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SYMMETRIZATION OF BRACE ALGEBRAS

MARILYN DAILY AND TOM LADA

ABSTRACT. We show that the symmetrization of a brace algebra structure yields the structure of a symmetric brace algebra. We also show that the symmetrization of the natural brace structure on $\bigoplus_{k\geq 1} \mathrm{Hom}\,(V^{\otimes k}, V)$ coincides with the natural symmetric brace structure on $\bigoplus_{k\geq 1} \mathrm{Hom}\,(V^{\otimes k}, V)^{as}$, the direct sum of spaces of antisymmetric maps $V^{\otimes k} \to V$.

1. INTRODUCTION

Brace algebras were first studied in the context of multilinear operations on the Hochschild complex of an associative algebra [3, 2, 1]. Symmetric brace algebras, in which the brace operations possess the property of graded symmetry, were subsequently introduced in [5]. Just as one may construct L_∞ algebra structures by anti (skew) symmetrizing A_∞ algebra structures [4], we show in this note that the symmetrization of a brace algebra structure yields a symmetric brace algebra structure. We prove in Section 5 that one may define a symmetric brace operation $\langle\ ,\ \rangle$ on a graded vector space with a given non symmetric brace operation $\{\ ,\ \}$ by

$$f\langle g_1, \ldots, g_n\rangle := \sum_{\sigma\in S_n} \epsilon(\sigma) f\{g_{\sigma(1)}, \ldots, g_{\sigma(n)}\}\,.$$

The motivating example of a brace algebra is $\bigoplus_{k\geq 1} \mathrm{Hom}\,(V^{\otimes k}, V)$, and the fundamental example of a symmetric brace algebra is the subspace of anti symmetric maps, $\bigoplus_{k\geq 1} \mathrm{Hom}\,(V^{\otimes k}, V)^{as}$.

In Section 6, we show that these algebras are related by

$$\sum_{\sigma\in S_n} \epsilon(\sigma) as(f\{g_{\sigma(1)}, \ldots, g_{\sigma(n)}\}) = as(f)\langle as(g_1), \ldots, as(g_n)\rangle\,,$$

where $f, g_i \in \mathrm{Hom}\,(V^{\otimes *}, V)$, $as(f)(v_1, \ldots, v_k) := \sum_{\sigma\in S_k} (-1)^\sigma \epsilon(\sigma) f(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$, and $\epsilon(\sigma)$ is just the Koszul sign of the permutation.

In Sections 2 and 3, we review the definitions and fundamental examples of brace algebras and symmetric brace algebras respectively. Section 4 contains a collection of

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technical lemmas that are needed to prove the main theorems in the final two sections. Throughout this article, S_n will denote the symmetric group on n symbols.

2. Brace algebras

Definition 1. A *brace structure* on a graded vector space consists of a collection of degree 0 multilinear braces $x, x_1, \ldots, x_n \mapsto x\{x_1, \ldots x_n\}$ which satisfy the identity, $x\{\ \} = x$, and in which $x\{x_1,\ldots,x_n\}\{y_1,\ldots,y_r\}$ is equal to

$$\sum \epsilon \cdot x\{y_1,\ldots,y_{i_1},x_1\{y_{i_1+1},\ldots,y_{j_1}\},y_{j_1+1},\ldots,y_{i_n},x_n\{y_{i_n+1},\ldots,y_{j_n}\},y_{j_n+1},\ldots,y_r\}.$$

In the above formula, the sum is over all sequences $0 \le i_1 \le j_1 \le \ldots \le i_n \le j_n \le r$, and ϵ is the Koszul sign of the permutation which maps $(x_1,\ldots,x_n,y_1,\ldots,y_r)$ to

$$(y_1,\ldots,y_{i_1},x_1,y_{i_1+1},\ldots,y_{j_1},y_{j_1+1},\ldots,y_{i_n},x_n,y_{i_n+1},\ldots,y_{j_n},y_{j_n+1},\ldots,y_r).$$

The motivating example for a brace algebra structure is the space $\bigoplus \operatorname{Hom}(V^{\otimes k}, V)$ with the natural brace operation of degree $-n$ given by the composition

$$f\{g_1,\ldots,g_n\} = \sum_{k_0+\cdots+k_n=N-n} f\left(1^{\otimes k_0} \otimes g_1 \otimes 1^{\otimes k_1} \otimes \cdots \otimes g_n \otimes 1^{\otimes k_n}\right),$$

where $f \in \operatorname{Hom}(V^{\otimes N}, V)$. This operation arises from the endomorphism operad of V considered in [1]. This operation was also utilized in the context of the Hochschild complex of an associative algebra V in [3] and [2]. For maps of arbitrary degree p, we have

Example 2. Let V be a graded vector space and consider the graded vector space $B_*(V)$ where

$$B_s(V) := \bigoplus_{p-k+1=s} \operatorname{Hom}(V^{\otimes k}, V)_p$$

and where $\operatorname{Hom}(V^{\otimes k}, V)_p$ denotes the space of k-multilinear maps of degree p. Given $f \in \operatorname{Hom}(V^{\otimes N}, V)_p$ and $g_i \in \operatorname{Hom}(V^{\otimes a_i}, V)_{q_i}$, define

$$f\{g_1,\ldots,g_n\} \in \operatorname{Hom}(V^{\otimes r}, V)_{p+q_1+\cdots+q_n}$$

where $r = a_1 + \cdots + a_n + N - n$ by

$$f\{g_1,\ldots,g_n\} = \sum_{k_0+\cdots+k_n=N-n} (-1)^{\beta} f(1^{\otimes k_0} \otimes g_1 \otimes 1^{\otimes k_1} \otimes \cdots \otimes 1^{\otimes k_{n-1}} \otimes g_n \otimes 1^{\otimes k_n}),$$

where

$$\beta = \sum_{j<i} [a_i - 1]\,[k_j + a_j] + \sum_i (N-i)\, q_i + \sum_{j<i} q_i\, a_j.$$

Remark 3. In Example 2, suppose that there exists a collection of maps

$$\mu_k \in \operatorname{Hom}(V^{\otimes k}, V)_{k-2} \in B_{-1}(V).$$

If we let $\mu = \mu_1 + \mu_2 + \ldots$, then an A_∞ algebra structure on V may be described by the brace relation $\mu\{\mu\} = 0$ [5].

3. Symmetric brace algebras

Definition 4. An *n-unshuffle* of N elements is a partition $\sum_{i=1}^{n} a_i = N$ and a permutation $\gamma \in S_N$ such that

$$\gamma(1) < \cdots < \gamma(a_1), \gamma(1+a_1) < \cdots < \gamma(a_2+a_1), \ldots, \gamma\Big(1+\sum_{i=1}^{n-1} a_i\Big) < \cdots < \gamma(N).$$

If we wish to emphasize the partition, we call such a permutation an $(a_1|a_2|\ldots|a_n)$ unshuffle.

Definition 5. A *symmetric brace algebra* is a graded vector space together with a collection of degree zero multilinear braces $f\langle g_1, \ldots, g_n\rangle$ which are graded symmetric in $g_1, \ldots, g_n$. In a symmetric brace algebra, it is also required that $f\langle\ \rangle = f$, and that $f\langle g_1, \ldots, g_n\rangle\langle x_1, \ldots, x_r\rangle$ be equal to

$$\sum_{\substack{\gamma \text{ is } (n+1)\\ \text{unshuffle}}} \epsilon \cdot f\langle g_1\langle x_{\gamma(1)}, \ldots, x_{\gamma(a_1)}\rangle, \ldots, g_n\langle x_{\gamma(1+\sum_{i=1}^{n-1} a_i)}, \ldots, x_{\gamma(\sum_{i=1}^{n} a_i)}\rangle, x_{\gamma(1+\sum_{i=1}^{n} a_i)}, \ldots, x_{\gamma(r)}\rangle,$$

where ϵ is the Koszul sign of the permutation which maps $(g_1, \ldots, g_n, x_1, \ldots, x_r)$ to

$$\big(g_1, x_{\gamma(1)}, \ldots, x_{\gamma(a_1)}, g_2, \ldots, x_{\gamma(1+\sum_{i=1}^{n-1} a_i)}, \ldots, x_{\gamma(\sum_{i=1}^{n} a_i)}, g_n, x_{\gamma(1+\sum_{i=1}^{n} a_i)}, \ldots, x_{\gamma(r)}\big).$$

Just as with brace algebras, the fundamental example of a symmetric brace algebra is provided by the space of antisymmetric maps of degree p, $\bigoplus \mathrm{Hom}\,(V^{\otimes k}, V)_p^{as}$. To be precise, we have

Example 6. Let V be a graded vector space and $B_*(V)$ be the graded vector space given by

$$B_s(V) = \bigoplus_{p-k+1=s} \mathrm{Hom}\,(V^{\otimes k}, V)_p^{as},$$

Given $f \in \mathrm{Hom}\,(V^{\otimes N}, V)_p^{as}$ and $g_i \in \mathrm{Hom}\,(V^{\otimes a_i}, V)_{q_i}^{as}$, $1 \leq i \leq n$, define the symmetric brace

$$f\langle g_1, \ldots, g_n\rangle(x_1, \ldots, x_r) = (-1)^\delta \sum_{\substack{\gamma \text{ is an}\\ (a_1|a_2|\ldots|a_{n+1})\\ \text{unshuffle}}} \chi(\gamma) f(g_1 \otimes \cdots \otimes g_n \otimes 1^{\otimes N-n})(x_{\gamma(1)}, \ldots, x_{\gamma(r)}),$$

where

$$\delta = \sum_{i}^{n} (N-i)q_i + \sum_{j<i} q_i a_j + \sum_{j<i} a_i a_j + \sum_{i} (n-i)a_i,$$

and $\chi(\gamma) := \mathrm{sgn}\,(\gamma)e(\gamma)$ is the antisymmetric Koszul sign of the permutation γ.

Remark 7. Suppose that in Example 6 we have maps

$$l_k \in \mathrm{Hom}\,(V^{\otimes k}, V)_{k-2}^{as} \in B_{-1}(V).$$

If we let $l = l_1 + l_2 + \ldots$, then an L_∞ algebra structure on V is given by the symmetric brace relation $l\langle l\rangle = 0$.

4. Some lemmas

Although the expressions in this paper involve many sums, permutations, and antisymmetrizations, we will be able to simplify things considerably with the help of the following lemmas. Lemma 8 provides a decomposition of $as(f)$ which will be useful later.

Lemma 8. *$as(f) = f \circ \Phi_{nm} \circ \Psi_n \circ \Theta_m \quad \forall\ f \in \operatorname{Hom}(V^{\otimes n+m}, V)$, where*

$$\Theta_m(y_1, \ldots, y_n, z_1, \ldots z_m) = \sum_{\pi \in S_m} \chi(\pi)(y_1, \ldots, y_n, z_{\pi(1)}, \ldots z_{\pi(m)}),$$

$$\Psi_n(y_1, \ldots, y_n, z_1, \ldots z_m) = \sum_{\sigma \in S_n} \chi(\sigma)(y_{\sigma(1)}, \ldots, y_{\sigma(n)}, z_1, \ldots z_m),$$

$$\Phi_{nm}(y_1, \ldots, y_n, z_1, \ldots, z_m) = \sum_{k_0+\cdots+k_n=m} (-1)^{\eta}\,(z_1, \ldots, z_{k_0}, y_1, z_{1+k_0}, \ldots, y_n, z_{1+k_0} + \cdots + k_{n-1}, \ldots, z_m)\ ,$$

and $\eta = \sum_{i=1}^{n} \{y_i\,[z_1 + \cdots + z_{(k_0+k_1+\cdots+k_{i-1})}] + (n-i)k_i\}$.

Proof. Since Ψ_n does all permutations of the first n inputs, Θ_m provides all permutations of the last m inputs, and Φ_{nm} distributes the last n variables between the first m in every possible way, the composition is clearly a sum of all permutations of the original $n+m$ variables. A moment's reflection also reveals that the sign of each summand in the composition is the Koszul sign together with the sign of the permutation. □

Lemma 9 states that if we sum over all (signed) $(a_1|\ldots|a_n)$ unshuffles, and then sum over all (signed) permutations of the a_i variables in each piece, then this is equivalent to just summing over all signed permutations of the original $a_1 + \cdots + a_n$ variables.

Lemma 9. *If $N = a_1 + \cdots + a_n$, then $\sum_{\pi \in S_N} \chi(\pi)\left(x_{\pi(1)}, \ldots, x_{\pi(N)}\right)$ is equal to*

$$\sum_{\substack{\gamma \text{ is} \\ (a_1|\ldots|a_n) \\ \text{unshuffle}}} \chi(\gamma) \sum_{\pi_1 \in S_{a_1}} \chi(\pi_1) \cdots \sum_{\pi_{a_n} \in S_{a_n}} \chi(\pi_n)\big(x_{\gamma(\pi(1))}, \ldots, x_{\gamma(\pi_1(a_1))}, x_{\gamma(\pi_2(1)+a_1)}, \ldots, x_{\gamma\left(\pi_n(a_n)+\sum_{i=1}^{n-1} a_i\right)}\big)\,.$$

Proof. Clearly, the right hand side is the sum of distinct permutations of the x terms with the correct sign. Furthermore, since there are $\frac{N!}{(a_1)!\ldots(a_n)!}$ unshuffles γ and $(a_i)!$ permutations π_i, there are $N!$ summands in the right hand side, which agrees with the number of summands on the left hand side. □

Lemma 10. *Suppose $k_0 + a_1 + k_1 + \cdots + a_n + k_n = r$, $\sigma \in S_n$, and $\pi \in S_r$. Let $A = a_1 + \ldots a_n$, denote $X_i = x_{\pi(1+a_1+\cdots+a_{i-1})}, \ldots, x_{\pi(a_1+\cdots+a_i)}$, and also denote $X_\pi = x_{\pi(1+A)}, \ldots, x_{\pi(k_0+A)}, X_{\sigma(1)}, x_{\pi(1+k_0+A)}, \ldots, X_{\sigma(n)}, x_{\pi(1+k_0+\cdots+k_{n-1}+A)}, \ldots, x_{\pi(r)}$. Then we* *can define* $\hat{\pi} \in S_r$ *by*

$$\hat{\pi}(i) = \begin{cases} \pi\Big(i + A - \sum_{j\le m} a_{\sigma(j)}\Big) & \text{if } \sum_{j<m} k_j + \sum_{j\le m} a_{\sigma(j)} < i \\ & \quad \le \sum_{j\le m} k_j + \sum_{j\le m} a_{\sigma(j)}; \\ \pi\Big(i - \sum_{j<m} k_j + \sum_{j<\sigma(m)} a_j\Big) & \text{if } \sum_{j<m} k_j + \sum_{j<m} a_{\sigma(j)} < i \\ & \quad \le \sum_{j<m} k_j + \sum_{j\le m} a_{\sigma(j)}. \end{cases}$$

Furthermore, given this notation,

$$X_\pi = x_{\hat{\pi}(1)}, \dots, x_{\hat{\pi}(r)} \quad \text{and} \quad \epsilon(\hat{\pi}) = \epsilon(\pi)(-1)^{\alpha_1} \quad \text{and} \quad \chi(\hat{\pi}) = \chi(\pi)(-1)^{\alpha_2},$$

where

$$\alpha_1 = \sum_{i<j \ \& \ \sigma(i)>\sigma(j)} |X_{\sigma(i)}|\,|X_{\sigma(j)}| + \sum_{i=1}^{n} |X_{\sigma(i)}| \left[x_{\pi(1+A)} + \cdots + x_{\pi(k_0+\cdots+k_{i-1}+A)}\right]$$

and

$$\alpha_2 = \alpha_1 + \sum_{i<j \ \& \ \sigma(i)>\sigma(j)} a_{\sigma(i)} a_{\sigma(j)} + \sum_{j<i} a_{\sigma(i)} k_j .$$

Proof. Careful examination of the definition of $\hat{\pi}$ reveals that the first formula moves "free" strings of the form $x_{\pi(1+k_0+\cdots+k_{i-1})}, \dots, x_{\pi(k_0+\cdots+k_i)}$ into place (for $0 \le m \le n$), and the second formula relocates the strings $X_{\sigma(i)}$ (for $1 \le m \le n$). Thus $X_\pi = x_{\hat{\pi}(1)}, \dots, x_{\hat{\pi}(r)}$.

Furthermore, when $x_{\pi(1)}, \dots, x_{\pi(r)}$ are permuted to yield $x_{\hat{\pi}(1)}, \dots, x_{\hat{\pi}(r)}$, the Koszul sign is $(-1)^{\alpha_1}$, where the first sum in α_1 comes from σ permuting the X_i strings, and the second sum comes from moving the "free" strings into place. Finally, the additional sums in α_2 count the transpositions, yielding the correct antisymmetric Koszul sign. □

Lemma 11. *Suppose that* $\sigma \in S_n$ *permutes* $\{v_1 \dots v_n\}$ *and* $\{w_1 \dots w_n\}$. *Then*

$$(1) \qquad \sum_{i>j} v_i w_j + \sum_{i<j \ \& \ \sigma(i)>\sigma(j)} \{w_{\sigma(i)} v_{\sigma(j)} + v_{\sigma(i)} w_{\sigma(j)}\} + \sum_{i>j} v_{\sigma(i)} w_{\sigma(j)} \equiv 0 \pmod 2;$$

$$(2) \qquad \sum_{i<j \ \& \ \sigma(i)>\sigma(j)} \{v_{\sigma(i)} + v_{\sigma(j)}\} \equiv \sum_i (i-1) v_i + \sum_i (i-1) v_{\sigma(i)} \pmod 2.$$

Proof. To prove the first assertion, we note that

$$\sum_{i<j \ \& \ \sigma(i)>\sigma(j)} \{v_{\sigma(i)} w_{\sigma(j)} + w_{\sigma(i)} v_{\sigma(j)}\} + \sum_{i>j} v_{\sigma(i)} w_{\sigma(j)} = \sum_{i<j \& \sigma(i)>\sigma(j)} v_{\sigma(i)} w_{\sigma(j)} + \sum_{i>j \& \sigma(i)<\sigma(j)} v_{\sigma(i)} w_{\sigma(j)} + \sum_{i>j} v_{\sigma(i)} w_{\sigma(j)},$$

which is congruent $\pmod 2$ to

$$\sum_{i<j \& \sigma(i)>\sigma(j)} v_{\sigma(i)} w_{\sigma(j)} + \sum_{i>j \& \sigma(i)>\sigma(j)} v_{\sigma(i)} w_{\sigma(j)} = \sum_{\sigma(i)>\sigma(j)} v_{\sigma(i)} w_{\sigma(j)} = \sum_{i>j} v_i w_j .$$

To prove the second statement, suppose that all w_i are odd. Then

$$\sum_{i<j \ \& \ \sigma(i)>\sigma(j)} \left\{v_{\sigma(i)} + v_{\sigma(j)}\right\} \equiv \sum_{i<j \ \& \ \sigma(i)>\sigma(j)} \left\{v_{\sigma(i)}w_{\sigma(j)} + w_{\sigma(i)}v_{\sigma(j)}\right\}$$
$$\equiv \sum_{i>j} \left\{v_i w_j + v_{\sigma(i)}w_{\sigma(j)}\right\}$$

(by the first assertion). Since all w-terms are odd, this is congruent to

$$\sum_{j=1}^{n} \sum_{i=j+1}^{n} (v_i + v_{\sigma(i)}) = \sum_j (j-1)v_j + \sum_j (j-1)v_{\sigma(j)} \,. \qquad \square$$

5. Symmetrization of brace algebras

Given a (non-symmetric) brace structure $\{\ ,\ \}$ on a graded vector space, we can define a symmetric brace structure $\langle\ ,\ \rangle$ via

$$f\langle g_1, \ldots, g_n\rangle := \sum_{\sigma \in S_n} \epsilon(\sigma) f\{g_{\sigma(1)}, \ldots, g_{\sigma(n)}\} \,.$$

Clearly, this satisfies the first symmetric brace axiom, since $f\langle\ \rangle = f\{\ \} = f$. We show in Theorem 15 that it satisfies the second symmetric brace axiom given in Definition 5, so this does in fact induce a symmetric brace structure. First, however, we need the following two lemmas, which are analogous to Lemmas 8 and 9.

Lemma 12. $\sum_{\rho \in S_{n+m}} \epsilon(\rho)\, f\{x_{\rho(1)}, \ldots, x_{\rho(n)}\} = \tilde{f}_n \circ \theta_m(x_1, \ldots, x_{n+m})$, *where* $\Theta_m(y_1, \ldots, y_n, z_1, \ldots z_m) = \sum_{\pi \in S_m} \epsilon(\pi)(y_1, \ldots, y_n, z_{\pi(1)}, \ldots z_{\pi(m)})$ *and*

$$\tilde{f}_n(y_1, \ldots, y_n, z_1, \ldots, z_m)$$
$$= \sum_{\sigma \in S_n} \epsilon(\sigma) \sum_{k_0+\cdots+k_n=m} (-1)^{\eta} f\left\{z_1, \ldots, z_{k_0}, y_{\sigma(1)}, z_{1+k_0}, \ldots, y_{\sigma(n)}, z_{1+k_0+\cdots+k_{n-1}}, \ldots, z_m\right\} ,$$

with a Koszul sign given by $\eta = \sum_{i=1}^{n} y_{\sigma(i)} \left[z_1 + \cdots + z_{(k_0+k_1+\cdots+k_{i-1})}\right]$.

Lemma 13. *If* $N = a_1 + \cdots + a_n$, *then* $\sum_{\pi \in S_N} \epsilon(\pi) \left(x_{\pi(1)}, \ldots, x_{\pi(N)}\right)$ *is equal to*

$$\sum_{\substack{\gamma \text{ is} \\ (a_1|\ldots|a_n) \\ \text{unshuffle}}} \epsilon(\gamma) \sum_{\pi_1 \in S_{a_1}} \epsilon(\pi_1) \ldots \sum_{\pi_{a_n} \in S_{a_n}} \epsilon(\pi_n) \Big(x_{\gamma(\pi(1))}, \ldots, x_{\gamma(\pi_1(a_1))}, x_{\gamma(\pi_2(1)+a_1)},$$
$$\ldots, x_{\gamma\left(\pi_n(a_n)+\sum_{i=1}^{n-1} a_i\right)} \Big) \,.$$

Remark 14. Although a brace structure allows operators g which accept an arbitrary number of inputs, it will be convenient in the proof of the following theorem to let g^a denote the restriction of g which accepts only exactly a inputs.

Theorem 15. *Given a (non-symmetric) brace structure* $\{\ ,\ \}$ *on a graded vector space, define* $\langle\ ,\ \rangle$ *via*

$$f\langle g_1,\dots,g_n\rangle := \sum_{\sigma\in S_n}\epsilon(\sigma)f\{g_{\sigma(1)},\dots,g_{\sigma(n)}\}\,.$$

Then $f\langle g_1,\dots,g_n\rangle\langle x_1,\dots,x_r\rangle$ *is equal to*

$$\sum_{\substack{\gamma \text{ is } (n+1)\\ \text{unshuffle}}} \epsilon\cdot f\langle g_1\rangle x_{\gamma(1)},\dots,x_{\gamma(a_1)}\rangle,\dots,g_n\langle x_{\gamma\left(1+\sum_{i=1}^{n-1}a_i\right)},$$
$$\dots,x_{\gamma\left(\sum_{i=1}^{n}a_i\right)}\rangle,x_{\gamma\left(1+\sum_{i=1}^{n}a_i\right)},\dots,x_{\gamma(r)}\rangle\,,$$

where ϵ *is the Koszul sign of the permutation which maps* $(g_1,\dots,g_n,x_1,\dots,x_r)$ *to*

$$\left(g_1,x_{\gamma(1)},\dots,x_{\gamma(a_1)},g_2,\dots,x_{\gamma\left(1+\sum_{i=1}^{n-1}a_i\right)},\dots,x_{\gamma\left(\sum_{i=1}^{n}a_i\right)},g_n,x_{\gamma\left(1+\sum_{i=1}^{n}a_i\right)},\dots,x_{\gamma(r)}\right)\,.$$

Proof. First, we will look at the right hand side.
If we temporarily denote

$$\begin{aligned} h_k &= g_k\langle x_{\gamma(1+a_1+\dots+a_{k-1})},\dots,x_{\gamma(a_1+\dots+a_k)}\rangle \\ &= \sum_{\pi_k\in S_{a_k}}\epsilon(\pi_k)\ g_k\left\{x_{\gamma(\pi_k(1)+a_1+\dots+a_{k-1})},\dots,x_{\gamma(\pi_k(a_k)+a_1+\dots+a_{k-1})}\right\}, \end{aligned}$$

and denote $A=\sum_{i=1}^{n}a_i$, then the right hand side is equal to

$$\sum_{\substack{a_1+\dots+a_{n+1}=r\ \&\\ \gamma \text{ is } (a_1|\dots|a_{n+1}) \text{ unshuffle}}} (-1)^{\nu}\ \epsilon(\gamma)\ f\langle h_1,\dots,h_n,x_{\gamma(1+A)},\dots,x_{\gamma(a_{n+1}+A)}\rangle\,,$$

where $\nu=\sum_{i=2}^{n}g_i[x_{\gamma(1)}+\dots+x_{\gamma(a_1+\dots+a_{i-1})}]$ is a Koszul sign. After applying Lemma 12, this is equal to

$$\sum_{\substack{a_1+\dots+a_{n+1}=r,\\ \gamma \text{ is unshuffle}}}(-1)^{\nu}\,\epsilon(\gamma)\,\tilde{f}_n\Big(\sum_{\pi_{n+1}\in S_{a_{n+1}}}\epsilon(\pi_{n+1})(h_1,\dots,h_n,x_{\gamma(\pi_{n+1}(1)+A)},\dots,x_{\gamma(\pi_{n+1}(a_n+1)+A)}\Big)\,,$$

where $\tilde{f}_n$ is as defined in Lemma 12. Now, we will pull all of the x terms back out, in order to apply Lemma 13. Note that the Koszul signs from this transformation merely cancel out $(-1)^{\nu}$. We then have the following long formula:

$$\begin{aligned} \sum_{(a_i),\gamma}\epsilon(\gamma)\sum_{\pi_1\in S_{a_1}}\epsilon(\pi_1)\dots\sum_{\pi_{(n+1)}\in S_{a_{n+1}}}&\epsilon(\pi_{n+1})\tilde{f}_n\left(g_1^{a_1},\dots,g_n^{a_n},1^{a_{n+1}}\right)\\ &\times\left(x_{\gamma(\pi_1(1))},\dots,x_{\gamma(\pi_1(a_1))},x_{\gamma(\pi_2(1)+a_1)},\right.\\ &\left.\dots,x_{\gamma(\pi_n(A)},x_{\gamma(\pi_{n+1}(1)+A)},\dots x_{\gamma(\pi_{n+1}(a_n+1)+A)}\right)\,. \end{aligned}$$

Now, though, we can apply Lemma 13, which yields the much shorter formula,

$$\sum_{(a_i)}\sum_{\pi\in S_r}\epsilon(\pi)\,\tilde{f}_n\left(g_1^{a_1},\dots,g_n^{a_n},1^{a_{n+1}}\right)\left(x_{\pi(1)},\dots,x_{\pi(r)}\right)\,.$$

Before continuing, we need to pull all of the x terms back inside. In order to make our expressions a bit shorter, let X_i denote the input to g_i. In other words, define

$$X_i = x_{\pi(1+a_1+\cdots+a_{i-1})}, \ldots, x_{\pi(a_1+\cdots+a_i)} \quad \text{for} \quad i \in \{1 \ldots n\}.$$

It will also be convenient to let $|X_i|$ denote the sum of the degrees of the variables in X_i. When we pull the x-terms inside and use the more concise notation just defined, the formula for the right hand side becomes

$$\sum_{(a_i)} \sum_{\pi \in S_r} \epsilon(\pi)\,(-1)^{\tilde{\nu}} \tilde{f}_n\big(g_1(X_1), \ldots, g_n(X_n), x_{\pi(1+A)}, \ldots, x_{\pi(r)}\big) ,$$

where $\tilde{\nu} = \sum_{j<i} g_i|X_j|$. After expanding $\tilde{f}_n$, the right hand side is equal to

$$\sum_{(a_i)} \sum_{\pi \in S_r} \epsilon(\pi)\,(-1)^{\tilde{\nu}} \sum_{\sigma \in S_n} \epsilon(\sigma) \sum_{k_0+\cdots+k_n=a_n+1} (-1)^{\eta} f\{x_{\pi(1+A)}, \ldots, x_{\pi(k_0+A)}, g_{\sigma(1)}(X_{\sigma(1)}), x_{\pi(1+k_0+A)}, \ldots, g_{\sigma(n)}(X_{\sigma(n)}), x_{\pi(1+k_0+\cdots+k_n+A)}, \ldots, x_{\pi(r)}\} .$$

Here, $\eta = \sum_{i=1}^{n} \big(g_{\sigma(i)} + |X_{\sigma(i)}|\big)\,\big[x_{\pi(1+A)} + \cdots + x_{\pi(k_0+\cdots+k_{i-1}+A)}\big]$ and $\epsilon(\sigma) = (-1)^{\lambda}$, where $\lambda = \sum_{i<j \ \&\ \sigma(i)>\sigma(j)} \big(g_{\sigma(i)} + |X_{\sigma(i)}|\big)\big(g_{\sigma(j)} + |X_{\sigma(j)}|\big)$.

Now, we will look at the left hand side. $f\langle g_1, \ldots, g_n\rangle\langle x_1, \ldots, x_r\rangle$ is equal to $\sum_{\sigma \in S_n} \epsilon(\sigma) f\{g_{\sigma(1)}, \ldots, g_{\sigma(n)}\}\langle x_1, \ldots, x_r\rangle$, which is equal to

$$\sum_{\sigma \in S_n} \epsilon(\sigma) \sum_{\pi \in S_r} \epsilon(\pi)\ f\{g_{\sigma(1)}, \ldots, g_{\sigma(n)}\}\{x_{\pi(1)}, \ldots, x_{\pi(r)}\} .$$

If we apply Definition 1 and let $g_i^{a_i}$ denote the restriction of g_i which accepts exactly a_i inputs, then the left hand side is equal to

$$\sum_{\sigma \in S_n} \epsilon(\sigma) \sum_{\pi \in S_r} \epsilon(\pi) \sum_{k_0+\cdots+k_n+a_1+\cdots+a_n=r} f\{1^{k_0}, g_{\sigma(1)}^{a_{\sigma}(1)}, 1^{k_1}, \ldots, g_{\sigma(n)}^{a_{\sigma}(n)}, 1^{k_n}\}(x_{\pi(1)}, \ldots, x_{\pi(r)}) .$$

After applying Lemma 10, this is equal to

$$\sum_{\sigma \in S_n} \epsilon(\sigma) \sum_{(k_i,a_i)} \sum_{\pi \in S_r} \epsilon(\pi)\ (-1)^{\alpha_1} f\{1^{k_0}, g_{\sigma(1)}^{a_{\sigma}(1)}, 1^{k_1}, \ldots, g_{\sigma(n)}^{a_{\sigma}(n)}, 1^{k_n}\} \times \big(x_{\pi(1+A)}, \ldots, x_{\pi(k_0+A)}, X_{\sigma(1)}, x_{\pi(1+k_0+A)}, \ldots, X_{\sigma(n)}, x_{\pi(1+k_0+\cdots+k_n+A)}, \ldots, x_{\pi(r)}\big) ,$$

where α_1 is given in Lemma 10. Finally, when the x-terms are moved inside, the left hand side is equal to

$$\sum_{\sigma \in S_n} \epsilon(\sigma) \sum_{(k_i,a_i)} \sum_{\pi \in S_r} \epsilon(\pi)(-1)^{\alpha_1+\mu} f\{x_{\pi(1+A)}, \ldots, x_{\pi(k_0+A)}, g_{\sigma(1)}(X_{\sigma(1)}), x_{\pi(1+k_0+A)}, \ldots, g_{\sigma(n)}(X_{\sigma(n)}), x_{\pi(1+k_0+\cdots+k_n+A)}, \ldots, x_{\pi(r)}\} .$$

Here, $\mu = \sum_i g_{\sigma(i)}[x_{\pi(1+A)} + \cdots + x_{\pi(k_0+\cdots+k_{i-1}+A)}] + \sum_{j<i} g_{\sigma(i)}|X_{\sigma(j)}|$ and $\epsilon(\sigma) = (-1)^\zeta$, where $\zeta = \sum_{i<j \ \&\ \sigma(i)>\sigma(j)} g_{\sigma(i)}g_{\sigma(j)}$.

Now that the terms on both sides are easy to compare, it is clear that the two sides are equal if and only if $\tilde{\nu} + \lambda + \eta + \zeta + \alpha_1 + \mu \equiv 0 \pmod 2$.

After making the most obvious cancellations, we see that $\tilde{\nu} + \lambda + \eta + \zeta + \alpha_1 + \mu$ is congruent to

$$\sum_{j<i} g_i|X_j| + \sum_{i<j \ \&\ \sigma(i)>\sigma(j)} \left(g_{\sigma(i)}|X_{\sigma(j)}| + g_{\sigma(j)}|X_{\sigma(i)}|\right) + \sum_{j<i} g_{\sigma(i)}|X_{\sigma(j)}|,$$

which is congruent to zero $\pmod 2$ by Lemma 11. □

6. Symmetrization of the brace structure on $\bigoplus_{k\geq 1} \mathrm{Hom}\,(V^{\otimes k}, V)$

In this section, we will demonstrate a nice relationship between the the brace defined in Example 2 and the symmetric brace defined in Example 6, by showing that the symmetrization of the non symmetric brace structure on $Hom(V^{\otimes k}, V)$ is equal to the symmetric brace of the anti-symmetrized maps. Specifically, we have

Theorem 16. $\sum_{\sigma\in S_n} \epsilon(\sigma)\, as\,(f\{g_{\sigma(1)}, \ldots, g_{\sigma(n)}\}) = as(f)\langle as(g_1), \ldots, as(g_n)\rangle$.

Proof. First, we will manipulate the right hand side. Using the symmetric brace structure defined in Example 6, $as(f)\langle as\,(g_1), \ldots, as\,(g_n)\rangle(x_1, \ldots, x_r)$ is equal to

$$(-1)^\delta \sum_{\substack{\gamma \text{ is an} \\ (a_1|a_2|\ldots|a_n+1) \\ \text{unshuffle}}} \chi(\gamma) as\,(f)\left(as\,(g_1) \otimes \cdots \otimes as\,(g_n) \otimes 1^{\otimes N-n}\right)(x_{\gamma(1)}, \ldots, x_{\gamma(r)}),$$

where δ is given in Example 6.

When we substitute the x terms using the Koszul convention and suppress the tensor notation, this is equal to

$$(-1)^\delta \sum_\gamma \chi(\gamma)(-1)^\nu as\,(f)(h_1, \ldots, h_n, x_{\gamma(1+\sum_{i=1}^n a_i))}, \ldots, x_{\gamma(r)}),$$

where

$$\nu = \sum_{i=2}^n q_i[x_{\gamma(1)} + \cdots + x_{\gamma(a_1+\cdots+a_{i-1})}]$$

and

$$\begin{aligned} h_k &= as\,(g_k)\left(x_{\gamma(1+a_1+\cdots+a_{k-1})}, \ldots, x_{\gamma(a_1+\cdots+a_k)}\right) \\ &= \sum_{\pi_k\in S_{a_k}} \chi(\pi_k) g_k\left(x_{\gamma(\pi_k(1)+a_1+\cdots+a_{k-1})}, \ldots, x_{\gamma(\pi_k(a_k)+a_1+\cdots+a_{k-1})}\right). \end{aligned}$$

If we denote $A = \sum_{i=1}^{n} a_i$ and apply Lemma 8, this is equal to

$$\sum_{\gamma} \chi(\gamma)(-1)^{\delta+\nu} f \circ \Phi_{na} \circ \Psi_n\Big(\sum_{\pi_{a_n+1} \in S_{a_{n+1}}} \chi(\pi_{n+1})(h_1, \dots, h_n, x_{\gamma(\pi_{a_n+1}(1)+A)}, \dots, x_{\gamma(\pi_{a_n+1}(a_n+1)+A)}\Big) .$$

Now, we will pull all of the x terms back out, in order to apply Lemma 9. Note that the Koszul signs from this transformation merely cancel out $(-1)^{\nu}$. We then have the following long formula,

$$(-1)^{\delta} \sum_{\gamma} \chi(\gamma) \sum_{\pi_1 \in S_{a_1}} \chi(\pi_1) \dots \sum_{\pi_{(a_n+1)} \in S_{a_{n+1}}} \chi(\pi_{n+1}) f \circ \Phi_{na} \circ \Psi_n(g_1, \dots, g_n, 1^{a_n+1})$$
$$\big(x_{\gamma(\pi_1(1))}, \dots, x_{\gamma(\pi_1(a_1))}, x_{\gamma(\pi_2(1)+a_1)}, \dots, x_{\gamma(\pi_{a_n}(A)}, x_{\gamma(\pi_{a_n+1}(1)+A)}, \dots, x_{\gamma(\pi_{a_n+1}(a_n+1)+A)}\big) .$$

Now, though, we can apply Lemma 9, which yields the much shorter formula,

$$(-1)^{\delta} \sum_{\pi \in S_r} \chi(\pi) f \circ \Phi_{na} \circ \Psi_n(g_1, \dots, g_n, 1^{a_n+1}) \big(x_{\pi(1)}, \dots, x_{\pi(r)}\big) .$$

Before continuing, we need to pull all of the x terms back inside. In order to make our expressions a bit shorter, let X_i denote the input to g_i, and let X_{n+1} denote the free x terms (letting $a_{n+1} = N-n$). In other words, define

$$X_i = x_{\pi(1+a_1+\dots+a_{i-1})}, \dots, x_{\pi(a_1+\dots+a_i)} .$$

It will also be convenient to let $|X_i|$ denote the sum of the degrees of the variables in X_i. When we pull the x-terms inside and use the more concise notation just defined, the formula for the right hand side becomes

$$(-1)^{\delta} \sum_{\pi \in S_r} \chi(\pi)(-1)^{\tilde{\nu}} f \circ \Phi_{n,N-n} \circ \Psi_n\big(g_1(X_1), \dots, g_n(X_n), X_{n+1}\big) ,$$

where $\tilde{\nu} = \sum_{j<i} q_i |X_j|$. After expanding Ψ_n, the right hand side is equal to

$$(-1)^{\delta} \sum_{\pi \in S_r} \chi(\pi)(-1)^{\tilde{\nu}} f \circ \Phi_{n,N-n}\Big(\sum_{\sigma \in S_n} \chi(\sigma)\big(g_{\sigma(1)}(X_{\sigma(1)}), \dots, g_{\sigma(n)}(X_{\sigma(n)}), X_{n+1}\big)\Big) .$$

In the above expression, $\chi(\sigma)$ is equal to $(-1)^{\lambda}$, where

$$\lambda = \sum_{\substack{i<j\le n, \\ \sigma(i)>\sigma(j)}} \big[\big(q_{\sigma(i)} + |X_{\sigma(i)}|\big)\big(q_{\sigma(j)} + |X_{\sigma(j)}|\big) + 1\big] .$$

Now, if we expand $\Phi_{n,N-n}$, we get

$$\sum_{\substack{\pi \in S_r, \sigma \in S_n, \\ k_0+\dots+k_n=N-n}} \chi(\pi)(-1)^{\delta+\tilde{\nu}+\lambda+\eta} f\big(x_{\pi(1+A)}, \dots, x_{\pi(k_0+A)}, g_{\sigma(1)}\big(X_{\sigma(1)}\big), x_{\pi(1+k_0+A)}, \dots, g_{\sigma(n)}\big(X_{\sigma(n)}\big), x_{\pi(1+k_0+\dots+k_n+A)}, \dots, x_{\pi(r)}\big) ,$$

where

$$\eta = \sum_{i=1}^{n} \left\{ \left(q_{\sigma(i)} + |X_{\sigma(i)}|\right)\left(x_{\pi(1+A)} + \cdots + x_{\pi(k_0+\cdots+k_{i-1}+A)}\right) + (n-i)k_i \right\}.$$

Now, we will work with the left hand side of the equation. Using the brace defined in Example 2, $\sum_{\sigma\in S_n} \epsilon(\sigma) as\left(f\{g_{\sigma(1)},\ldots,g_{\sigma(n)}\}\right)(x_1,\ldots,x_r)$ is equal to

$$\sum_{\sigma\in S_n} \varepsilon(\sigma) as\Big(\sum_{k_0+\ldots k_n=N-n} (-1)^{\beta} f(1^{\otimes k_0}\otimes g_{\sigma(1)}\otimes 1^{\otimes k_1}\otimes\cdots\otimes 1^{\otimes k_{n-1}}\otimes g_{\sigma(n)}\otimes 1^{\otimes k_n})\Big)(x_1,\ldots,x_r),$$

where β is given in Example 2. Note also that the Koszul sign $\epsilon(\sigma)$ must be calculated using the degree of g_i as an element of the symmetric brace algebra (so $|g_i| = q_i + a_i - 1$). Thus $\epsilon(\sigma) = (-1)^{\zeta}$, where

$$\zeta = \sum_{i<j \ \& \ \sigma(i)>\sigma(j)} (q_{\sigma(i)} + a_{\sigma(i)} - 1)(q_{\sigma(j)} + a_{\sigma(j)} - 1).$$

If we now antisymmetrize by taking all signed permutations of the x's, and suppress the tensor notation, this is equal to

$$\sum_{\sigma\in S_n} \sum_{k_0+\ldots k_n=N-n} (-1)^{\beta+\zeta} f\left(1^{k_0}, g_{\sigma(1)}, 1^{k_1}, \ldots, 1^{k_{n-1}}, g_{\sigma(n)}, 1^{k_n}\right)\Big(\sum_{\pi\in S_r} \chi(\pi)\left(x_{\pi(1)},\ldots,x_{\pi(r)}\right)\Big).$$

After applying Lemma 10, the left hand side is equal to

$$\sum_{\substack{\pi\in S_r,\sigma\in S_n,\\ k_0+\cdots+k_n=N-n}} (-1)^{\beta+\zeta+\alpha_2}\chi(\pi) f\left(1^{k_0}, g_{\sigma(1)}, 1^{k_1}, \ldots, g_{\sigma(n)}, 1^{k_n}\right)\left(x_{\pi(1+A)},\ldots,x_{\pi(k_0+A)}, X_{\sigma(1)},\right.$$
$$\left. x_{\pi(1+k_0+A)}, \ldots, X_{\sigma(n)}, x_{\pi(1+k_0+\cdots+k_n+A)}, \ldots, x_{\pi(r)}\right),$$

where α_2 is given in Lemma 10.

Finally, when the variables are moved inside, the left hand side is equal to

$$\sum_{\substack{\pi\in S_r,\sigma\in S_n,\\ k_0+\cdots+k_n=N-n}} (-1)^{\beta+\zeta+\alpha+\mu}\chi(\pi) f\left(x_{\pi(1+A)}, \ldots, x_{\pi(k_0+A)}, g_{\sigma(1)}(X_{\sigma(1)}), x_{\pi(1+k_0+A)},\right.$$
$$\left. \ldots, g_{\sigma(n)}(X_{\sigma(n)}), x_{\pi(1+k_0+\cdots+k_n+A)}, \ldots, x_{\pi(r)}\right),$$

where $\mu = \sum_i q_{\sigma(i)}[x_{\pi(1+A)} + \cdots + x_{\pi(k_0+\cdots+k_{i-1}+A)}] + \sum_{j<i} q_{\sigma(i)}|X_{\sigma(j)}|$.

Since the right hand side is equal to

$$\sum_{\substack{\pi\in S_r,\sigma\in S_n,\\ k_0+\cdots+k_n=N-n}} \chi(\pi)(-1)^{\delta+\tilde{\nu}+\lambda+\eta} f\left(x_{\pi(1+A)}, \ldots, x_{\pi(k_0+A)}, g_{\sigma(1)}(X_{\sigma(1)}), x_{\pi(1+k_0+A)},\right.$$
$$\left. \ldots, g_{\sigma(n)}(X_{\sigma(n)}), x_{\pi(1+k_0+\cdots+k_n+A)}, \ldots, x_{\pi(r)}\right),$$

we see that the two sides are equal if and only if

$$\beta + \zeta + \alpha_2 + \mu + \delta + \tilde{\nu} + \lambda + \eta \equiv 0 \pmod 2.$$

After cancelling the most obvious terms, $\beta+\zeta+\alpha_2+\mu+\delta+\tilde{\nu}+\lambda+\eta$ is congruent to

$$\sum_i (n-i)a_{\sigma(i)} + \sum_i (N-i)q_{\sigma(i)} + \sum_{j<i} q_{\sigma(i)}a_{\sigma(j)}$$
$$+ \sum_{i<j \ \&\ \sigma(i)>\sigma(j)} \left[q_{\sigma(i)}a_{\sigma(j)} + q_{\sigma(i)} + a_{\sigma(i)}q_{\sigma(j)} + a_{\sigma(i)} + q_{\sigma(j)} + a_{\sigma(j)}\right]$$
$$+ \sum_{j<i} q_{\sigma(i)}|X_{\sigma(j)}| + \sum_i (N-i)q_i + \sum_{j<i} q_i a_j + \sum_i (n-i)a_i$$
$$+ \sum_{j<i} q_i|X_j| + \sum_{i<j \ \&\ \sigma(i)>\sigma(j)} \left[q_{\sigma(i)}|X_{\sigma(j)}| + |X_{\sigma(i)}|q_{\sigma(j)}\right].$$

After applying Lemma 11, this is congruent to

$$\sum_i \left\{(n-i)a_{\sigma(i)} + (N-i)q_{\sigma(i)} + (i-1)\left[a_i + a_{\sigma(i)} + q_i + q_{\sigma(i)}\right] + (N-i)q_i + (n-i)a_i\right\},$$

which is equal to $\sum_i \left\{(n-1)\left[a_{\sigma(i)}+a_i\right] + (N-1)\left[q_{\sigma(i)}+q_i\right]\right\} \equiv 0 \pmod 2$. □

As a corollary, we obtain Theorem 3.1 of [4]:

Corollary 17. *The anti-symmetrization $l := as(\mu)$ of an A_∞-algebra structure μ yields an L_∞-algebra structure.*

Proof. Given $\mu\{\mu\} = 0$ (recall Remarks 3 and 7) we have

$$0 = as(\mu\{\mu\}) = as(\mu)\langle as(\mu)\rangle = l\langle l\rangle.$$ □

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