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Smooth Measures, the Malliavin Calculus and Approximations in Infinite Dimensional Spaces

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1. Introduction

In this paper we discuss approximations of infinite dimensional mappings by smooth mappings. We study two types of approximations: convolutions with smooth measures and spherical means. The main results are Theorem 4 and its corollary on approximations of functions on uniformly convex spaces. Recent researches in the theory of differentiable measures on infinite dimensional spaces and the Malliavin calculus have brought new ideas in solving traditional problems in this field and stimulated further investigations. Recall some known things about approximations in Banach spaces.

We shall start with the results on smoothing of real valued functions on Banach spaces and then briefly discuss the situation with infinite dimensional maps.

Let X be a Banach space with the unit ball U, f — a uniformly continuous real function on X or U. It is well known that f can be approximated by Lipschitz functions uniformly on X (or on U), see [1]. If X is separable, then such approximations can be taken to be differentiable in the sense of Hadamard (see [2]). Recall that for Lipschitz mappings Hadamard and Gateaux differentiabilities are equivalent. The problem of approximations by Fréchet differentiable functions is more delicate. There exist spaces, where even the norm can not be approximated uniformly on the ball by Fréchet differentiable functions (see [3–5]). If X is superreflexive, then each Lipschitz function on U is approximated by functions with Lipschitz Fréchet derivatives [6, 7], but even in the Hilbert space the class $C_u^2(U)$ of functions with uniformly continuous second Fréchet derivative is not dense in the space $C_u(U)$, see [6].

Remark 1. Recently I. Tsarkov has proved that any Lipschitz function f on the unit ball in the Hilbert space can be approximated uniformly by functions with

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uniformly bounded continuous sècond Fréchet derivatives (thus, uniform closures of $C_b^2(U)$ and $C_u^2(U)$ do not coincide).

Another similar problem which often arises in infinite dimensional analysis is nonexistence of smooth functions with bounded support on many important spaces (see [4], [5]). Such functions are called bump functions and now we recall some basic facts about their properties.

1) There are no nontrivial Fréchet differentiable bump functions on $C[0, 1]$.

2) X is superreflexive if and only if there exists a nontrivial bump function with uniformly continuous Fréchet derivative [8].

3) On c_0 there exists a nontrivial C^∞ bump function (c_0 has an equivalent norm which is C^∞ and even analytic outside of the origin).

4) For spaces X, not containing c_0 isomorphically, the existence of a C^1 bump function with locally uniformly continuous derivative implies superreflexivity. If this derivative is locally Lipschitzian then X has type 2, see [9].

5) If X and X^* have nontrivial bump functions with locally Lipschitz derivatives then X is Hilbert [9].

6) The existence of a nontrivial bump function with Lipschitzian derivative is equivalent to the existence of an equivalent norm with Lipschitzian derivative on the sphere, see [9].

7) If X admits a nontrivial C^∞ bump function then either X contains an isomorphic copy of c_0 or X contains an isomorphic copy of l^{2k} for some k [10].

8) The following assertions are equivalent [11]: a) X doesn't contain c_0 and for every k there exists nontrivial C^k bump function; b) there exists C^∞ bump function on X with uniformly continuous derivative; c) there exists a continuous polynomial P on with $P(0) = 0$ and $P(x) > 1$ for $\|x\| = 1$.

The situation with approximations of maps with infinite dimensional ranges is even more complicated. According to [7] there exist uniformly continuous mappings with values in separable reflexive Banach spaces which are not approximated by Lipschitz mappings (moreover, there exists a separable reflexive E such that for each infinite dimensional Banach space X one can find uniformly continuous $F : X \to E$ without Lipschitz approximations). According to a private communication of S. V. Konyagin there exist Hölder maps from separable Banach spaces X to Hilbert space which can not be approximated by Lipschitzian maps. But Lipschitz approximations are possible for uniformly continuous maps F between Hilbert spaces (see [12]). It is also known [13] that uniformly continuous maps with values in a Hilbert space Z can be approximated by maps satisfying the Hölder condition of the order $1/2$. Moreover, for each superreflexive separable Banach space Z there exists $\alpha = \alpha(Z)$ such that every uniformly continuous map f with values in Z is approximated by Hölder maps of the order α and for $Z = L^p$, $p > 1$, it is possible to take $\alpha = \min(1/2, 1/p)$, see [7]. S. V. Konyagin has proved recently that the last number can not be improved. This problem is closely connected with the problem of extension of a map G which is defined on a closed subset in X and takes values in some Banach space Z, with the preservation of the modulus of continuity (see [12, 13]). If $F: X \to Z$ is already Lipschitzian then it can be approximated by better maps F_n. For example, if X is separable, then F_n can be chosen Hadamard differentiable and Lipschitzian with the same constant (see Theorem 2 below). For Hilbert spaces X, Z maps F_n can be taken Fréchet differentiable with bounded derivatives [7]. If boundedness of derivatives of approximating functions is not required then much more can be obtained. As shown in [3], if there exists a continuous polynomial P on X with $P(0) = 0$, strictly positive on the unit sphere in X then for any continuous map $F: U \to X$ and any strictly positive continuous function $r: U \to R$ there exists a real analytic map $G: U \to X$ with $\|F(x) - G(x)\| \leqq r(x)$. A map G is called analytic in [3], if for each point x there exist continuous homogeneous forms J_n of the degree n such that $G(x + h) - G(x) = \sum_n J_n(h)$, where the series converges uniformly in some neighbourhood of the origin in X. Meanwhile, for many applications (for example, to the stochastic analysis) it is necessary to have approximations by sufficiently many times differentiable function wtih the controlled growth of norms of derivatives. Moreover, sometimes it is necessary to approximate discontinuous mappings. So the natural thing to do in this situation would be to discuss some other concepts of approximation. Recall that it was too rigid notion of Fréchet differentiability to cause the described difficulties. Now we are going to discuss a weaker form of differentiability.

2. Subspace-differentiable approximations and convolutions with smooth measures

Let X be a locally convex space (LCS). An LCS E is called continuously embedded in X, if E is a linear subspace in X and the natural embedding $E \to X$ is continuous. A map F from X to an LCS Z is called differentiable in a point x along the subspace E, if the map $h \mapsto F(x + h)$ is differentiable, provided that we fix a notion of differentiability for E. Usually it will be Fréchet differentiability and E will be a Banach space. However, other definitions of differentiability with respect to the system of sets are possible. Recall that a map F from an LCS E to an LCS Z is called differentiable in a point x with respect to the system of sets $\mathscr{A}$ in E if there exists a continuous linear map $D: E \to Z$ such that

$$\lim_{t \to 0} \frac{F(x + th) - F(x)}{t} - D(h) = 0$$

uniformly in $h \in S$, $S \in \mathscr{A}$ being fixed.

The definitions of Gateaux, Hadamard and Fréchet are covered by this definition if we choose for $\mathscr{A}$ the classes of finite, compact, bounded subsets correspondently.

Recall that a Radon measure m on an LCS X is called differentiable along a vector

$h \in X$, if for every Borel set $A \subset X$ there exists

$$\lim_{t \to 0} \frac{m(A + th) - m(A)}{t} \tag{1}$$

It is known that in this case there exists a measure $d_h m$ on $\mathscr{B}(X)$ such that the limit in (1) coincides with $d_h m(A)$ and besides $d_h m \ll m$. The measure $d_h m$ is called the derivative of m along h and its Radon - Nikodym derivative with respect to m is denoted by $\varrho_h(m)$. The property above is equivalent to the differentiability of the map $t \mapsto m_{th}$ from the real line to the Banach space $M(X)$ of Radon measures on X, equipped with the variation – norm. Here $m_{th}(A) = m\,(A + th)$. The n-fold differentiability of a measure m along a continuously embedded subspace $H \subset X$ is defined in a natural way as the n-fold differentiability of the map $h \mapsto m_h$, $H \to M(X)$. The basic facts concerning differentiability of measures can be found in [14]. A Radon measure m on X is called Skorohod-differentiable along a vector $h \in X$ if for every $f \in C_b(X)$ the function

$$t \mapsto \int f(x - th)\, m(dx)$$

is differentiable. This property is equivalent to the following: all functions $t \to m(A + th)$ are Lipschitz (see [14]).

Denote by $D(m)$ and $D_C(m)$ correspondently the sets of all vectors along which the measure m is differentiable and Skorohod differentiable. The spaces $D(m)$ and $D_C(m)$ possess the natural structures of Banach spaces compactly embedded in the original space X [14]. For this aim we can take the norm $\|h\| = \|d_h m\|$.

Lemma 1. Let $F: X \to Z$ be a locally Lipschitz map between normed spaces. $E \subset X$ be a dense linear subsapce. If F is Gateaux differentiable along finite dimensional subspaces $L \subset E$ at a point x then F is Hadamard differentiable at x.

Proof. We can assume that F is Lipschitzian with the constant 1. It is sufficient to establish Gateaux differentiability of F (see [15]). Let $h \in X$. Choose a sequence $h_n \subset E$ tending to h. By Lipschitz condition uniformly in $t \neq 0$

$$\lim_{n \to \infty} \frac{F(x + th_n) - F(x)}{t} = \frac{F(x + th) - F(x)}{t}$$

Hence, by the differentiability along h_n, F is differentiable along h. Moreover, the map $D: h \to F'(x)(h)$ is linear on E and satisfies the condition $\|D(h)\| \leq \|h\|$ on X. Thus, D is a bounded operator.

Lemma 2. 1) If a measure m on X is k times differentiable along vectors from a linear subspace E (or k times differentiable along continuously embedded E) and $F: X \to Z$ is uniformly continuous with $\|F\| \in L^1(m)$, $\|F\| \in L^1(d_{h_1} \dots d_{h_j} m)$, $h_i \in E$, $j \leq k$ (correspondingly, norms of $\|F\|$ in these L^1-spaces are uniformly bounded for

$\|h_i\| \leqq 1$), then the map $G: x \to \int F(z + x)\, m(dz)$ has the same properties and

$$G'(x)(h) = \int F(z + x)\, d_h m(dz)\,.$$

The proof is simple and uses compositions with linear functionals on Z and the mean value theorem.

Theorem 1. *Let E be a Banach space, continuously embedded in an LCS X and $E \subset D_C(m)$, where m is a nonzero Radon measure on X. Then for every $k \in N$ there exists a probability Radon measure v_k on X with a compact support, k times differentiable along E. Any uniformly continuous map F from X to a Banach space Z can be uniformly approximated by maps $F_n: X \to Z$ which are uniformly continuous and k times Fréchet differentiable along E. If X is Banach and F satisfies the condition $\|F(a) - F(b)\| \leqq \|a - b\|^\alpha$ then for every $\varepsilon > 0$ there exists k times Fréchet differentiable along E map G with the same Hölder condition such that $\|F(x) - G(x)\| \leqq \varepsilon$,*

$$\|D_E^j G\| \leqq \text{const.}\, \varepsilon^{-j/\alpha}\,.$$

Proof. The first statement was actually proved in [14] where the measure v with a compact support and $D_C(m) \subset D(v)$ was constructed. Taking $v_1 = v * v$ we obtain the measure which is twice differentiable along directions from $D(v)$. Moreover,

$$\|d_h^2(v_1)_b\| = \|d_h^2 v_1\| = \|d_h v * d_h v\| \leqq \|d_h v\|^2 = \|h\|^2$$

and $((v_1)_{th} - v_1)/t - d_h v_1$ belongs to the closed convex hull of the set $\frac{1}{2}t^2 d_h^2(v_1)_{sh}$, $s \in [0, 1]$. This ensures the Fréchet differentiability along $D(v)$. Then we apply analogous arguments to convolutions of k copies of the measure v_1: $v_k = v_1 * \ldots * v_1$. Now it is sufficient to prove the second statement for $E = D_C(m)$ and so we can apply the first statement also for this case. Take the corresponding measure v and define the approximation by the trivial formula

$$G(x) = \int F(x + \delta s)\, v(ds)$$

which is usually used (see [1, 2, 14, 15, 16]). Here $\delta = \delta(\varepsilon)$ is to be chosen according to the uniform continuity, for example, $\delta(\varepsilon) = \varepsilon^{1/\alpha}$, if F is Hölder. Now apply Lemma 2.

Theorem 2. *Let $F: X \to Z$ be a Lipschitz map, X being separable. Then for every $\varepsilon > 0$ there exists an Hadamard differentiable Lipschitzian (with the same constant as F) mapping $G: X \to Z$ such that for all x*

$$\|F(x) - G(x)\| \leqq \varepsilon\,.$$

Proof. We can assume that F is Lipschitzian with the constant 1. Find a probability measure m on X with the compact support in the ball $B(0, \varepsilon)$ and infinitely differentiable along dense subspace $E \subset X$ (it is possible, see [14]). Let $G(x) = \int F(x + z)\, m(dz)$. Obvious estimates and application of lemmas complete the proof.

All known examples of pairs (E, X) for which Lipschitzian functions on X can be approximated by $k \pm 3$ times differentiable along E functions with bounded derivatives are constructed by means of differentiable measures (on the space X or its isometric extensions). It is unknown whether it is always so. We conclude this section with a concrete example of subspaces of differentiability in the space $C[0, 1]$. The following result was obtained in [17], for the proof see [14].

Denote by H_ω the Banach space of all functions on $[0, 1]$ satisfying the condition $|f(t) - f(s)| \leqq C(f)\, \omega(|t - s|)$, ω being a modulus of continuity (this space has the natural Banach norm).

Theorem 3. 1) If ω satisfies the condition

$$\omega(t) = 0(t/(\log(1/t))^\beta)\,, \quad \beta > 3/2\,,$$

then there exists a probability measure m on $C[0, 1]$ which is infinitely differentiable along H_ω.

2) If $H_\omega \subset D_C(m)$ for some nonzero measure m on $C[0, 1]$ then

$$\sum_{n=1}^{\infty} \omega(1/n)/\sqrt{n} < \infty\,.$$

In particular, the first statement is not true for $\beta = 1$.

It remains open whether the last condition is equivalent to the existence of a measure, differentiable along H_ω

Remark 2. There exists an example of a probability measure m on c_0 (or $C[0, 1]$) with a compact support, infinitely differentiable along a dense subspace E such that no Gaussian measure g exists with $E \subset D(g)$, see [14]. Thus, Gaussian measures are not the best in this aspect.

The problem of characterization of subspaces of differentiability of measures (introduced in [18]) is open. Such subspaces with the norm indicated above are isomorphic to subspaces in $L^1[0, 1]$ and all spaces l^p, $1 \leqq p \leqq 2$, can be represented in this form. It is unknown for what subspaces of $L^1[0, 1]$ this is true. Properties of subspaces of differentiability for measures are investigated in [14, 17]. V. Bentkus and A. Rachkauskas elaborated a method of estimating the rate of convergence in the central limit theorem in Banach spaces based on the use of such subspaces (see [19, 20]). The idea to use the subspace-differentiability plays an important role in the Malliavin calculus (see [14]). Recall that solutions of stochastic differential equations can be regarded as functionals of the Wiener process which are differentiable along the reproducing kernel of the Wiener measure on the space of trajectories but in general they are not even continuous on the whole space.

3. Approximations by spherical means

Another natural way of approximating was suggested by A. V. Uglanov [21]. The main idea is the following. Assume we have a bounded set $V \subset X$ with nonempty interior such that for all measures m, infinitely differentiable along dense subspaces, functions $x \mapsto m(V + x)$ are in C^k (have k continuous Fréchet derivatives). Fix one of these measures m positive on open sets in the unit ball U and infinitely differentiable along E. If $f\colon U \to R$ is Lipschitzian and infinitely differentiable along E (with bounded derivatives) then maps

$$f_\varepsilon(x) = \int_{\varepsilon V + x} f(z)\, m(dz)/m(\varepsilon V + x)$$

can serve as C^k approximations of f. Now we combine both types of approximation: first approximate $g \subset C_u(U)$ by Lipschitzian h and h approximate by f of the indicated type using convolutions, then take f_ε. Clearly in many spaces there are no measures m and bounded sets V with the property that $x \mapsto m(V + x)$ is Fréchet differentiable not vanishing identically, since $m(V + x) \to 0$ when $\|x\| \to 0$. However, for some spaces this plan can be carried out. Below we develop some ideas of A. V. Uglanov [21] (he considered spaces L^p), but instead of his methods of studying surface measures we apply the Malliavin calculus.

Remark 3. Let g be a nondegenerate Gaussian measure on a Banach space X, $R(x, t) = g(B(x, t))$, $B(x, t)$ being a closed ball of the radius $t > 0$, centred in x. As noted in [22, 23] for every a and $r > 0$ there exists $C = C(a, r)$ such that $|R(a, t) - R(a, s)| \leqq C|t - s|$ if $s, t > r$. This implies local Lipschitzness in both variables since the symmetric difference of $B(x, r)$ and $B(y, r)$ is contained in $B(x, r + q) \setminus B(x, r - q)$, $q = \|x - y\|$. Thus $F\colon x \mapsto R(x, t)$ is Hadamard differentiable by Lemma 1 since $D(g)$ is dense (more complicated proof of Gateaux differentiability of F is given in [24]). It is clear from the above that F need not (in $C[0, 1]$ even can not) be Fréchet differentiable, in particular its Gateaux derivative need not be continuous. It would be interesting to examine analogous questions for other smooth measures on general spaces.

Now let F be a measurable function possessing almost everywhere a Gateaux derivative F' with locally integrable norm, $S = \{x\colon F(x) = 0\}$.

Definition. A locally finite measure σ is called a local surface measure for a measure m if each $s \in S$ has a neighbourhood V such that measures

$$\eta_t(B) = (2t)^{-1} \int_{B \cap V \cap \{-t < F < t\}} \|F'\|\, m\,, \quad t \downarrow 0$$

converges weakly to $\sigma(\cdot \cap V)$.

We call σ a surface measure for m and denote it by m^S if σ has bounded variation. If $\{S_n\}$ is a countable union of surfaces of considered type then we define the surface measure m^S on $S = \bigcup S_n$ as the sum of the series $\sum m^{S_n}$ converging weakly.

Proposition 1. *If $D(m)$ is dense, F has locally bounded Gateaux derivative and partial derivatives $\partial_h F$, $h \in D(m)$, are continuous and, at each $s \in S$, not identically zero, then the local surface measure on S exists.*

Proof. For $s \in S$ find a ball $B = B(s, r)$, a vector $h \in D(m)$ and $c_1, c_2 > 0$ such that $c_1 < 1/\partial_h F < c_2$ on B. Take a function $\varphi: X \to [0, 1]$ with bounded derivative $\partial_h \varphi$ and $\varphi = 1$ on $B_1 = B(s, 3r/4)$, $\varphi = 0$ on $X \setminus B$. The measure $v = \varphi m$ is differentiable along h and $c_1 < 1/\partial_h F < c_2$ on the support of v Hence the function $t \mapsto v(x: F(x) < t)$ has a bounded continuous derivative p. This follows from the existence of differentiable conditional measuies on lines $y + Rh$, see [14, 25]. If f is a bounded Borel function on X with bounded $\partial_h f$ then the same arguments can be applied to the measure $\eta = fv$. Moreover,

$$\frac{|\eta(x: -t < F < t)|}{2t} \leqq \|p\|_\infty \sup |f| ,$$

p being a density of the measure $|v| \circ F^{-1}$ which is also bounded since $D(v) \subset \subset D(|v|)$. The set of such functions f is uniformly dense in $C_u(B)$. So

$$\lim_{t \to 0} \frac{1}{2t} \int_{-t<F<t} fv$$

exists for all $f \in C_u(B)$. The space of Radon measures on B is weakly sequentially complete and this gives us the weak limit λ of the family

$$\frac{1}{2t} v(\cdot \cap \{-t < F < t\}) , \quad t \to 0 . \quad \text{Let} \quad \sigma = \|F'\| \lambda .$$

Remark 4. It is possible to derive from the proof above that for any $f \in C(X)$: $(fm)^S = fm^S$. Besides, if $S(t, r) = F^{-1}(t) + ra$, $a \in X$, then surface measures $\sigma(t, r)$ on $S(t, r)$ converge locally weakly to σ, $t, r \to 0$.

Notation. Write $m^S = \sigma$ if $\|\sigma\| < \infty$. If λ in the proof has bounded variation we denote it by λ^S and call a nonnormalized surface measure. Denote by $n(x)$ the unit normal to S at a point x, i.e. if S is locally $F^{-1}(0)$ then $n(x) = F'(x)/\|F'(x)\|$.

Note that λ^S varies if we replace F by const F while m^S depends only on the geometry of the surface.

Proposition 2. *If m is differentiable along a vector field $v: X \to X$ (in the sense of [14, 25]), F is twice differentiable along v, $\partial_v^2 F/(\partial_v F)^2 \in L^1(m)$, $1/\partial_v F \in L^1(d_v m)$ and $\|F'\|$ is bounded and continuous (or $\|F'\|$, $\partial_v \|F'\|$, $\|F'\| \varrho_v(m)$, $\partial_v^2 F/(\partial v F)^2$, $1/\partial_v F \in L^2(m)$) then there exist bounded surface measures m^{S_t} on $S_t = F^{-1}(t)$.*

Proof. According to [14, 25] $m \circ F^{-1}$ possesses a bounded continuous (even absolutely continuous) density p. This gives the first assertion. To prove the second repeat the same for the measure $v = \|F'\| m$ (note that $d_v v = \partial_v \|F'\| m + \|F'\| \varrho_v(m) m$).

Remark 5. Sometimes it is useful to define a surface measure in another way replacing $\|F'\|$ by $\|DF\|_H$ where the derivative and its norm are taken with respect to some continuously embedded Banach space $H \subset X$. In a similar way surface measures on surfaces of codimension $n > 1$ can be defined [26]. This modification is especially useful for introducing surface measures on level surfaces of functions from Sobolev spaces over abstract Wiener spaces since such functions need not be even continuous. The same definitions have sense for surface measures on manifolds of general type considered in [14, 25]. A. V. Uglanov defined surface measures in another but equivalent way; his results can be proved by these methods.

The next result, which appeared in [27], is very important for the integration theory because it connects surface integrals with integrals over the space.

Proposition 3. *Let V be an open set with a boundary S which locally has a form, indicated in Proposition 1, n be a continuous normal. If $D(m)$ is dense, m^S_∞ is finite, $\{a_i\} \subset D(m)$, the series $G = \sum_1^\infty (\partial^2_{a_i}\varphi + \partial a_i\varphi\, \varrho_{a_i}(m))$ converges in $L^1(m)$, the series $\partial_v\varphi = \sum_1^\infty \partial_{a_i}\varphi(n, a_i)$ converges in $L^1(m^S)$, then*

$$\int_S \partial_v\varphi m^S = \int_V Gm\,.$$

Proof. It is sufficient to prove equalities

$$\int_S \partial_{a_i}\varphi(n, a_i)\, m^S = \int_V (\partial^2_{a_i}\varphi + \partial_{a_i}\varphi\, \varrho_{a_i}(m))\, m\,.$$

But the right-hand integral coincides with $\int_V d_{a_i}v$, where $v = \partial_{a_i}\varphi m$, while the left-hand one equals to $\int_S (n, a_i)\, v^S$. So we have to prove the formula $\int_S (n, a_i)\, v^S = \int_V d_{a_i}v$. We can assume that v has a compact support K since it can be approximated in variation by measures $v_j = \varphi_j v$, φ_j being functions with compact support with derivatives along $D(v)$, see [14]. This makes possible to consider the following case: $V = \{x = z + se: z \in Q, s < F(z)\}$, where $X = Z + Ra$, $\|a\| = 1$, Z is a hyperplane, Q is a ball in Z, $F|_Q \geqq 0$. In this case $(n, a) = \partial_a F/\|F'\|$. Hence for the surface $S_1 = \{x: z \in Q,\ s = F(z)\}$ we have

$$\int_{S_1} (n, a)\, v^S = \lim_{t\to 0} \frac{(\partial_a F v)\,(0 \leqq F \leqq t)}{t}\,.$$

It is not difficult to show that this expression equals to $\lim_{t\to 0} v(V_t)/t$, where V_t denotes the region $Q + [0, t]\, a$. Besides,

$$v^S(Q) = \lim_{t\to 0} \frac{1}{t}\, v(x: z \in Q,\ s \leqq t)\,.$$

Now we use the relationship

$$d_a\, v(V) = \lim_{t\to 0} (v(V + ta) - v(V))/t =$$

$$= \lim_{t\to 0} (v(x: z \in Q,\ s \leqq t) + v(V_t))/t\,.$$

Proposition 4. *Assume that conditions of Proposition 3 are fulfilled for surfaces $V + ta$, $0 \leqq t \leqq r$, where $a \in X$ is fixed. Then for such t*

$$m(V + ta) - m(V) = \int_0^t \int_{S+ra} (n, a)\, m^{S+ra} dr = \int_0^t \int_{S+ra} \partial_a F \lambda^{S+ra} dr .$$

Proof. Let $a \in D(m)$. Then the derivatives of both terms coincide as we have seen above in the proof of Proposition 3 (the derivative of the left-hand term is $d_a m(V + ta)$ which equals to the integral

$$\int_{S+ta} (n, a)\, m^{S+ta} .$$

Since $D(m)$ is dense in X we can use Remark 4.

Remark 6. Similar arguments are valid in a more general situation (see [21]).

A modulus of convexity of a Banach space B is defined by the formula

$$\delta_B(\varepsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} ;\ \|x\|, \|y\| \leqq 1,\ \|x - y\| \geqq \varepsilon \right\} .$$

The norm $\|\cdot\|$ is said to be uniformly convex of the order q if $\delta_B(\varepsilon) \geqq c\varepsilon^q$ for some $c > 0$. Each superreflexive space admits a norm with the modulus of convexity $\delta(\varepsilon) \sim \varepsilon^q$, $q > 1$ [28]. The following statement was mentioned in [29] without proof.

Lemma 3. *The following assertions are equivalent: 1) a norm p is uniformly convex of the order k; 2) there exists $C > 0$ such that $p'(b + te) \geqq Ct^{k-1}$ for all $t \in (0, 1)$ and b, e with $p(b) = 1$, $p(e) = 1$, $p'(b)(e) \geqq 0$ where p' states for lower derivative.*

Proof. Note that a function $t \mapsto p(x + ta)$ has a continuous derivative except at most countable set. Assume that for some b, e we have $p'(b)(e) \geqq 0$ but $p'(b + t_1 e) . . (e) \leqq dt_1^{k-1}$, $d = ck(3/2)^{k-1}$ for some $t_1 \in (0, 1/2)$. By the above argument we can consider that $p'(b + te)(e) \leqq 2dt^{k-1}$ for all t in the interval $(t_2 - r, t_2 + r)$. We can assume also that $p'(b + te)(e) \geqq 0$ for $t \in [t_2, t_2 + r]$, $r < 1/4$, since the function $t \mapsto p'(b + te)(e)$ has only one zero. Replacing b by $b + t_2 e$ we come to the case $t_2 = 0$. Find $q \in (0, 1)$ with $\|qb + re\| = 1$. It is clear that $q > 3/4$. Then $(p = \|\cdot\|)$:

$$1 - \|qb + re/2\| = \int_0^{r/2} p'(qb + te)(e)\, dt =$$

$$= \int_0^{r/2} p'(qb + qte/q)(e)\, dt = \int_0^{r/2} p'(b + te/q)(e)\, dt \leqq$$

$$\leqq 2d \int_0^{r/2} t^{k-1}/q^{k-1} dt = \frac{2d}{kq^{k-1}} \left(\frac{r}{2}\right)^k < cr^k .$$

On the other hand $1 - \|qb + re/2\| \geqq cr^k$. This contradiction proves the implication 1) $\Rightarrow$ 2). Now assume that 2) is true and $\|x\| = \|y\| = 1$, $\|x - y\| = \varepsilon < 1/4$, $e = (x - y)/(\|x - y\|)$. Denote by b the point on the segment, connecting x and y, with the minimal norm. Then $p'(b)(e) = 0$. Note that $p'(b/\|b\|)(e) = p'(b)(e) = 0$ and $\|b\| \geqq 1 - \varepsilon \geqq 3/4$, from where $p'(b \pm te)(e) = p'(b/\|b\| \pm te/\|b\|)(e) \geqq$

$\geqq C(t/\|b\|)^{k-1} \geqq C(4/3)^{k-1} t^{k-1}$. If $b \in [(x+y)/2, y]$, $\delta = \|(x+y)/2 - b\|$, then

$$1 - \frac{\|x+y\|}{2} = \int_{\delta}^{\varepsilon/2+\delta} p'(b+te)(e)\,dt \geqq \frac{3C}{4k}\left(\frac{2}{3}\right)^k \varepsilon^k .$$

Theorem 4. *Let B be a Banach space with a norm p, which is uniformly convex of some order d and has k Lipschitzian Fréchet derivatives on the unit sphere. If a measure m on B is infinitely differentiable along vectors from infinite dimensional linear subspace D and together with all derivatives possesses all moments, then the function $M: t \mapsto m(x: p(x) < t)$ is k times differentiable and $M^{(k)}$ is absolutely continuous. Besides, the function $Q: x \mapsto m(U + x)$ has k continuous Fréchet derivatives, U being a ball. Moreover, the map $(t, x) \mapsto m(tU + x)$ is in C^k.*

Proof. 1) We shall use the results [14, 25] where a variant of the Malliavin's method is described. According to these results for the proof of the first statement it is sufficient to construct a vector field $v: X \to X$ along which the function p and the measure m are k times differentiable with $1/(\partial_v p)^q \in L^1(v)$, $v = \partial_v^l p \,.\, d_v^r m$, $q \in N$, $l, r \leqq k$. Let $v(x) = \sum_1^n \partial_{e_i} p(x)\, e_i$, where $\|e_i\| = 1$, $e_i \in D$ are linearly independent and n will be chosen later. All differentiability conditions hold and the only thing to verify is the inclusion $(\partial_v p)^{-q} \in L^1(v)$. For this end it is enough to obtain estimates

$$|v|\,(x: G(x) \leqq \varepsilon) \leqq \beta_r \varepsilon^r , \quad G = \partial_v p = \sum_1^n (\partial_{e_i} p)^2 .$$

Denote by Y the topological complement to $L = \operatorname{span}(e_1, \ldots, e_n)$ in X. By [14, 25] there exist smooth conditional measures v^y on subspaces $y + L$, $y \in Y$. Affine spaces $y + L$ are equipped with Lebesgue measures λ_y translated from L, L being identified with R^n: $e_i \to (0, \ldots, 1, \ldots, 0)$. Denote by $m(y)$ the point in $y + L$ where p attains the minimum. Note that for all $z \in L$ we have

$$G(z + m(y)) \geqq \beta(|z|/\|m(y)\|)^{2k-2} , \quad \beta > 0 ,$$

$|\cdot|$ being the new norm on L taken from R^n. Indeed, $p'(m(y) + e)(e) \geqq 0$, hence $p'(m(y)/\|m(y)\| + e)(e) \geqq 0$ and $p'(m(y)/\|m(y)\| + te)(e) \geqq ct^{k-1}$. Thus $p'(m(y) + t\|m(y)\|\, e)(e) \geqq ct^{k-1}$ and for $\|e\| = 1$ $p'(m(y) + se)(e) \geqq c(s/\|m(y)\|)^{k-1}$. Since norms $\|\cdot\|$ and $|\cdot|$ are equivalent on L $p'(m(y) + se)(e) \geqq c_1(s/\|m(y)\|)^{k-1}$. From this equivalence we have also

$$\sum_1^n (\partial_{e_i} p(m(y) + se))^2 \geqq (c_2(s/\|m(y)\|)^{k-1})^2 ,$$

since

$$p'(x)(e) = \sum_1^n \alpha_i\, \partial_{e_i} p(x) , \quad e = \sum_1^n \alpha_i e_i ,$$

$$|\sum_1^n \alpha_i\, \partial_{e_i} p| \leqq \sqrt{(\sum_1^n \alpha_i^2)} \,.\, \sqrt{(\sum_1^n (\partial_{e_i} p)^2)} = |e| \sqrt{G} .$$

Finally we get the following estimate:

$$G(m(y) + z) \geqq c_3|z|^{2k-2}/\|m(y)\|^{2k-2}.$$

Since $\|m(y)\| \leqq \|y\|$ we have

$$\{x \in L + y: G(x) \leqq \varepsilon\} \subset \{x \in L + y: (|x - m(y)|/\|y\|)^{2k-2} \leqq \varepsilon/\beta\} =$$
$$= \{x \in L + y: |x - m(y)| \leqq (\varepsilon/\beta)^{1/(2k-2)} \|y\|\}.$$

The $|v^y|$-measure of the last set is majorized by

$$\|f^y\|_\infty \lambda_y(x: |x - m(y)| \leqq (\varepsilon/\beta)^{1/(2k-2)} \|y\|) \leqq$$
$$\leqq \|f^y\|_\infty c_n \|y\|^n (\varepsilon/\beta)^{n/(2k-2)},$$

where f^y denotes the density of v^y with respect to λ_y. Accodding to [14, 25]

$$\|f^y\|_\infty \leqq \|d_{e_1} \dots d_{e_n} v^y\|.$$

This gives the estimate

$$|v| (G \leqq \varepsilon) = \int_Y v^y(G \leqq \varepsilon) \, \sigma(dy) \leqq$$
$$\leqq \varepsilon^{n/(2k-2)} \int_Y \tilde{c}_n \|y\|^n \|d_{e_1} \dots d_{e_n} v^y\| \, \sigma(dy),$$

where σ denotes the projection of $|v|$ on Y. By the closed graph theorem there exists $d > 0$ with $\|a + y\| \geqq d\|y\|$, $a \in L$. Hence

$$\int_Y \|y\|^n \|d_{e_1} \dots d_{e_n} v^y\| \, \sigma(dy) \leqq$$
$$\leqq d^{-n} \int_Y \int_{L+y} \|z\|^n |d_{e_1} \dots d_{e_n} v^y| (dz) \, \sigma(dy) =$$
$$= d^{-n} \int_X \|x\|^n |d_{e_1} \dots d_{e_n} v| (dx) < \infty$$

The last inequality holds by the condition since

$$v = \partial_v^l p \, . \, d_v^r m = \sum_{j \leqq r} \varphi_j d_{i_1} \dots d_{i_j} m$$

(d_{i_j} states for the derivative along e_{i_j}), φ_j are polynomials of p and $\partial^j_{i_1 \dots i_j} p$. Now we can make our choice of n and write $n = (2k - 2) r$. This completes the proof of the first statement. 2) The second statement follows from the first and Propositions 3, 4. Indeed, the first derivative of $t \mapsto m(U + ta)$ coincides with

$$\int_{S+ta} \partial_a p \lambda^{S+ta} = \int_{S+ta} \sum_1^N (\partial_{a_i} p)^2 \frac{\partial_a p}{\sum_1^N (\partial_{a_i} p)^2} \lambda^{S+ta} =$$
$$= \int_{S+ta} (\sum_1^N \partial^2_{a_i} p + \partial a_{a_i} p \, \varrho_{a_i}(v)) \, v,$$

where $a_1, \dots, a_N \in D$ are linearly independent, $v = (\partial_a p / \sum_1^N (\partial_{a_i} p)^2) \, m$. From the reasonings of part 1) it is clear that for sufficiently large N the measure ν is correctly defined. By the induction we get the desirable statement. From the arguments above it is possible to obtain continuity of partial derivatives of investigated map in both variables, which completes the proof.

Corollary. *If X satisfies the condition of Theorem 4 then $C^k(X)$ is dense in $C_u(X)$.*

Remark 7. Theorem 4 permits also to approximate Lipschitz maps from X to Banach spaces using the same formula.

Remark 8. a) If we are interested only in the boundedness of the density of distribution of the norm then we can omit the condition of differentiability (for Gaussian measures it is proved in [23]). b) For stable measures the first statement of Theorem 4 was proved in [29]. c) In some cases the described approximations are the best possible. For example, according to [4] the results [21] for spaces L^p can not be improved.

4. Additional remarks

Introduce the following conditions on a Banach space X. $C^kN - X$ admits a norm $\|\cdot\|$ which is C^k on $X \setminus \{0\}$;

$C^kN - X$ admits a norm $\|\cdot\|$ which is C^k on $X \setminus \{0\}$;

$C^kB - X$ has a nontrivial bump function $f \in C^k$;

$C^kD - C^k(U)$ is dense in $C_u(U)$, U being a ball.

In section 1 we have mentioned some results concerning the connections between these properties. It is interesting to investigate other cases. Certainly, there are evident connections: $C^kN \Rightarrow C^kB$, $C^kD \Rightarrow C^kB$. To prove the second one find $g \in C^k$ with $0 < g(x) < 1/4$ for $\|x\| = 1$ and $g(0) > 1$. Let $f(x) = g(x) - 1/2$ if $\|x\| \leqq 1$ and $g(x) \geqq 1/2$, $f(x) = 0$ in other cases. It is easy to verify that f^{2k} belongs to C^k. The same argument shows that C^kB is equivalent to the existence of a nontrivial C^k function tending to zero at the infinity. Theorem 4 ensures C^kD for sufficiently smooth norms $\|\cdot\|$ which are uniformly convex of some order. This theorem enables also to obtain the implication $C^kB \Rightarrow C^kD$ if one can construct $f \in C^k$ such that $f^{-1}(0)$ is a "good" surface. It might be of interest to characterize geometrical properties of the sphere in a Banach space X in terms of differential properties of functions $t \mapsto m(tU)$ for smooth measures. We conclude by mentioning some relevant questions.

1. It is interesting to study possible supports and critical points of bump functions of different classes. Such functions in infinite dimensions can have surprising properties. For example, S. A. Shkarin [30] constructed a function f on Hilbert space H possessing uniformly bounded Fréchet derivatives of all orders such that f is vanishing outside of the unit ball while $f' \neq 0$ in the open unit ball. Below we give his example

of a continuous polynomial P of the fourth degree on H, vanishing on the unit sphere and having no critical points in the open unit ball (similar example was constructed also by Yu. Prostov).

Example 1. Let $H = L^2[-1, 1]$, $Ax(t) = tx(t)$, $f = c(1 - t^2)^2$, $|c| < 1/20$, $P(x) = (1 - (x, x))[(Ax, x) + (f, x)]$.

2. A group of interesting questions is connected with results on extension of smooth mappings in infinite dimensional spaces and different versions of the Whitney theorem (constructing a map with prescribed restrictions of derivatives on some closed subsets), see [31].

3. It may be of interest to consider a problem of uniform approximation of maps $F: U \to U$ by maps with fixed points or with invariant measures. Some discussion on this subject can be found in [32], where there is an example of a polynomial diffeomorphism F of the closed unit ball U in a real Hilbert space which has no invariant measure (in particular, has no fix points). It remains open whether a polynomial map $F: U \to U$ must have ε-fixed points x_ε for all $\varepsilon > 0$ (i.e. $\|F(x_\varepsilon) - x_\varepsilon\| < \varepsilon$).

We don't know any example of a polynomial map $F: U \to U$ without fixed points but with invariant measures. Another problem — differential properties of invariant measures of smooth transformations in infinite dimensional spaces (for example, corresponding to infinite dimensional dynamic systems).

At last note that in this paper we have not touched the theory of Sobolev spaces over infinite dimensional manifolds which is now the object of intensive studies.

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