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Jozef Komorník

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Asymptotic Decomposition of Smoothing Positive Operators

J. KOMORNÍK

Bratislava*)

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A very mild criterion of asymptotic periodicity, established in [4] for Markov operators, is generalized for positive power bounded linear operators on L^1.

Let P be linear operator on $L^1 = L^1(X, \Sigma, \mu)$ where μ is a σ – finite measure. Recall (cf. [5], [10], [7]) that P is said to be

(i) *positive* if it preserves the conus $L^1_+ = \{f \in L^1 : f \geqq 0\}$;

(ii) *Markov* if it preserves the set of densities

$$D = \{f \in L^1_+ : \|f\| = 1\} ;$$

(iii) *power bounded* if the inequality

$$\|P^n\| \leqq M \tag{1}$$

holds for some $M \in R$ and all $n \in N$;

(iv) *weakly almost periodic* if for every $a \in L^1$ the trajectory $\{P^n a\}_{n \in N}$ is weakly precompact;

(v) *asymptotically periodic* if there exists $r \in N$ and a finite subset $E = \{g_1, \ldots, g_r\} \subset L^1_+$ such that $P(E) = E$ and the convex envelope

$$F = \text{co}\,(E) = \{\sum_{i=1}^{r} \lambda_i g_i : 0 \leqq \lambda_i; \sum_{i=1}^{r} \lambda_i = 1\} \tag{2}$$

satisfies

$$\lim_{n \to \infty} d(P^n f, F) = 0 \quad \text{for all} \quad f \in D\,, \tag{3}$$

where $d(h, F) = \inf\{\|h - g\| : g \in F\}$.

Note that the elements $g_1, \ldots, g_r$ satisfying (2) and (3) can be chosen as the vertices of the polygon F (that may be degenerated). Moreover, there obviously exist a per-

*) Department of Applied Mathematics, Comenius University, Mlynska Dolina, 842 15 Bratislava, Czechoslovakia

mutation α of the set $\{1, \ldots, r\}$ and a number $q \leqq r!$ such that

(4) $$Pg_i = g_{\alpha(i)} \quad \text{and} \quad P^q g_i = g_i \quad \text{for} \quad i = 1, \ldots, r\,.$$

This fact easily yields that for every $f \in L^1$ there exists a strong limit

(5) $$Q(f) = \lim_{n\to\infty} P^{nq} f = \sum_{i=1}^{r} \lambda_i(f)\, g_i$$

where $\lambda_1, \ldots, \lambda_r$ are uniquely determined positive linear functionals on L^1.

The main result of [4], that is to be generalized in this paper, provides a very mild sufficient condition for asymptotic periodicity of Markov operators. It is so-called smoothing property that can be generalized for positive power bounded linear operators as follows.

Definition 1. A positive power bounded linear operator P on L^1 is called *smoothing* if there exist a set $K \subset \Sigma$, $\mu(K) < \infty$ and positive numbers ε and δ such that

(6) $$\liminf \int_{K-B} P^n f \,\mathrm{d}\mu > \varepsilon \quad \text{for all} \quad f \in D \quad \text{and} \quad B \in \Sigma\,, \quad \mu(B) < \delta\,.$$

Theorem. *A smoothing positive power-bounded linear operator is asymptotically periodic.*

Proof. First we prove that P is weakly almost periodic. Arguing in the same way as in [12] or [3] we deduce that for any $f \in D$ the sequence

(7) $$A_n f = \sum_{i=0}^{n-1} P^i f$$

is precompact in so-called w^*-topology of the second dual W of the space L^1. Any cluster point z of this sequence determines an additive measure μ_z on Σ by

(8) $$\mu_z = z(1_A)$$

that can be uniquely decomposed (cf. [3]) as a sum

(9) $$\mu_z = \mu_g + \mu_a$$

where μ_g is a $\sigma -$ additive measure, $\mu_g \ll \mu$, and μ_a is a purely additive measure. Moreover, the Radon-Nikodym derivative $g = \mathrm{d}\mu_g/\mathrm{d}\mu$ satisfies

(10) $$Pg \leqq g \quad \text{and} \quad \|g\| \geqq \varepsilon$$

where ε is a constant satisfying (6).

The sequence $P^n g$ is nonincreasing, hence it converges strongly to a $P -$ invariant function $g_0 \leqq g$.

Suppose that $c = \|g - g_0\| > 0$. The density

$$h = (g - g_0)/c$$

obviously satisfies

$$\lim_{n\to\infty} \|P^n h\| = 0 ,$$

which contradicts (6). Therefore, $g = g_0 = Pg$.

The function $g/\|g\|$ is a P-invariant density.

Repeating the arguments of [8] we get that there exists a set $G \in \Sigma$ that is a so-called maximal support of P-invariant densities. This means that there exists a P-invariant density g_0 such that

$$G = \operatorname{supp} g_0 = \{x\colon g_0(x) > 0\} \tag{11}$$

and for any P-invariant density h the set difference

$$(\operatorname{supp} h - G)$$

has measure 0.

Note that the subspace $L^1_G = \{1_G h\colon h \in L^1\}$ is P-invariant and that the restriction P_G of the operator P to L^1_G is weakly almost periodic. According to Mean Ergodic Theorem there exist strong limits

$$Ah = \lim_{n\to\infty} A_n h , \quad h \in L^1_G . \tag{12}$$

It is straightforward to observe (cf. [3]) that for any $f \in L^1_+$ the sequence $A(1_G P^n f)$ is nondecreasing and thus it converges strongly to the limit

$$Af = \lim_{n\to\infty} A(1_G P^n f) . \tag{13}$$

Note that (12) and (13) define a positive power bounded linear operator on L^1.

Now we are going to prove that for every $f \in D$ we have

$$Af = g , \tag{14}$$

where g is the function defined above and satisfying (9) and (10).

Let $f \in D$ and $h \in L^\infty$, $h \geqq 0$ be given. Let z be a cluster point of the sequence (7). The properties of w^*-topology yield that there exists a subsequence $\{n_k\}$ such that

$$z(h) = \lim_{k\to\infty} (h, A_{n_k} f) .$$

For any given n, and $k \in N$ the inequalities

$$(A_{n_k} 1_G P^n f, h) \leqq (A_{n_k} P^n f, h) \leqq (A_{n_k} f, h) + \frac{n}{n_k} M\|f\| \, \|h\|$$

clearly hold (where M is a constant satisfying (1)). Therefore, $(A(1_G P^n f), h) \leqq$ $\leqq (z, h)$ for every $n \in N$.

This, together with (13), clearly implies that $Af \leqq z$ in W. However, μ_g is the maximal σ-additive measure that is not greater than z. Therefore, $Af \leqq g$.

To prove the converse inequality consider the operator A' defined on

$$L_G^\infty = \{1_G f: f \in L^\infty\}$$

as the dual operator to the restriction of the operator A to the subspace L_G^1. Let $f \in D$ and $h \in L_G^\infty$, $h \geqq 0$ be given. Obviously, $A'h \in L_G^\infty$, thus there exists a subsequence $\{n_k\}$ satisfying

$$z(A'h) = \lim_{k\to\infty} (A'h, A_{n_k} f) = \lim_{k\to\infty} (1_G A'h, A_{n_k} f) =$$

$$= \lim_{k\to\infty} (h, A(1_G A_{n_k} f)) = (h, Af).$$

However, $A'h \geqq 0$, hence

$$z(A'h) \geqq (A'h, g) = (h, Ag) = (h, g).$$

Therefore, $g \leqq Af$, which completes the proof of (14).

From (10) and (14) we conclude that the operator A, defined by (12) and (13) satisfies

$$\|Ah\| \geqq \varepsilon \|f\| \quad \text{for} \quad f \in L_+^1, \tag{15}$$

(where ε is a constant satisfying (6)).

From (13) we obtain that

$$Af = AP^n f \quad \text{for} \quad f \in L^1 \quad \text{and} \quad n \in N$$

as well as that

$$\lim_{n\to\infty} \|AP^n f - A(1_G P^n f)\| = \lim_{n\to\infty} \|A(1_{X-G} P^n f)\| = 0.$$

Combining this with (15) we get that

$$\lim_{n\to\infty} \|P^n f - 1_G P^n f\| = 0 \quad \text{for} \quad f \in L_+^1, \tag{16}$$

which clearly implies that P is weakly almost periodic.

Now we can repeatedly use to reduction method that was successfuly applied in [6], [9] and [4] and prove asymptotic periodicity of the operator P using the fact that its restriction P_G is isometrically isomorphic to the unity preserving operator P_g defined on $L^1(\mu_g)$ by

$$\bar{P}(f) = g^{-1} P_G(fg). \tag{17}$$

Therefore, it suffices to prove asymptotic periodicity of $\bar{P}$. The that $\bar{P}$ is a smoothing operator can be obtained in the same simple way as the corresponding result of [8]. Hpwever, $\bar{P}$ need not be a Markov on $L^1(\mu_g)$. Fortunately, it is a Markov operator on the space $L^1(\bar{\mu})$ that contains the same elements as $L^1(\mu_g)$ but its metric is determined by the measure $\bar{\mu} = \mu_{hg}$, where the function h is the strong limit (in $L^1(\mu_g)$) of the sequence

$$A'_n 1_G = 1/n \sum_{i=1}^{n} \bar{P}'(1_G), \tag{18}$$

where $\bar{P}'$ is the dual to the operator $\bar{P}$, $\bar{P}'$ is a power bounded operator on $L^\infty(\mu_g) \subset L^1(\mu_g)$. Hence it can be uniquely extended to a weakly almost periodic Markov operator on L_g^1, which ascertain that the sequence (18) has a strong limit h. Moreover, from (1) and (6) we obtain the inequalities

$$\varepsilon \leqq h \leqq M. \tag{19}$$

This immediatelly implies that $\bar{P}$ is a smoothing Markov operator on $L^1(\bar{\mu})$. According to [4], $\bar{P}$ is an asymptotically periodic operator on $L^1(\bar{\mu})$.

The repeated application of the inequality (19) yields that $\bar{P}$ is an asymptotically periodic operator on $L^1(\mu_g)$.

References

[1] DUNFORD, N. and SCHWARTZ, J. T., Linear Operators, Part I, 4th printing (1967) Interscience Publishers, New York.

[2] KOMORNÍK, J., Asymptotic periodicity of the iterates of weakly constrictive Markov operators, Tohoku Math. Journ., 38 (1986), 15—27.

[3] KOMORNÍK, J., Asymptotic periodicity of the iterates of Markov operators. Bernoulli Congress Proceedings, UNU Press, Utrecht 1988, 321.

[4] KOMORNÍK, J. and LASOTA A.: Asymptotic decomposition of Markov operators, Bull. Pol. Ac.: Math. 35 (1987), 321.

[5] KRENGEL, U., Ergodic Theorems, deGruyter, 1985.

[6] LASOTA, A. et al., Asymptotic periodicity of the iterates of Markov operators. Trans. Amer. Math. Soc., 286 (1984), 751.

[7] LASOTA, A. and MACKEY, M. C.: Probabilistic properties of deterministic systems. Cambridge Univ. Press, 1985.

[8] LASOTA, A. and SOCAŁA, J., Asymptotic properties of constrictive Markov operators, Bull. Pol. Ac.: Math., 35 (1987), 71.

[9] LASOTA, A. and YORKE J. A.: Statistical periodicity of deterministic systems. Časop. Pěst. Mat. 111 (1986), 1.

[10] MIKLAVČIČ, M.: Asymptotic periodicity of the iterates of positivity preserving operators. Trans. Amer. Math. Soc., 307 (1988), 469.

[11] SOCAŁA, J.: On existence of invariant measures for Markov operators, Am. Polon. Math. [to appear].

[12] STRAUBE, E., On existence of invariant absolutly continuous measure. Comm. Math. Phys. 81, 27.