathcal{V}(4)) = \dim(\mathcal{V}'(4)) = 5$, and $\mathsf{G}(4)$ is the pentagon. An $\mathcal{A}ss$-structure on a category $\mathcal{C}$ is the same as a multiplication on $\mathcal{C}$ with an associativity isomorphism as it was discussed in the introduction. The coherence of $(\mathcal{C}, \square, a)$ in the sense of the above definitions coincides with Mac Lane's definition, and Theorem 15 gives Mac Lane's celebrated coherence result.

Example 17. Consider the algebraic structure consisting of a vector space V and two bilinear maps $\circ, \bullet : V \otimes V \to V$ which satisfy

$$\begin{aligned} a \circ (b \circ c) &= (a \circ b) \circ c, & a \circ (b \bullet c) &= (a \circ b) \bullet c, \\ a \bullet (b \circ c) &= (a \bullet b) \circ c, & a \bullet (b \bullet c) &= (a \bullet b) \bullet c. \end{aligned}$$

These algebras were introduced in [13] and called *nonsymmetric Poisson algebras.* The corresponding operad $\mathcal{K}$ is Koszul, quadratic self-dual (see again [13]) and it has a quadratic presentation $\mathcal{K} = \langle E; R \rangle$ with $\dim(E) = 2$ and $\dim(R) = 4$. We easily calculate that $\dim(\mathcal{K}(4)) = 8$, thus, by Theorem 7, $\dim(\mathsf{C}_{\mathcal{K}}) = \dim(\mathcal{K}^!(4)) = \dim(\mathcal{K}(4)) = 8$, with a basis consisting of eight pentagons shown in Figure 4.

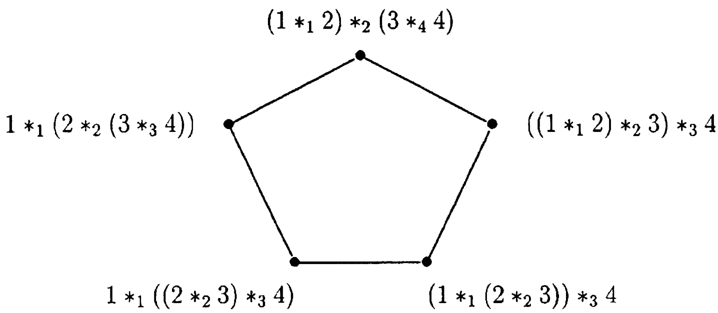


FIGURE 4. The graph $\mathsf{G}(4)$ for nonsymmetric Poisson algebras. The triple $(*_1, *_2, *_3)$ runs through all eight possible combinations $(\circ, \circ, \circ)$, $(\bullet, \circ, \circ)$, $(\circ, \bullet, \circ)$, $(\circ, \circ, \bullet)$, $(\circ, \bullet, \bullet)$, $(\bullet, \circ, \bullet)$, $(\bullet, \bullet, \circ)$ and $(\bullet, \bullet, \bullet)$.

A $\mathcal{K}$-structure on $\mathcal{C}$ consists of two covariant bifunctors $\square_1, \square_2 : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and four natural transformations

$$\begin{aligned} a_{11} &: \square_1(\mathbb{1} \times \square_1) \to \square_1(\square_1 \times \mathbb{1}), & a_{12} &: \square_1(\mathbb{1} \times \square_2) \to \square_2(\square_1 \times \mathbb{1}), \\ a_{21} &: \square_2(\mathbb{1} \times \square_1) \to \square_1(\square_2 \times \mathbb{1}), & a_{22} &: \square_2(\mathbb{1} \times \square_2) \to \square_2(\square_2 \times \mathbb{1}). \end{aligned}$$

This structure is a variant of a *weakly distributive category* [1, Definition 1.1]. These categories are important for linear logic; we intend to discuss the applications of our theory to this direction in another paper. By Theorem 15, such a $\mathcal{K}$-structure is coherent if and only if the eight pentagonal diagrams corresponding to those in Figure 4 commute.

Example 18. Let us discuss the following bizarre objects introduced by Loday in [8]. The importance of this example is that some coherence constraints *will not be the pentagons*.

By a *digebra* we mean a vector space V together with two bilinear operations $\circ$ and $\bullet$ which satisfy the following axioms:

$$x \circ (y \circ z) = (x \circ y) \circ z = x \circ (y \bullet z),$$
$$(x \bullet y) \circ z = x \bullet (y \circ z),$$
$$(x \circ y) \bullet z = x \bullet (y \bullet z) = (x \bullet y) \bullet z\,.$$

Let $\mathcal{D}$ be the corresponding non-Σ operad. It has a quadratic presentation $\mathcal{D} = \langle E; R\rangle$ with $\dim(E) = 2$ and $\dim(R) = 5$, and formula (13) gives that $\dim(\mathcal{V}(4)) = 40$ and $\dim(\mathcal{V}'(4)) = 50$.

As it was proven in [8], the operad $\mathcal{D}$ is Koszul. It is easy to compute that $\dim(\mathcal{D}(n)) = n$, and Proposition 8 says that $\dim(\mathsf{C}_{\mathcal{D}}) = 14$. The graph $\mathsf{G}(4)$ is complicated (it has 40 vertices and 50 edges), but we know, by Proposition 14, that there exist 14 closed cycles in $\mathsf{G}(4)$ which generate the coherence constraints. These cycles are shown on Figure 5.

6. Operads and their algebras in a category of modules

Recall [15] that a $\mathcal{P}$-algebra structure on a (graded) $\mathbf{k}$-vector space U is an operad map $A : \mathcal{P} \to \mathrm{End}(U)$ from the operad $\mathcal{P}$ to the endomorphism operad $\mathrm{End}(U)$ of U, where all structures are considered in the category $\mathtt{Vect}_{\mathbf{k}}$ or $\mathtt{gr\text{-}Vect}_{\mathbf{k}}$ with multiplication given by the (graded) tensor product over $\mathbf{k}$, $\otimes := \otimes_{\mathbf{k}}$. If U is a left V-module over a $\mathbf{k}$-algebra V, then $\mathrm{End}(U)(n)$ has a natural left V- right $V^{\otimes n}$-module structure. This allows us to consider generalized $\mathcal{P}$-algebras satisfying axioms that may involve coefficients from V and its tensor powers. In this section we study the way in which the structure of the operad $\mathcal{P}$ imposes conditions on V which are necessary for there to be a reasonable concept of generalized $\mathcal{P}$-algebras.

An example of this type was given in the introduction, where we considered Φ-associative algebras and derived the pentagon identity on the associator for the bialgebra V.

Fix a unital, associative, local $\mathbf{k}$-algebra V, with augmentation $\epsilon : V \to \mathbf{k}$ (i.e., each $v \in V$ with $\epsilon(v) \neq 0$ is invertible) and an operad $\mathcal{P} = \langle E; R\rangle$. In order to take tensor products of V-modules we must assume that, for each n with $E(n) \neq 0$, we are given a $\mathbf{k}$-linear algebra homomorphism (a 'diagonal') $\Delta^n : V \to V^{\otimes n}$ such that

$$(\epsilon^{\otimes n})\Delta^n(v) = \epsilon(v)\,, \text{ for each } v \in V, \tag{14}$$

where we identify, in the left hand side of (14), $\mathbf{k}$ with $\bigotimes_{\mathbf{k}}^n \mathbf{k}$. When $\mathcal{P}$ is quadratic, then $\Delta = \Delta^2 : V \to V \otimes V$ and $(V, \cdot, \Delta, 1, \epsilon)$ is an ordinary augmented unital associative, non necessarily coassociative, bialgebra.

For $\mathcal{P} = \langle E; R\rangle$ and $n \geq 1$ we have the basic exact sequence of $\mathbf{k}$-vector spaces, which defines $\mathcal{P}$ as a quotient of the free operad generated by E,

$$0 \longrightarrow \mathrm{Ker}(\pi)(n) \longrightarrow \mathcal{F}(E)\langle R\rangle(n) \xrightarrow{\pi} \mathcal{F}(E)(n) \longrightarrow \mathcal{P}(n) \longrightarrow 0\,. \tag{15}$$

This identifies $\mathcal{P}(n)$ to $\mathrm{Coker}(\pi)(n)$. We now define an analog of this sequence which brings V into play and allows us to consider generalized coherence conditions. Let us discuss the non-Σ case first.

We already observed in Section 4 that elements of $\mathcal{F}(E)(n)$ are represented by a sum of planar rooted trees with vertices labeled by elements of E. Such a labeled

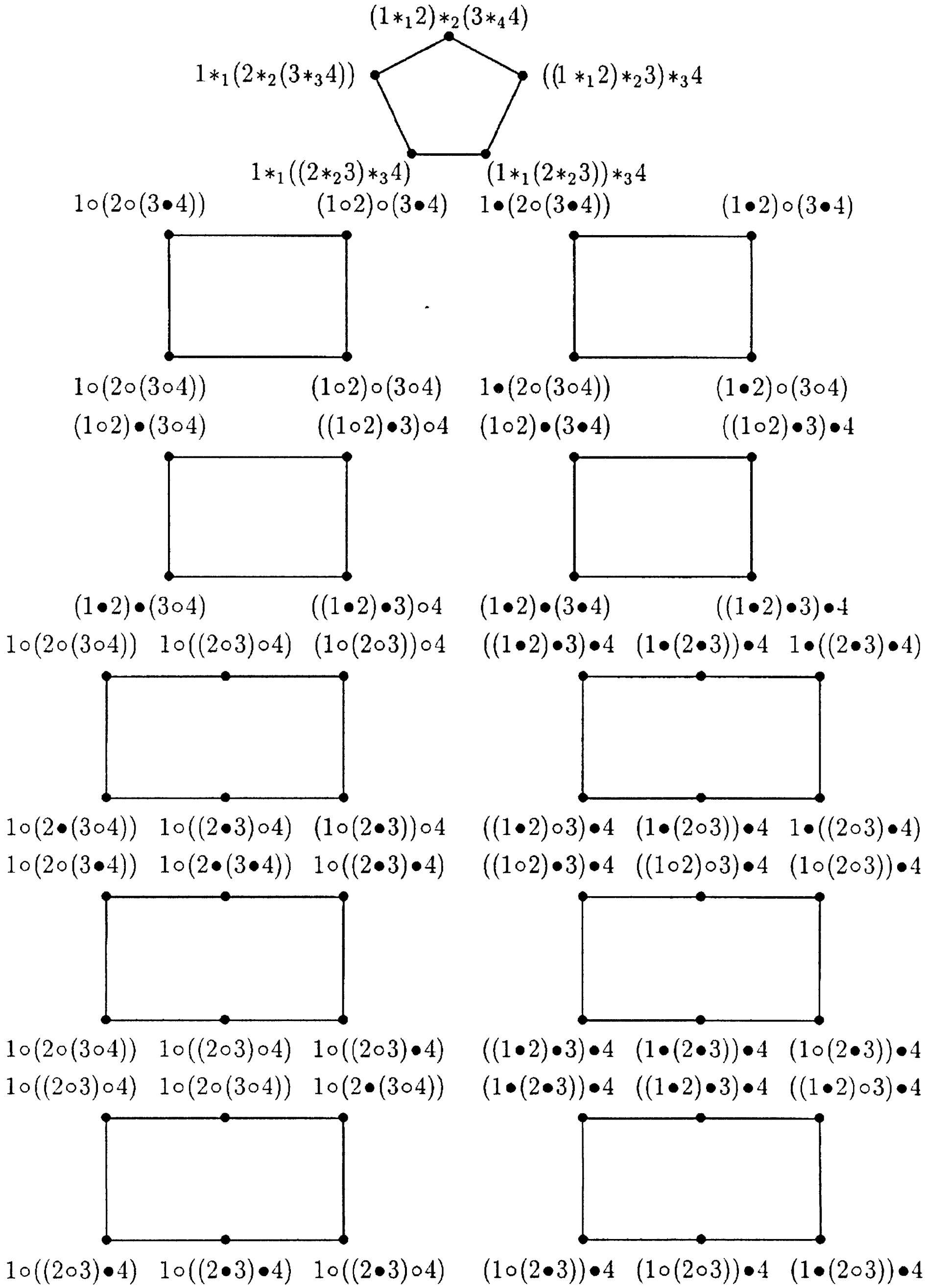


FIGURE 5. The graph G(4) for digebras. It consists of four pentagons (the triple $(*_1, *_2, *_3)$ runs through $(\circ,\circ,\circ)$, $(\bullet,\circ,\circ)$, $(\bullet,\bullet,\circ)$ and $(\bullet,\bullet,\bullet)$), four squares and six hexagons.

tree t determines uniquely a bracketing $b = b_t$ of n indeterminates or, equivalently, an iterated comultiplication denoted $\Delta^{n,b}$. Form the *V-relative free operad* $\mathcal{F}_V(E)$ as follows:

$$\mathcal{F}_V(E) = \bigoplus_{n\geq 1} \mathcal{F}_V(E)(n)\,, \text{ with } \mathcal{F}_V(E)(n) := \mathcal{F}(E)(n) \otimes_{\mathbf{k}} V^{\otimes n}\,. \tag{16}$$

Each $\mathcal{F}_V(E)(n)$ has an obvious right $V^{\otimes n}$-module structure. We give it a left V-module structure by defining

$$v \cdot (t \otimes \vec{u}) = t \otimes \Delta^{n,b_t}(v)\vec{u}\,,\ \vec{u} = u_1 \otimes \ldots \otimes u_n \in V^{\otimes n}\,. \tag{17}$$

Given $t \otimes \vec{u} \in \mathcal{F}_V(E)(n)$, where $\vec{u} = u_1 \otimes \ldots \otimes u_n$, the 'operadic' composition of this element with the tensor product of n elements $t_i \otimes \vec{v}_i \in \mathcal{F}_V(E)(a_i)$, $i = 1, \ldots n$, is defined as

$$\begin{aligned}&\gamma((t\otimes\vec{u}); (t_1\otimes\vec{v}_1)\otimes\cdots\otimes(t_n\otimes\vec{v}_n)) := \\ &\quad := t(t_1\otimes\cdots\otimes t_n)\otimes\Delta^{a_1,b_{t_1}}(u_1)\vec{v}_1\otimes\ldots\otimes\Delta^{a_n,b_{t_n}}(u_n)\vec{v}_n\,,\end{aligned} \tag{18}$$

where $t(t_1 \otimes \cdots \otimes t_n)$ is the composition in $\mathcal{F}(E)$. Heuristically, we can say that the composition moves the interior coefficients v_i across the tree t_i using the comultiplication $\Delta^{a_i,b_{t_i}}$. Clearly this defines on $\mathcal{F}_V(E)$ the structure of a non-Σ operad. In degree n it is a free right $V^{\otimes n}$-module on the k-linear space $\mathcal{F}(E)(n)$.

In the symmetric case we define the right action of the symmetric group on $\mathcal{F}_V(E)$ by

$$[t \otimes (u_1 \otimes \cdots \otimes u_n)]\sigma = t\sigma \otimes (u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(n)})\,,$$

for $t \otimes \vec{u} \in \mathcal{F}_V(E)(n)$ and $\sigma \in \Sigma_n$.

To guarantee the consistency of this action with the left V-module structure (17) we define the comultiplication for the tree $t\sigma$ by

$$\Delta^{n,b_{t\sigma}}(v) := \sigma^{-1}(\Delta^{n,b_t}(v))\,. \tag{19}$$

Assume now that $R_V = \{R_V(n)\}_{n\geq 1}$ is a subcollection of $\mathcal{F}_V(E)$ such that each $R_V(n)$ is left V- right $V^{\otimes n}$-submodule. In the symmetric case we moreover require that $R_V(n)$ is Σ_n-closed.

Then we form the operadic ideal $(R_V) \subset \mathcal{F}_V(E)$ and define the *V-relative operad* $\mathcal{P}_V$ by

$$\mathcal{P}_V := \mathcal{F}_V(E)/(R_V)\,. \tag{20}$$

Observe that each $\mathcal{P}_V(n)$ is a left V- right $V^{\otimes n}$-module.

To formulate the concept of an operad algebra from this point of view we use the usual endomorphism operad with $\mathrm{End}(U)(n) = \mathrm{Hom}_{\mathbf{k}}(U^{\otimes n}, U)$ considered as a left V- and right $V^{\otimes n}$-module in the standard way:

$$\begin{aligned}(v \cdot \alpha)(u_1 \otimes \cdots \otimes u_n) &= v \cdot (\alpha(u_1 \otimes \cdots \otimes u_n)\,,\\ (\alpha \cdot (v_1 \otimes \cdots \otimes v_n))(u_1 \otimes \cdots \otimes u_n) &= \alpha(v_1 \cdot u_1 \otimes \cdots \otimes v_n \cdot u_n)\,,\end{aligned}$$

for $\alpha \in \mathrm{End}(U)(n)$, $u_1 \otimes \ldots \otimes u_n \in U^{\otimes n}$, $v \in V$ and $v_1 \otimes \ldots \otimes v_n \in V^{\otimes n}$. A $\mathcal{P}_V$-algebra structure on the V-module U is given by a 'V-relative' operad map, i.e. a family of left V- right $V^{\otimes n}$-module maps (equivariant, in the Σ-case)

$$A(n) : \mathcal{P}_V(n) \longrightarrow \mathrm{End}(U)(n)\,,\ n \geq 1\,,$$

that are compatible with the operadic compositions.

Example 19. Suppose that U has a $\mathbf{k}$-linear multiplication $* : U \otimes U \to U$ and assume that $(U, *)$ is a V-module algebra, that is,

$$v(a * b) = v_{(1)}(a) * v_{(2)}b\,, \text{ for } v \in V,\ a, b \in U\,,$$

where, as before, $v_{(1)} \otimes v_{(2)}$ stands for $\sum_i v_{(1)i} \otimes v_{(2)i} = \Delta(v)$ and Δ is not necessarily coassociative. We want to consider associativity in the category of V-modules. Suppose that U is a left $V = (V, \cdot)$-module and replace $a * (b * c) = (a * b) * c$ by

$$\Phi(a * (b * c)) = ((a * b) * c) \tag{21}$$

where $\Phi \in V^{\otimes 3}$ is an invertible element and $\Phi(a * (b * c)) := \sum(\Phi_1 a * (\Phi_2 b * \Phi_3 c))$, if $\Phi = \sum \Phi_1 \otimes \Phi_2 \otimes \Phi_3$.

These algebras are algebras over the V-relative operad Ass_Φ defined as follows. In the notation introduced in Example 2, let $R_V = R_{Ass,\Phi}$ be a right $V^{\otimes 3}$-submodule of $\mathcal{F}_V(E)(3)$ generated by

$$\tilde{r}_{Ass,\Phi} = (1(23)) \cdot \Phi - ((12)3) \in \mathcal{F}_V(E)(3)\,.$$

Then $Ass_\Phi := \mathcal{F}_V(E)/(R_{Ass,\Phi})$ is the V-relative operad describing algebras that satisfy (21).

The augmentation map $\epsilon : V \to \mathbf{k}$ induces, for each n, an augmentation (denoted by the same symbol) $\epsilon : V^{\otimes n} \to \mathbf{k}$. The right tensoring with $\mathbf{k}$ over $V^{\otimes n}$ thus makes sense and it defines a right-exact functor from the category of right $V^{\otimes n}$-modules to the category of $\mathbf{k}$-vector spaces. We denote this functor by $\epsilon(-)$. We also have, for any right $V^{\otimes n}$-module M, a canonical $\mathbf{k}$-linear epimorphism $\epsilon_M : M \to \epsilon(M)$ given by $\epsilon_M(m) := m \otimes_{V^{\otimes n}} 1 \in \epsilon(M)$.

Lemma 20. *For each $n \geq 1$, there is a canonical isomorphism*

$$\epsilon(\mathcal{F}_V(E)(n)) \cong \mathcal{F}(E)(n)\,.$$

The canonical map $\epsilon_{\mathcal{F}_V(E)} := \{\epsilon_{\mathcal{F}_V(E)(n)} : \mathcal{F}_V(E)(n) \to \mathcal{F}(E)(n)\}_{n\geq 1}$ is a homomorphism of $\mathbf{k}$-operads.

Proof. The first statement of the lemma is obvious. The second part immediately follows from definition (18) of the operadic structure on $\mathcal{F}_V(E)$ and compatibility (14) of ϵ and Δ's. □

Let us introduce the central notion of this section.

Definition 5. Assume that each $R_V(n)$ is a left V-submodule of $\mathcal{F}_V(E)(n)$, is Σ_n-closed and is free as a right $V^{\otimes n}$-module. Let $\mathcal{P}_V = \mathcal{F}_V(E)/(R_V)$ be as in (20) and define $R := \epsilon(R_V) \subset \mathcal{F}(E)$. Then we call the V-relative operad $\mathcal{P}_V$ a V-relativization or V-quantization of the $\mathbf{k}$-operad $\mathcal{P} := \mathcal{F}(E)/(R)$. Let $\{\tilde{r}_1^n, \ldots, \tilde{r}_{s(n)}^n\} \subset \mathcal{F}_V(E)(n)$ be a $V^{\otimes n}$-basis of $R_V(n)$, $n \geq 3$. We require further that $(r_1^n, \ldots, r_{s(n)}^n)$, with $r_i^n := \epsilon(\tilde{r}_i^n)$, $1 \leq i \leq s(n)$, are independent over $\mathbf{k}$ and thus form a basis for $R(n) = \epsilon(R_V(n))$.

Define the map $\pi_V : \mathcal{F}_V(E)\langle R\rangle \to \mathcal{F}_V(E)$ of $\mathcal{F}_V(E)$-modules by $\pi_V(r_i) := \tilde{r}_i$, $1 \leq i \leq s$. We have the following V-relative analog of (15):

$$0 \longrightarrow \mathrm{Ker}(\pi_V)(n) \longrightarrow \mathcal{F}_V(E)\langle R\rangle(n) \xrightarrow{\pi_V} \mathcal{F}_V(E)(n) \longrightarrow \mathcal{P}_V(n) \longrightarrow 0\,. \tag{22}$$

Now apply the functor $\epsilon(-)$ on this sequence. Since clearly $\epsilon(\mathcal{F}_V(E)\langle R\rangle(n)) \cong \mathcal{F}(E)\langle R\rangle(n)$, $\epsilon(\mathcal{F}_V(E)(n)) \cong \mathcal{F}(E)(n)$ and $\epsilon(-)$ is right exact, we obtain the exact sequence

$$\mathcal{F}(E)\langle R\rangle(n) \xrightarrow{\pi} \mathcal{F}(E)(n) \longrightarrow \epsilon(\mathcal{P}_V) \longrightarrow 0$$

where $\pi(r_i) = r_i$, $1 \leq i \leq s$. This implies the existence of the canonical $\mathbf{k}$-isomorphism $\epsilon(\mathcal{P}_V(n)) \cong \mathcal{P}(n)$. The universal property of the kernel gives the canonical map

$$\rho : \epsilon(\mathrm{Ker}(\pi_V)(n)) \to \mathrm{Ker}(\pi(n))\,. \tag{23}$$

Let $F : X \to Y$ be a map of two $V^{\otimes n}$-modules and let $f : \epsilon(X) \to \epsilon(Y)$ be a map of $\mathbf{k}$-vector spaces. We say that $F : X \to Y$ is a *map over* f if $\epsilon F = f\epsilon$.

Definition 6. Let $\mathcal{P}_V$ be a V-quantization of an operad $\mathcal{P}$. We say that $\mathcal{P}_V$ is coherent if $\mathcal{P}_V(n)$ is, for each degree n, isomorphic to $\mathcal{P}(n) \otimes V^{\otimes n}$ as a right $V^{\otimes n}$-module over the canonical isomorphism $\epsilon(\mathcal{P}_V) \cong \mathcal{P}(n)$.

Coherence in this sense measures the regularity of the behavior of the operadic ideal generated by R_V in $\mathcal{F}_V(E)$.

Recall (Definition 1) that the collection of coherence constraints C of the operad $\mathcal{P}$ is defined as the indecomposables of the quotient $\mathrm{Ker}(\pi)/\mathcal{O}$, where $\mathcal{O}$ is the module of 'obvious relations' of the operad $\mathcal{P}$. Let p be the projection $\mathrm{Ker}(\pi) \twoheadrightarrow \mathsf{C}$. The main statement of this section reads:

Theorem 21. *The V-quantization $\mathcal{P}_V$ of $\mathcal{P}$ is coherent if and only if the composition*

$$\Xi := \epsilon(\mathrm{Ker}(\pi_V)) \xrightarrow{\rho} \mathrm{Ker}(\pi) \overset{p}{\twoheadrightarrow} \mathsf{C} \tag{24}$$

is an epimorphism.

Proof. The inclusion $\mathbf{k} \to V$, $\alpha \mapsto \alpha \cdot 1$, induces an embedding $\mathcal{F}(E) \hookrightarrow \mathcal{F}_V(E)$ which is clearly a map of operads. This induces on $\mathcal{F}_V(E)\langle R\rangle$, and thus also on $\mathrm{Ker}(\pi_V)$, an $\mathcal{F}(E)$-module structure. It is easy to conclude from Lemma 20 that $\epsilon(\mathrm{Ker}(\pi_V))$ is also a natural $\mathcal{F}(E)$-module and that the map $\rho : \epsilon(\mathrm{Ker}(\pi_V)) \to \mathrm{Ker}(\pi)$ is an $\mathcal{F}(E)$-homomorphism.

Claim 22. *The map $\Xi : \epsilon(\mathrm{Ker}(\pi_V)) \to \mathsf{C}$ of (24) is an epimorphism if and only if the canonical map $\rho : \epsilon(\mathrm{Ker}(\pi_V)) \to \mathrm{Ker}(\pi)$ is onto.*

Proof of the claim. If ρ is onto, then clearly Ξ must be an epimorphism, too. To prove the opposite implication, observe that a map $f : X \to Y$ of $\mathcal{F}(E)$-modules is an epimorphism if and only if the composite of f with the canonical projection $Y \twoheadrightarrow Q_{\mathcal{F}(E)}(Y)$ to the space of indecomposables is onto. This means that if Ξ is an epimorphism, then the composition $\epsilon(\mathrm{Ker}(\pi_V)) \xrightarrow{\rho} \mathrm{Ker}(\pi) \longrightarrow \mathrm{Ker}(\pi)/\mathcal{O}$ is an epimorphism, too. As it is easy to verify, the map ρ is always an epimorphism on the space of obvious relations, $\mathcal{O} \subset \mathrm{Im}(\rho)$, this means that the map ρ is onto. The claim is proved.

Summing up our observations, we have, for each $n \geq 1$, the sequence (22) of right $V^{\otimes n}$-modules

$$0 \longrightarrow \mathrm{Ker}(\pi_V)(n) \longrightarrow \mathcal{F}_V(E)\langle R\rangle(n) \xrightarrow{\pi_V} \mathcal{F}_V(E)(n) \longrightarrow \mathcal{P}_V(n) \longrightarrow 0$$

and a sequence of $\mathbf{k}$-modules

$$0 \longrightarrow \mathrm{Ker}(\pi)(n) \longrightarrow \mathcal{F}(E)\langle R\rangle(n) \xrightarrow{\pi} \mathcal{F}(E)(n) \longrightarrow \mathcal{P}(n) \longrightarrow 0\,.$$

We also know that $\epsilon(\mathcal{F}_V(E)\langle R\rangle(n)) \cong \mathcal{F}(E)\langle R\rangle(n)$, $\epsilon(\mathcal{F}_V(E)(n)) \cong \mathcal{F}(E)(n)$ and that $\epsilon\pi_V = \pi\epsilon$. To finish the proof of Theorem 21, it is, by Claim 22, enough to show that, for each $n \geq 1$,

(25) $\mathcal{P}_V(n) \cong \mathcal{P}(n) \otimes V^{\otimes n}$ over the canonical isomorphism $\epsilon(\mathcal{P}_V(n)) \cong \mathcal{P}(n)$ if and only if the canonical map $\rho : \epsilon(\mathrm{Ker}(\pi_V(n))) \to \mathrm{Ker}(\pi(n))$ is an epimorphism.

This will clearly follow from the following lemma, in which we put $W := V^{\otimes n}$.

Lemma 23. *Suppose we have an exact sequence*

$$0 \longrightarrow S \xrightarrow{\eta} C \xrightarrow{\pi} B \xrightarrow{\tau} A \longrightarrow 0 \tag{26}$$

of $\mathbf{k}$*-modules, and an exact sequence*

$$0 \longrightarrow S' \xrightarrow{\eta'} C' \xrightarrow{\pi'} B' \xrightarrow{\tau'} A' \longrightarrow 0 \tag{27}$$

of right W-modules. Suppose that the modules C' and B' are W-free, $\epsilon(C') \cong C$, $\epsilon(B') \cong B$, and that, under these identifications, $\pi = \epsilon(\pi')$.

The canonical map $\rho : \epsilon(S') \to S$ is always a monomorphism and the following two conditions are equivalent:

(i) *the map $\rho : \epsilon(S') \to S$ is an epimorphism,*
(ii) *$A' \cong A \otimes W$ over the canonical isomorphism $\epsilon(A') \cong A$.*

Proof of the lemma. To prove that the map ρ is a monomorphism, observe that, since C' is a free W-module, the induced map $\epsilon(S') \to \epsilon(C')$ is monic. Because ρ is the composition of this map with the identification $\epsilon(C') \cong C$, it must be a monomorphism as well.

For the rest of the proof we may clearly assume there is a $\mathbf{k}$-vector space D such that (26) is of the form

$$0 \longrightarrow S \xrightarrow{\eta} D \oplus S \xrightarrow{\pi} D \oplus A \xrightarrow{\tau} A \longrightarrow 0$$

with $\eta(r) = (0, r)$, $\pi(d, r) = (d, 0)$ and $\tau(d, a) = a$. Then (27) is necessarily of the form

$$0 \longrightarrow S' \xrightarrow{\eta'} D \otimes W \oplus S \otimes W \xrightarrow{\pi'} D \otimes W \oplus A \otimes W \xrightarrow{\tau'} A' \longrightarrow 0$$

with π' represented by the matrix

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix}, \tag{28}$$

where $x \in \mathrm{Hom}_W(D \otimes W, D \otimes W)$, $y \in \mathrm{Hom}_W(S \otimes W, D \otimes W)$, $z \in \mathrm{Hom}_W(D \otimes W, A \otimes W)$ and $w \in \mathrm{Hom}_W(S \otimes W, A \otimes W)$. The assumption $\epsilon(\pi') = \pi$ translates to

$$(\mathbb{1}_D \otimes \epsilon)x = \mathbb{1}_D \otimes \epsilon,\ (\mathbb{1}_D \otimes \epsilon)y = 0,\ (\mathbb{1}_A \otimes \epsilon)z = 0 \text{ and } (\mathbb{1}_A \otimes \epsilon)w = 0$$

and completeness of W implies that the map x is invertible.

In order to finish the proof, it is clearly enough to show that either (i) or (ii) is equivalent to the existence of isomorphisms ϕ and ψ of right W-modules such that the following diagram commutes:

$$\begin{array}{ccc} D\otimes W\oplus S\otimes W & \xrightarrow{\pi\otimes 1\!\!1_W} & D\otimes W\oplus A\otimes W \\ \phi\uparrow & & \psi\uparrow \\ D\otimes W\oplus S\otimes W & \xrightarrow{\pi'} & D\otimes W\oplus A\otimes W \end{array} \tag{29}$$

The existence of ϕ and ψ and the commutativity of (29) clearly implies (i) and (ii) so it is enough to prove the converse. Assuming either (i) or (ii) we need to prove that there exist ϕ, ψ satisfying the equation

$$\psi^{-1}\circ(\pi\otimes 1\!\!1_W)\circ\phi = \pi'\,. \tag{30}$$

Let us suppose (i), that the map ρ is an epimorphism. Put

$$\phi := \begin{pmatrix} x & y \\ 0 & 1\!\!1_{S\otimes W} \end{pmatrix} \text{ and } \psi := \begin{pmatrix} 1\!\!1_{D\otimes W} & 0 \\ -zx^{-1} & 1\!\!1_{A\otimes W} \end{pmatrix}.$$

The map ψ is clearly invertible and

$$\begin{aligned} \psi^{-1}\circ(\pi\otimes 1\!\!1)\circ\phi &= \begin{pmatrix} 1\!\!1_{D\otimes W} & 0 \\ zx^{-1} & 1\!\!1_{A\otimes W} \end{pmatrix}\begin{pmatrix} 1\!\!1_{D\otimes W} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x & y \\ 0 & 1\!\!1_{S\otimes W} \end{pmatrix} \\ &= \begin{pmatrix} x & y \\ z & zx^{-1}y \end{pmatrix}. \end{aligned}$$

To prove that this composition gives π' we need to show that $w = zx^{-1}y$. Let $\{s_1, \dots, s_n\}$ be a $\mathbf{k}$-basis for S. By the surjectivity of ρ, we know that for each s_i there exists a corresponding $s_i' \in S'$, such that $\rho(\epsilon(s_i')) = s_i$. Since $S' \subset D\otimes W\oplus S\otimes W$, we can represent, for $1\leq i\leq n$, s_i' as $r_i' = d_i' + \sum_{1\leq j\leq n} s_j\otimes w_{ij}$ with $w_{ij}\in W$ and $d_i'\in D\otimes W$. Then $\rho(\epsilon(s_i')) = s_i$ implies $\epsilon(w_{ij}) = \delta_{ij}$, and the completeness of W assures that the matrix (w_{ij}) is invertible. Thus S' contains elements of the form $s_i'' = d_i'' + s_i\otimes 1$. For $s'\in S'$ there exist w_i such that $s' - \sum s_i'' w_i \in S'\cap(D\otimes W)$. The invertibility of the map x implies that $S'\cap(D\otimes W) = 0$, so the elements $\{s_i''\}_{1\leq i\leq n}$ span S' as a W-module. They are clearly linearly independent over W so they form a basis. The definition of S' as the kernel of π' gives the matrix equation

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix}\begin{pmatrix} u \\ 1\!\!1_{S\otimes W} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{31}$$

where $u\in \mathrm{Hom}_W(S\otimes W, D\otimes W)$ is defined by $u(s_i\otimes 1) = d_i''$. It follows from (31) that

$$xu + y = 0\,, \text{ and } zu + w = 0\,.$$

Solving the first equation, $u = -x^{-1}y$, and substituting in the second we obtain $w = zx^{-1}y$.

Suppose now (ii). This means that there exists a W-isomorphism $\xi : A' \to A\otimes W$ such that $(1\!\!1_A\otimes\epsilon)\xi = \epsilon$. Consider the composition $\xi\circ\tau' : D\otimes W\oplus A\otimes W \to A\otimes W$ represented by the matrix (r, s), $r : D\otimes W\to A\otimes W$ and $s : A\otimes W\to A\otimes W$. By the assumption on ξ, $(1\!\!1_A\otimes\epsilon)s = 1\!\!1_A\otimes\epsilon$. So s is invertible and we can replace ξ by $s^{-1}\circ\xi =: \overline{\xi}$. The matrix representing the composition $\overline{\xi}\circ\tau$ is $(\overline{r}, 1\!\!1_{1\!\!1\otimes W})$.

Now $\tau' \circ \pi' = 0$, so $\bar{\xi} \circ \tau' \circ \pi' = 0$ and representing π' as in (28) we conclude that

$$(\bar{r}, 1\!\!1_{A\otimes W}) \begin{pmatrix} x & y \\ z & w \end{pmatrix} = (0,0).$$

Then

$$\begin{pmatrix} 1\!\!1_{D\otimes W} & 0 \\ \bar{r} & 1\!\!1_{A\otimes W} \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1\!\!1_{D\otimes W} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x & y \\ 0 & 1\!\!1_{R\otimes W} \end{pmatrix}$$

and we may in (29) take

$$\phi := \begin{pmatrix} x & y \\ 0 & 1\!\!1_{R\otimes W} \end{pmatrix} \text{ and } \psi := \begin{pmatrix} 1\!\!1_{D\otimes W} & 0 \\ \bar{r} & 1\!\!1_{A\otimes W} \end{pmatrix}.$$

The lemma, and thus also Theorem 21, is proved. □

Example 24. Let us investigate the coherence of the operad Ass_Φ which we introduced in Example 19. Clearly, R_{Ass_Φ} is a free right $V^{\otimes 3}$-module. Definition (17) of the left V-action gives

$$\begin{aligned} v \cdot \tilde{r}_{Ass_\Phi} &= (1(23)) \cdot (1\!\!1 \otimes \Delta)\Delta(v) \cdot \Phi - ((12)3) \cdot (\Delta \otimes 1\!\!1)\Delta(v) \\ &= \tilde{r}_{Ass_\Phi} \cdot (\Delta \otimes 1\!\!1)\Delta(v) - (1(23))[\Phi \cdot (\Delta \otimes 1\!\!1)\Delta(v) - (1\!\!1 \otimes \Delta)\Delta(v)\Phi]. \end{aligned}$$

We see that R_{Ass_Φ} is left V-closed if, for each $v \in V$,

$$(1\!\!1 \otimes \Delta)\Delta(v) \cdot \Phi = \Phi \cdot (\Delta \otimes 1\!\!1)\Delta(v).$$

We already know that the coherence constraints C of the operad $Ass = \epsilon(Ass_\Phi)$ form an one dimensional $\mathbf{k}$-vector space $\mathsf{C} = \mathsf{C}(4)$ with the generator corresponding to the pentagon. Theorem 21 in this case says that the operad Ass_Φ is coherent if an only if the map $\rho : \epsilon(\mathrm{Ker}(\pi_{Ass_\Phi})(4)) \to \mathrm{Ker}(\pi_{Ass})(4)$ is onto. Since, by Lemma 23, the map ρ is a monomorphism, the later is true if and only if $\epsilon(\mathrm{Ker}(\pi_{Ass_\Phi})(4)) \neq 0$.

The 'quantized' map $\pi_{Ass_\Phi} : \mathcal{F}_V(E)\langle R_{Ass_\Phi}\rangle(4) \to (R_{Ass_\Phi})(4)$ is described by the following matrix with entries in $W = V^{\otimes 4}$ (notation of Example 2):

(32)

	a	**b**	**c**	**d**	**e**
$\pi_{Ass_\Phi}(\mathbf{1})$	-1	$(\Delta\otimes 1\!\!1^2)(\Phi)$	0	0	0
$\pi_{Ass_\Phi}(\mathbf{2})$	0	-1	$(1\!\!1^2\otimes\Delta)(\Phi)$	0	0
$\pi_{Ass_\Phi}(\mathbf{3})$	0	0	$(1\otimes\Phi)$	-1	0
$\pi_{Ass_\Phi}(\mathbf{4})$	0	0	0	$(1\!\!1\otimes\Delta\otimes 1\!\!1)\Phi$	-1
$\pi_{Ass_\Phi}(\mathbf{5})$	-1	0	0	0	$(\Phi\otimes 1)$

Consider the kernel of this map. It follows from a very special form of the matrix for $\pi_{Ass_\Phi}(4)$ that $\pi_{Ass_\Phi}(\mathbf{1}\cdot x_0 + \mathbf{2}\cdot x_1 + \cdots + \mathbf{5}\cdot x_4) = 0$ can be expanded to the following system of equations for $x_0, \ldots, x_4 \in W$:

$$\alpha_i x_{\bar{i}} = x_{\bar{i}+1} \quad \text{for } \bar{i} \in \mathbb{Z}_5,$$

with $\alpha_0 = (\Delta \otimes 1\!\!1^2)(\Phi)$, $\alpha_1 = (1 \otimes \Phi)^{-1}(1\!\!1^2 \otimes \Delta)(\Phi)$, $\alpha_2 = (1\!\!1 \otimes \Delta \otimes 1\!\!1)(\Phi^{-1})$, $\alpha_3 = (\Phi \otimes 1)^{-1}$ and $\alpha_4 = -1$. We see that

$$\mathrm{Ker}(\pi_{Ass_\Phi})(4) \cong \{x \in W;\ (\alpha_4 \cdots \alpha_0)x = x\}.$$

Clearly $\epsilon(\mathrm{Ker}(\pi_{Ass_\Phi})(4)) \neq 0$ if and only if there exists $x \in W$ such that $\epsilon(x) \neq 0$ and $(\alpha_4 \cdots \alpha_0)x = x$. By the completeness of W, such x is invertible and $(\alpha_4 \cdots \alpha_0)$ must equal 1, which is the standard pentagon identity for Φ:

$$(1 \otimes \Phi)(\mathbb{1} \otimes \Delta \otimes \mathbb{1})(\Phi)(\Phi \otimes 1) = (\mathbb{1}^2 \otimes \Delta)(\Phi)(\Delta \otimes \mathbb{1}^2)(\Phi). \tag{33}$$

We see that the coherence of the V-relative operad Ass_Φ means that the object $V = (V, \cdot, \Delta, 1, \Phi)$ is a quasi-bialgebra in the sense of [2]. A natural example of an Ass_Φ-algebra is the dot-construction of [14].

Example 25. Next consider the coherence of the generalized Lie operad. Let E and ζ have the same meaning as in Example 9 and let $t \in \mathcal{F}(E)(3)$ be the element $\zeta(\zeta \otimes \mathbb{1})$, corresponding to the bracketing [[12]3] and s the element $\zeta(\mathbb{1} \otimes \zeta)$ corresponding to the bracketing [1[23]]. Define a symmetry condition on ζ by

$$\zeta \circ T_{21} \otimes \mathcal{R} = -\zeta, \tag{34}$$

where

$$\mathcal{R} := \sum \mathcal{R}_1 \otimes \mathcal{R}_2 \in V^{\otimes 2}.$$

In other words, $E_V := \mathcal{F}_V(E)(2)$ is the free rank one right $V^{\otimes 2}$-module which is the quotient of the rank two module with basis $\{\zeta, \zeta \circ T_{21}\}$ by the submodule with basis $\{\zeta \circ T_{21} \otimes \mathcal{R} + \zeta\}$. From identity (17) we have

$$v \cdot (\zeta \circ T_{21} \otimes \mathcal{R}) = \zeta \circ T_{21} \otimes \Delta^{op}(v)\mathcal{R}$$

and

$$-v \cdot \zeta = -\zeta \otimes \Delta(v) = \zeta \circ T_{21} \otimes \mathcal{R}\Delta(v).$$

Therefore the condition that $\mathcal{F}_V(E)(2)$ is a free right $V^{\otimes 2}$-module implies

$$\Delta^{op}(v) = \mathcal{R}\Delta(v)\mathcal{R}^{-1}. \tag{35}$$

Iterating (34), the free module condition implies Drinfel'd's triangularity condition on $\mathcal{R}$:

$$\mathcal{R}_{21} \cdot \mathcal{R} = 1. \tag{36}$$

Operadic composition of equation (34) with $\mathbb{1} \otimes \zeta$ implies the identity

$$t \circ T_{231} \otimes (\mathbb{1} \otimes \Delta)\mathcal{R} = -s. \tag{37}$$

The $(\mathcal{R}, \Phi)$-relative version of the Lie operad is defined by the basic tertiary relation

$$\tilde{r}_{Lie_{\mathcal{R},\Phi}} := t \otimes 1 - s \otimes \Phi + s \circ T_{213} \otimes \Phi_{213}\mathcal{R}, \tag{38}$$

which is our formalism for the 'quantum Jacobi identity'

$$[[x, y], z] = \Phi[x, [y, z]] - \Phi\mathcal{R}_{21}[y, [x, z]]. \tag{39}$$

Let $R_{Lie_{\mathcal{R},\Phi}}$ be the right $V^{\otimes 3}$-module generated by $\tilde{r}_{Lie_{\mathcal{R},\Phi}}$. Requiring $\epsilon(\tilde{r}_{Lie_{\mathcal{R},\Phi}}) = r_{Lie}$ (the Jacobi identity) means that

$$\epsilon(\Phi) = 1 \text{ and } \epsilon(\mathcal{R}) = 1. \tag{40}$$

Again from the identity (17) we have

$$\begin{aligned} v \cdot \tilde{r}_{Lie_{\mathcal{R},\Phi}} &= v \cdot (t \otimes 1 - s \otimes \Phi + s \circ T_{213} \otimes \Phi_{213}\mathcal{R}) \\ &= t \otimes (\Delta \otimes 1\!\!1)\Delta(v) - s \otimes (1\!\!1 \otimes \Delta)(\Delta(v))\Phi \\ &\quad + s \circ T_{213} \otimes T_{213}\left((1\!\!1 \otimes \Delta)\Delta(v)\right)\Phi_{213}\mathcal{R}. \end{aligned}$$

Together with the condition $R_V(3)$ be a free right $V^{\otimes 3}$-module this implies

$$(1\!\!1 \otimes \Delta)(\Delta(v))\Phi = \Phi(\Delta \otimes 1\!\!1)\Delta(v) \text{ and} \tag{41}$$

$$T_{213}(1\!\!1 \otimes \Delta)(\Delta(v))\Phi\mathcal{R}_{21}) = \Phi_{213}\mathcal{R}(\Delta \otimes 1\!\!1) \otimes \Delta(v). \tag{42}$$

Equation (42) follows from equation (41) and equation (35).

Next we investigate the symmetry condition (37). It is most convenient to study these relations in the form of the quantum Jacobi identity (39).

$$\begin{aligned} [[x,y],z] &= [\Phi_1 x, [\Phi_2 y, \Phi_3 z]] - [\Phi_1\mathcal{R}_2 y, [\Phi_2\mathcal{R}_1 x, \Phi_3 z]] \\ &= -[\Phi_1 x, [\mathcal{R}_2\Phi_3 z, \mathcal{R}_1\Phi_2 y]] - [\Phi_1\mathcal{R}_2 y, [\Phi_2\mathcal{R}_1 x, \Phi_3 z]] \\ &= -[\Phi_1\mathcal{R}_2 y, [\Phi_2\mathcal{R}_1 x, \Phi_3 z]] - [[(\Phi^{-1})_1\Phi_1 x, (\Phi^{-1})_2\mathcal{R}_2\Phi_3 z], (\Phi^{-1})_3\mathcal{R}_1\Phi_2 y] \\ &\quad -[\Phi_1\mathcal{R}_2(\Phi^{-1})_2\mathcal{R}_2\Phi_3 z, [\Phi_2\mathcal{R}_1(\Phi^{-1})_1\Phi_1 x, \Phi_3(\Phi^{-1})_3\mathcal{R}_1\Phi_2 y]. \\ &= -\Phi\mathcal{R}_{21}[y,[x,z]] - \Phi^{-1}\mathcal{R}_{32}\Phi_{132}[[x,z],y] - \Phi\mathcal{R}_{21}\Phi_{213}^{-1}\mathcal{R}_{31}\Phi_{231}[z,[x,y]]. \end{aligned} \tag{43}$$

From the symmetry relation (34) we conclude:

$$\begin{aligned} &(\Phi\mathcal{R}_{21} - (1\!\!1 \otimes \Delta)\mathcal{R}_{21}\Phi_{231}^{-1}\mathcal{R}_{13}\Phi_{213})[y,[x,z]] - \\ &\quad -((1\!\!1 \otimes \Delta)\mathcal{R}_{21} - \Phi\mathcal{R}_{21}\Phi_{213}^{-1}\mathcal{R}_{31}\Phi_{231})[z,[x,y]] = 0. \end{aligned}$$

In order that $(\mathcal{F}(E)_V/R_V)(3)$ be a free $V^{\otimes 3}$-module of rank 2 we must have the identity:

$$(\Delta \otimes 1\!\!1)(\mathcal{R}) = \Phi_{312}\mathcal{R}_{13}\Phi_{132}^{-1}\mathcal{R}_{23}\Phi. \tag{44}$$

This is the first Drinfel'd hexagon identity, see [2, 3.9a, 3.9b]. The second hexagon identity

$$(1\!\!1 \otimes \Delta)(\mathcal{R}) = \Phi_{231}^{-1}\mathcal{R}_{13}\Phi_{213}\mathcal{R}_{12}\Phi^{-1}. \tag{45}$$

follows from the first one by the triangularity $\mathcal{R}_{21} = \mathcal{R}_{12}^{-1}$.

Next we want to determine the conditions which guarantee the coherence of the operad $Lie_{\mathcal{R},\Phi} := \mathcal{F}_V(E)/(R_{Lie_{\mathcal{R},\Phi}})$ in the sense of Definition 6.

We saw in Example 9 that the coherence constraints for $Lie = \epsilon(Lie_{\mathcal{R},\Phi})$ form the one-dimensional trivial representation of Σ_4, it is thus enough to investigate when $\epsilon(\mathrm{Ker}(\pi_{Lie_{\mathcal{R},\Phi}})(4))$ is nontrivial.

After determining the appropriate coefficients in $V^{\otimes 4}$ given by extending the relation (38) to brackets with four terms, we obtain a 'Φ-matrix' representing the map

$$\pi_{Lie(\mathcal{R},\Phi)} : \mathcal{F}_V(E)\langle R_{Lie(\mathcal{R},\Phi)}\rangle(4) \to (R_{Lie,\Phi}) \subset \mathcal{F}_V(E)(4).$$

We will not write the full Φ-matrix here since it is to big to fit on a page (it is obtained by decorating the entries of the matrix in Figure 2); for our purposes it is enough to observe that the upper left 5×5 submatrix of this Φ-matrix is:

$$
(46)\qquad
\begin{array}{c|ccccc}
 & \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} & \mathbf{e} \\
\hline
\pi_{Lie_{\mathcal{R},\Phi}}(1) & 1 & -(\Delta \otimes 1\!\!1^2)(\Phi) & 0 & 0 & 0 \\
\pi_{Lie_{\mathcal{R},\Phi}}(2) & 0 & 1 & -(1\!\!1^2 \otimes \Delta)(\Phi) & 0 & 0 \\
\pi_{Lie_{\mathcal{R},\Phi}}(3) & 0 & 0 & -(1 \otimes \Phi) & 1 & 0 \\
\pi_{Lie_{\mathcal{R},\Phi}}(4) & 0 & 0 & 0 & -(1\!\!1 \otimes \Delta \otimes 1\!\!1)(\Phi) & 1 \\
\pi_{Lie_{\mathcal{R},\Phi}}(5) & 1 & 0 & 0 & 0 & -(\Phi \otimes 1)
\end{array}
$$

Observe that this matrix coincides, up to the sign reversal, with the matrix (32).

The next step is to describe $\mathrm{Ker}(\pi_{Lie_{\mathcal{R},\Phi}})(4)$ as a $V^{\otimes 4}$-module. It is clear from the form of the matrix that, as in Example 9, the system of equations for $\mathrm{Ker}(\pi_{Lie_{\mathcal{R},\Phi}})(4)$ has the form

$$\alpha_j x_{i(j)} = x_{i'(j)},\ 1 \leq j \leq 15,$$

where j is the index for a column, i.e. an edge of the graph, and $i(j), i'(j)$ are the two vertices adjacent to that edge, i.e., the two rows with non-zero entries in that column. The consistency conditions have the form

$$\beta_i x_i = x_i$$

where β_i is a product of the α's going around a closed path with initial and terminal vertex i, which implies the pentagon identity (33) as in the previous example. Thus we must have both the pentagon and hexagon identities, and coherence of the operad $Lie_{\mathcal{R},\Phi}$ implies Mac Lane coherence. The converse is clear since the coefficients of a row in the matrix for $\pi_{Lie_{\mathcal{R},\Phi}}$ represent natural transformations between the three bracketings appearing in a Jacobi relation, and Mac Lane coherence implies the uniqueness of the natural transformation connecting any two bracketings.

More precisely, let $T_{\mathbf{xy}}$ be the element of $V^{\otimes 4}$ representing the natural transformation from a bracketing $\mathbf{x}$ to a bracketing $\mathbf{y}$. If we fix one bracketing, such as $\mathbf{a}$ in the table in Figure 2, multiplying each row of $\pi_{Lie_{\mathcal{R},\Phi}}$ by a suitable factor we get a new matrix with the two non-zero entries in column $\mathbf{i}$ given by $\pm T_{\mathbf{ai}}$. Thus there is the same kind of dependency among the rows as in the classical case.

Theorem 26. *The operad $Lie_{\mathcal{R},\Phi}$ is coherent if and only if $(V, \cdot, \Delta, \Phi, \mathcal{R})$ is a triangular quasi-Hopf algebra in the sense of* [2].

Example 27. Let us consider algebras consisting of a graded vector space U_* and a trilinear degree -1 product $\{-,-,-\} : U_*^{\otimes 3} \to U_*$ satisfying, for all homogeneous $a, b, c, d, e \in U_*$,

$$(47)\qquad \{\{a,b,c\},d,e\} + (-1)^{|a|}\{a,\{b,c,d\},e\}(-1)^{|a|+|b|} + \{a,b,\{c,d,e\}\} = 0\,.$$

The above means that $(U_*, \{-,-,-\})$ is an $A(\infty)$-algebra [17] with all structure operations μ_n trivial except $\mu_3 = \{-,-,-\}$. Let $\mathcal{A}$ be the non-Σ operad describing these algebras. It is not quadratic, but it is homogeneous in the sense that $\mathcal{A} = \langle E; R\rangle$ with $E = E(3)$, the one dimensional space generated by the product $\{-,-,-\}$, and $R \subset \mathcal{F}(E)(5)$ generated by the left hand side of (47). For these homogeneous operads, it still makes sense to introduce their !-duals, and an easy calculation gives

$$\mathcal{A}^{!}(n) = \begin{cases} \downarrow^n \mathbf{k}, & \text{for } n = 1 \pmod 2\text{, while} \\ 0, & \text{otherwise,} \end{cases}$$

where $\downarrow^n \mathbf{k}$ is the one-dimensional graded vector space concentrated in degree $-n$. It can also be shown that $\mathcal{A}$ is Koszul in a suitably generalized sense, thus $\mathsf{C}_{\mathcal{A}} = \mathsf{C}_{\mathcal{A}}(7) = \mathcal{A}^!(7)$ is one-dimensional. Therefore the kernel of the map

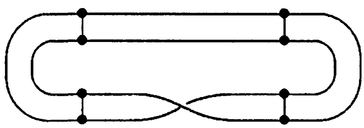
$$\pi(7) : \mathcal{F}(E)\langle R\rangle(7) \longrightarrow \mathcal{F}(E)(7)$$

is also of dimension one. In fact, $\dim \mathcal{F}(E)\langle R\rangle(7) = 8$, $\dim \mathcal{F}(E)(7) = 12$ and $\pi(7)$ is represented by the matrix

$$\begin{pmatrix} +1 & +1 & +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & +1 & +1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & +1 & +1 & +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & +1 & 0 & 0 & +1 & 0 & 0 & 0 & 0 & 0 & -1 \\ +1 & 0 & 0 & +1 & 0 & 0 & +1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & +1 & 0 & 0 & +1 & 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & 0 & +1 & +1 \end{pmatrix} \tag{48}$$

Since (48) has exactly two nontrivial entries in each column, the corresponding TA-structure is graphlike, with Tel-A-graph the 'Möbius strip:'

(49)



with 8 vertices corresponding to the rows of (48) and 12 edges corresponding to the columns of (48).

Let $V = (V, \cdot, 1, \epsilon)$ be a unital augmented algebra and let $\Delta = \Delta^3 : V \to V^{\otimes 3}$ be a homomorphism. Let us 'quantize' (47) by decorating it by some invertible $\Phi, \Psi \in V^{\otimes 3}$ to

$$\{\{a,b,c\},d,e\} + (-1)^{|a|}\Phi\{a,\{b,c,d\},e\} + (-1)^{|a|+|b|}\Psi\{a,b,\{c,d,e\}\} = 0\,.$$

It is an easy exercise to prove that the left V-invariance of this axiom implies that

$$(\mathbb{1}\otimes\Delta\otimes\mathbb{1})\Delta\cdot\Phi = \Phi\cdot(\Delta\otimes\mathbb{1}^2)\Delta \ \text{ and } (\mathbb{1}^2\otimes\Delta)\Delta\cdot\Psi = \Psi\cdot(\Delta\otimes\mathbb{1}^2)\Delta\,.$$

The coherence of the corresponding V-relative operad $\mathcal{A}_{\Phi,\Psi}$ then means that Φ and Ψ satisfy 6 equations corresponding to 6 generators of the fundamental group of (49). To get these equations would mean to evaluate the 'decorated' matrix for $\pi_{\Phi,\Psi}(7)$ from which these equations can be easily read off. We leave this to the reader.

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