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Remarks on Integration by Parts in Infinite Dimension

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0. Introduction

The theory of differentiable measures suggested by S. V. Fomin about 25 years ago now is playing more and more significant role in infinite dimensional analysis, stochastics and applications in mathematical physics. The survey of this theory was given in [BS]. This paper is devoted to some new results and directions connected with this theory. In section 2 we discuss integration by parts formulae and applications to the notion of smooth conditional expectation (introduced by Malliavin, Nualart, Ustünel and Zakai). Section 3 is devoted to the study of properties of logarithmic derivatives. In particular, it is motivated by recent researches in infinite dimensional stochastics and quantum field theory. In sections 2 and 3 we also discuss Cameron - Martin type formulae. Preliminary version of this paper appeared as [Bo9]. Main results were presented at the Winter School-92 in Strobl and I would like to thank J. B. Cooper, P. Müller, M. Schmuckenschläger, C. Stegall and W. Schachermayer for this opportunity. Valuable discussions with M. Röckner, J. Tišer, E. Mayer-Wolf and M. Zakai are gratefully acknowledged.

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1. Notations and terminology

Let X be a real locally convex Hausdorff space (LCS X) with a topological dual X^*. In this paper a measure μ on X means a bounded Radon measure (may be, signed) on the borelian σ-field $\mathscr{B}(X)$. For a measurable map $F: X \to Y$ we denote the image measure $A \to \mu(F^{-1}A)$ on Y by $\mu \circ F^{-1}$. In particular, $\mu_h(A) = \mu(A + h)$. If μ is equivalent to μ_{th} for all t then it is called quasi-invariant along h. The σ-field generated by a family $\mathscr{F}$ of functions is denoted by $\sigma(\mathscr{F})$.

Recall that a measure μ on X is called continuous (resp. Fomin differentiable) along a vector $h \in X$ if for each Borel $A \in X$ the function $t \to \mu(A + th)$ is continuous (resp. differentiable). If μ is differentiable along h then the formula

$$\mathrm{d}_h \mu(A) = \lim_{t \to 0} (\mu(A + th) - \mu(A))/t$$

defines the measure $\mathrm{d}_h\mu$ absolutely continuous with respect to μ. The density $\varrho_h(\mu) = \mathrm{d}_h\mu/\mu$ is called the logarithmic derivative of μ along h. Higher derivatives $\mathrm{d}_h^n\mu$ as well as mixed derivatives $\mathrm{d}_k\, \mathrm{d}_h\mu$ are defined in a natural way. Let E be a LCS continuously embedded into X and M be some locally convex space of measures on X. We say that μ is differentiable along E (in some sense) if the map $h \to \mu_h$ from E to M is differentiable in this sense. If E is Banach then always the Fréchet differentiability is meant. Note that the differentiability along a subspace is stronger than the differentiability along all vectors from this subspace.

Two other types of differentiability were introduced by Skorohod [Sk] and Albeverio - Høegh-Krohn [AHK].

We say that μ is Skorohod differentiable along h if for each bounded continuous function f on X the function $t \to \int f(x - th)\, \mu(\mathrm{d}x)$ is differentiable. In this case there exists a measure ν (called a weak derivative) such that for all bounded continuous f

$$\lim_{t \to 0} t^{-1} \int (f(x - th) - f(x)) \;\; \mu(\mathrm{d}x) = \int f(x) \;\; \nu(\mathrm{d}x) \tag{1.1}$$

This measure ν is denoted by the same symbol $\mathrm{d}_h\mu$ since Fomin differentiability implies Skorohod differentiability. The difference is that Skorohod derivative need not be absolute continuous with respect to μ. Note that Fomin differentiability is equivalent to Skorohod differentiability and continuity of the weak derivative along h. It is equivalent also to absolute continuity of the weak derivative with respect to μ.

Note that (1.1) can be rewritten as

$$\int \partial_h f \mu = - \int f \, \mathrm{d}_h\mu \, . \tag{1.2}$$

This permits to define differentiability of μ along a nonconstant vector field h on X as the existence of a measure $d_h\mu$ satisfying (1.2) for a suitable class of f. We discuss this below.

On the line continuity along nonzero h means existence of a density with respect to Lebesgue measure, Fomin differentiability means existence of absolutely continuous density p with $p' \in L^1(R^1)$, while Skorohod differentiability means the existence of the density of bounded variation.

We say that a nonnegative measure μ on X is Albeverio - Høegh-Krohn (AHK) differentiable along h (or L^2-differentiable) if there exists a measure λ such that $\mu_{th} \ll \lambda$, $\mu_{th} = f_t\lambda$ and the map $t \to f_t^{1/2}$ from R^1 to $L^2(\lambda)$ is differentiable. According to [Bo5] this differentiability is equivalent to Fomin differentiability with square integrability of $\varrho_h(\mu)$.

Let $D_C(\mu) = \{h\colon \mu \text{ is Skorohod differentiable along } h\}$, $D(\mu) = \{h\colon \mu \text{ is Fomin differentiable along } h\}$, $H(\mu) = \{h \in D(\mu)\colon \varrho_h(\mu) \in L^2(\mu)\}$. For further information see [BS], [Bo3], [YH].

A probability measure γ on X is called gaussian if all measures $\gamma \circ f^{-1}$, $f \in X^*$, are gaussian on the line (i.e. have gaussian densities or are Dirac measures). For each gaussian measure μ there exists a vector $a \in X$ and a centred gaussian measure γ such that $\mu = \gamma_a$. For centred γ denote by X_γ^* the closure of X^* in $L^2(\gamma)$, i.e. the space of measurable linear functionals. Note that in this case $D_C(\gamma) = D(\gamma) = H(\gamma)$ equals to the Cameron - Martin subspace (reproducing kernel) and can be identified with the dual to X_γ^* since $\varrho_h(\gamma) \in X_\gamma^*$ and each $f \in X_\gamma^*$ admits such representation. Let $\|h\|_\gamma = \|\varrho_h(\gamma)\|_{L^2}$.

With a gaussian measure γ one can associate Sobolev spaces $W^{p,r}$-completions of the class $\mathscr{FC}^\infty$ of cylindrical functions of the form $f = \varphi(g_1, \ldots, g_n)$, $\varphi \in C_b^\infty(R^n)$, $g_i \in X^*$, with respect to norms $\|f\|_{p,r} = \sum\limits_{k \leq r} \|\nabla^k f\|_{L^p(X,\mathscr{H}^k)}$, ∇ being the derivative along $H = H(\gamma)$, $\mathscr{H}^k$ being the space of k-linear mappings of the Hilbert - Schmidt type with the natural norm (see [IW]). Let $W^\infty = \bigcap\limits_{p,r} W^{p,r}$. In a similar way one defines $W^{p,r}(X, Y)$ for Hilbert Y.

2. Integration by parts formula in infinite dimensions

A. Constant directions.

Let μ be a measure differentiable along a vector field v and f be a function differentiable along this field. The equality

$$\int \partial_v f \mu = - \int f \, d_v\mu\,, \tag{2.1}$$

is called an integration by parts formula.

In the case of constant vector fields the first nontrivial formula of this type in infinite dimensional setting was proved in [ASF] for Fomin differentiable measures (on linear spaces) under assumption of the existence of both sides of (2.1) and some additional technical condition of "stable integrability". In this particular situation during some time the formula from [ASF] was remaining the most general, though in [Bo2] it was shown that the condition of differentiability of f can be replaced by the Lipschitz condition. In 1988 M. Khafisov [Kh1] noticed that using results on the existence of smooth conditional measures obtained by Yamasaki and Hora in [YH] one can reduce the problem to the one-dimensional case and thus proved (2.1) under the following conditions:

1. μ is Fomin differentiable along V, $\partial_v f$ exists μ-almost everywhere, $f \in L^1(\mathrm{d}_v\mu)$, $\partial_v f \in L^1(\mu)$, 2. for μ-a.a. $x \in X$ the function $t \to f(x + tv)$ is continuous and possesses the Lusin (N)-property (for example, is absolutely continuous or everywhere differentiable).

To get this result it suffices to use corresponding results [S] for functions on the line. For example, in [Kh1] the following theorem was used: if f has the (N)-property, g is absolutely continuous and both $\int f'g\, \mathrm{d}t$, $-\int fg'\, \mathrm{d}t$, exist, then they are equal.

2.1. Remark. In [Bo6], [BS] a particular case of Khafisov's result (for f absolutely continuous or everywhere differentiable on lines $x + R^1 v$) was mentioned with a more elementary proof. I would like to stress that this particular case was also indicated first by M. Khafisov (unfortunately, this was not pointed out explicitely in our earlier papers).

In all papers cited above the natural question about extending these results to Skorohod differentiability was left open. The following proposition answering this question follows the same ideas, but uses a little bit more refined one dimensional integration by parts formula [S, Theorem 2.5, Chap. 8, p. 355]:

$$(\mathscr{D}) \int_a^b Fg\, \mathrm{d}x = GF\Big|_a^b - (\mathrm{S}) \int_a^b G(x)\, \mathrm{d}F(\mathrm{d}x),$$

provided F is of bounded variation, g is $\mathscr{D}$- or $\mathscr{D}^*$-integrable on $[a, b]$ (under these conditions Fg is integrable in the same Denjoy sense), G being a primitive of g. In particularly, this formula covers the case where G is locally absolutely continuous or everywhere differentiable with $G \in L^1(\mathrm{d}F)$, $g \in L^1(F\, \mathrm{d}x)$.

2.2. Proposition. *Let μ be Skorohod differentiable along v, $f \in L^1(\mathrm{d}_v\mu)$, for μ-a.a. x functions $t \to f(x + tv)$ are differentiable or locally absolutely continuous, $\partial_v \in L^1(\mu)$. Then (2.1) holds.*

Proof. For measures on the line this follows from the formula above since μ admits a density F of bounded variation and $\mathrm{d}_v\mu = \mathrm{d}F$ (for extending that formula to the whole line it suffices to consider compositions $\varphi_n \circ G$ with $\varphi_n \in C_b^\infty(R)$,

$\varphi_n(t) = t$ if $|t| \leqq n$, $\varphi_n(t) = n+1$ if $t \geqq n+1$, $\varphi\ (-t) = -\varphi(t)$, $\sup|\varphi_n^{(m)}(t)| = K_m < \infty$). In the general case take a hyperplane Y complementary to R^1h and Skorohod differentiable conditional measures μ^y on lines $y + R^1h$, $y \in Y$, such that $\mu(B) = \int_Y \mu^y(B)\, \sigma(\mathrm{d}y)$, σ being the projection of $|\mu|$ on Y (see [BS], [YH]).

The exists an interesting connection between differentiability and quasi-invariance found by Skorohod [Sk]. His result can be extended to locally convex spaces as follows.

2.3. Proposition. *Let μ be a measure on a LCS X differentiable along h. If $\exp(|\varrho_h(\mu)|) \in L^\varepsilon(\mu)$ for some $\varepsilon > 0$,* then *μ is quasi-invariant along h and for $r_h = \mu_h/\mu$ we have:*

$$r_h(x) = \exp\left(\int_0^1 \varrho_h(x - sh)\, \mathrm{d}s\right) \tag{2.2}$$

Proof. In one-dimensional case one can use the arguments from [Sk]. In general case it suffices to choose differentiable and quasi-invariant conditional measures μ^y on lines $y + R^1h$, which is possible because $\exp(|\varrho_h(\mu^y)|) \in L^\varepsilon(\mu^y)$ a.e. This implies quasi-invariance of μ and the equality (2.2).

2.4. Remark. *The measure μ is h-quasi-invariant and (2.2) holds under the following weaker condition: μ has the logarithmic derivative $\varrho_h(\mu)$ such that $\int_0^1 \varrho_h(x - sh)\, \mathrm{d}s$ exists for a.a. x.*

Proof. Again it suffices to consider the one-dimensional case and $h = 1$. In this case μ possesses an absolutely continuous density f. Then we have: $f(x + t) = \exp(\int_0^t f'(x+s)/f(x+s)\ \mathrm{d}s)\, f(x)$. Indeed, choose some x with nonzero $f(x)$. Then due to integrability of f'/f the formula above holds for small t, since $(\ln f)' = f'/f$ near x. Both parts of this equality being continuious, we conclude in view of the integrability of f'/f that f is non vanishing.

Nevertheless, it should be noted that less general initial Skorohod's condition of expoinential integrability of ϱ_h can be easier for verification. In [AR] the formula (2.2) was proved under conditions: $\varrho_h(\mu) \in L^2(\mu)$ and μ admits conditional measures on lineso.n lines $y + R^1h$ with densities f^y such that $1/f^y$ are locally integrable (if $\varrho_h(\mu) \in L^{1+\varepsilon}(\mu)$ the condition $(1/f^y)^{1/\varepsilon} \in L^1_{loc}$ is sufficient). In particularly, this is true if $\varrho_h(\mu)$ is continuous (this was also proved in [B1] under additional conditions). There are interesting generalizations of the formula (2.2) for shifts along vector fields. We will discuss this question (connected also with Onsager - Machlup functionals) in the next section.

B. Vector fields.

Infinite dimensional integration by parts formulae involving vector fields appeared first for Gauss measures, in particular, in connection with stochastic integration in infinite dimension and Skorohad stochastic integral (see formulae and references in [DF], [NZ]). Researches in this direction has become especially active after celebrated Malliavin's works (see [B2], [BS]). It should be noted that implicitely formulae of these type were appearing even earlier in quantum field theory (sometimes at heuristic level) and in mathematical statistics (see [P1]–[P4]).

Recall that a collection $(X, \mathscr{B}, \mu, \mathscr{E})$ is called a measurable manifold [Bo6], if $(X, \mathscr{B}, \mu)$ is a measurable space equipped with an algebra $\mathscr{E} \subset \bigcap_{p>1} L^p(\mu)$, consisting of $\mathscr{B}$-measurable functions and satisfying the following condition: for all $f_1, \ldots, f_n \in \mathscr{E}$ and $\varphi \in C_p^\infty(R^n)$ one has $\varphi(f_1, \ldots, f_n) \in \mathscr{E}$.

Sometimes the following stronger condition is useful.

2.5. Definition. $\mathscr{E}$ is said to satisfy the condition (C) if whichever be an open $U \subset R^n$, $\psi \in C^\infty(U)$ and $F = (f_1, \ldots, f_n)$: $X \to U$ with $f_i \in \mathscr{E}$, $\partial^{(\alpha)}\psi(F) \in \bigcap L^p$ one has $\psi(F) \in \mathscr{E}$.

It isn't difficult to check that W^∞ satisfies this condition.

We say that a linear map v: $\mathscr{E} \to \mathscr{E}$ is a smooth vector field on X if, denoting $v(f)$ by $\partial_v f$, we have for all $\varphi \in C_p^\infty(R^n)$, $f_i \in \mathscr{E}$:

$$\partial_v \varphi(f_1, \ldots, f_n) = \sum_{i=1}^{n} \partial_{x_i} \varphi(f_1, \ldots, f_n)\, \partial_v f_i\,.$$

A measure λ on $(X, \mathscr{B})$ is called differentiable along the vector field v if $\mathscr{E} \subset L^1(\lambda)$ and there exists a measure $d_v\lambda$ such that for all bounded $f \in \mathscr{E}$ the following formula holds:

$$\int \partial_v f \lambda = -\int f \, \mathrm{d}_v \lambda\,. \tag{2.3}$$

If $d_v\lambda \ll \lambda$ then the corresponding Radon - Nikodym derivative is denoted by $\varrho_v(\lambda)$ (or just by ϱ_v if there is no risk of confusion) and is called a logarithmic derivative of λ along v.

It is possible to extend the notions of a vector field and corresponding differentiability to include also the case of not necessarily smooth fields v: $\mathscr{E} \to L^0(\lambda)$. Namely, in (2.3) one has to take only f with $\partial_v f \in L^1$.

A map $F = (f_1, \ldots, f_n)$, $f_i \in \mathscr{E}$, will be said nondegenerate, if there exist smooth vector fields $v_1, \ldots, v_n$, such that

$$\det\left((\partial_{v_i} f_j)_{i,j=1}^n\right)^{-1} \in \mathscr{E}\,.$$

2.6. Definition [M1]. Let $\mathscr{A}$ be subalgebra in $\mathscr{E}$, $E^{\mathscr{A}}$ being the conditional expectation corresponding to the σ-algebra $\sigma(\mathscr{A})$. We say that $E^{\mathscr{E}}$ is smooth if $E^{\mathscr{A}}(\mathscr{E}) \subset \mathscr{E}$.

Formally this definition differs from that of [M1], where the equality $E^{\mathscr{A}}(\mathscr{E}) = \mathscr{A}$ was claimed. But in the case of the definition above one can always replace the initial algebra $\mathscr{A}$ by its extension $\mathscr{A}_1 = \mathscr{E} \cap L^2(\mathscr{A})$ and thus obtain equalities $\sigma(\mathscr{A}) = \sigma(\mathscr{A}_1)$, $E^{\mathscr{A}_1}(\mathscr{E}) = E^{\mathscr{A}}(\mathscr{E}) = \mathscr{A}_1$. If $\mathscr{A}$ was closed in $\mathscr{E}$, then $\mathscr{A}_1$ is also closed.

2.7. Remark. It would be interesting to investigate the question when the closeness of $\mathscr{A}$ implies the equality $\mathscr{A} = \mathscr{E} \cap L^2(\mathscr{A})$. In general this is not true as the following easy example shows. Let $\mathscr{E} = C_b(R)$, $\mathscr{A} = \{f \in \mathscr{E} : \exists \lim_{|t| \to 0} f(t)\}$. Then $\mathscr{A}$ is closed in $\mathscr{E}$, $\sigma(\mathscr{E}) = \sigma(\mathscr{A}) = \mathscr{B}(R)$, but $E^{\mathscr{A}}\mathscr{E} = \mathscr{E}$ is bigger than $\mathscr{A}$.

An important example of a smooth conditional expectation is delivered by $\mathscr{A}$, generated by a nondegenerate map $F = (f_1, \ldots, f_n)$. This has been proved for functionals on the Wiener space by Ustunel and Zakai in [UZ] (see the proof of their Proposition 2.10) and by Malliavin in [M1]. Below we prove the same in a general setting.

2.8. Remark. Assume that $\mathscr{E}$ is equipped with a structure of a Fréchet (or barreled) space such that the natural inclusion $\mathscr{E} \to L^2(\mu)$ is continuous and let $\mathscr{A}$ be closed in $\mathscr{E}$. Then the map $E^{\mathscr{A}} : \mathscr{E} \to \mathscr{E}$ is automatically continuous, provided $E^{\mathscr{A}}$ is smooth (and $E^{\mathscr{A}} : \mathscr{E} \to \mathscr{A}$ is continuous, provided $E^{\mathscr{A}}$ is smooth in the Malliavin's sense). Indeed, since the map $E^{\mathscr{A}} : \mathscr{E} \to L^2(\mu)$ is continuous, we can apply the closed graph theorem. In particular, this is the case under conditions in [M1], so in Definition 4.2 in [M1] the map $E^{\mathscr{A}}$ is automatically continuous.

2.9. Proposition. *Let $F = (f_1, \ldots, f_n)$ be nondegenerate and $\mathscr{A} = \sigma(\{f_i\})$. Then $E^{\mathscr{A}}\mathscr{E} \subset \mathscr{E}$, provided $\mathscr{E}$ satisfies the condition (C).*

Proof. Let $\psi \in \mathscr{E}$ and denote by R the Radon - Nikodym derivative of $\nu = (\psi\mu) \circ F^{-1}$ with respect to $\lambda = \mu \circ F^{-1}$. Using the Mallivin's method it was proved in [Bo6] (see also [BS]) that measures ν and λ admit densities p_ν and p_λ belonging to the Schwartz space $\mathscr{S}(R^n)$. Hence $R(x) = p_\nu(x)/p_\lambda(x)$ on $U = \{|p_\lambda| > 0\}$ and $R(x) = 0$ on $R^n \setminus U$. Notice that $E^{\mathscr{A}}\psi = R \circ F$. So it suffices to show that $R \circ F \in \mathscr{E}$. Now we verify the inclusion $\partial^{(\alpha)} R \circ F \in \bigcap L^p$. First consider the case $\partial^{(\alpha)} = \partial_{x_i} = \partial$. Let $f \in C_0^\infty(U)$. Then $\partial R(F) f(F)$ is a bounded and measurable function which coincides with $\partial T(F) f(F)$ for some $T \in C_0^\infty(U)$ where $T = R$ in a neighbourhood of *supp f*. Hence for all $\xi \in \mathscr{E}$ we have:

$$E(\partial R(F) f(F) \; \xi) = E[\partial(RF)(F) \; \xi - R(F) \, \partial f(F) \; \xi] =$$
$$= E[\partial(Rf)(F) \; \xi] - E(\psi \, \partial f(F) \; \xi) .$$

The first integral in the r.h.s. can be represented as follows:

$$E[\partial(Rf)(F) \; \xi] = E\left[\sum_{j,k} \xi \gamma^{ij} \sigma_{jk} \, \partial_k(Rf)(F)\right] = E\left[\sum_j \xi \gamma^{ij} \, \partial_{v_j}((Rf) \circ F)\right] =$$
$$= \sum_j \int (Rf)(F) \; \mathrm{d}_{v_j}(\gamma^{ij} \xi \mu) = E[f(F) \, R(F) \, Q] ,$$

where $Q \in \mathscr{E}$ doesn't depend on f. In analogous way one can represent $E(\psi\, \partial f(F)\, \xi]$. So we obtain the equality

$$E[\partial R(F)\, f(F)\ \ \xi] = E[f(F)\, R(F)\, Q] + E[f(F)\, G] =$$
$$= E[f(F)\ \ \psi E^{\mathscr{A}} Q] + E[f(F)\, G]\,,$$

where $Q, G \in \mathscr{E}$ don't depend on f. For $\partial^{(\alpha)} R$ we apply this formula several times starting with $\xi = 1$. Thus we get:

$$E[\partial^{(\alpha)} R(F)\, f(F)] = E(f(F)\, R(F)\, V] + E[f(F)\, W] =$$
$$= E[f(F)\ \ \psi E^{\mathscr{A}} V] + E[f(F)\, W] = E[f(F)\, Z]\,, \qquad V, W, Z \in \ \mathscr{E}.$$

By Holder inequality the r.h.s. is majorated by $\|f(F)\|_p \cdot \|Z\|_q$. This implies that $\partial^{(\alpha)} R(F) \in \bigcap L^p$ and hence $R(F) \in \mathscr{E}$.

A vector field v on X is called $\mathscr{A}$-basic [M1] if $\partial_v(\mathscr{E}^{\mathscr{A}}) \subset \mathscr{E}^{\mathscr{A}}$.

2.10. Proposition. *Assume that v is $\mathscr{A}$-basic and $E^{\mathscr{A}}$ is smooth. Then the following Malliavin's identity [M1] holds:*

$$\partial_v(E^{\mathscr{A}} f) = E^{\mathscr{A}}(\partial_v f) + E^{a}(f\ \varrho_v(\mu)) - (E^{\mathscr{A}} f)\,(E^{\mathscr{A}}\ \varrho_v(\mu))\,, \qquad f \in \mathscr{E}. \quad (2.4)$$

Proof. Both sides of (2.4) being $\sigma(\mathscr{A})$-measurable, it suffices to prove the following equality for all $g \in \mathscr{A}$:

$$\int g\, \partial_v(E^{\mathscr{A}} f)\ \mu = \int g[E^{\mathscr{A}}(\partial_v f) + E^{\mathscr{A}}(f\ \varrho_v(\mu)) - (E^{\mathscr{A}} f)\,(E^{\mathscr{A}}\ \varrho_v(\mu))]\ \mu\,. \quad (2.5)$$

Applying the integration by parts formula to the function $gE^{\mathscr{A}} f$ and using $\sigma(\mathscr{A})$-measurability of $\partial_v g$ we get:

$$\int g\, \partial_v(E^{\mathscr{A}} f)\ \mu = -\int \partial_v g(E^{\mathscr{A}} f)\ \mu - \int g(E^{\mathscr{A}} f)\ \mathrm{d}_v\mu =$$
$$= -\int \partial_v g f\, \mu - \int g E^{\mathscr{A}} f\ \varrho_v(\mu)\, \mu\,.$$

On the other hand, the right-hand side of (2.5) equals to

$$\int [g\, \partial_v f + gf\ \ \varrho_v(\mu) - g(E^{\mathscr{A}} f)\ \ \varrho_v(\mu)]\ \mu\,,$$

and now it suffices to notice that applying the integration by parts formula again we have:

$$\int \partial_v g f\, \mu = -\int g\, \partial_v f\, \mu - \int gf\ \ \varrho_v(\mu)\, \mu\,.$$

C. Examples of vector fields and nonlinear Cameron - Martin formula.

There are two main ways to construct vector fields of differentiability. The first one is to form vector fieds of the type $v(x) = \sum f_n(x)\, a_n$ on the space equipped with some measure μ differentiable along a_n, where f_n are differentiable in a suitable sense. The second way (considered below) — fields generated by flows of transformations. Certainly, there is also a possibility to take nonlinear images of fields of these two types.

Consider the following situation appearing in the generalized Girsanov's transformations and in Bismut's approach to the Malliavin calculus (see [B2], [BS], [NZ], [Sm] for references). It should be noted that first such situation was investigated by T. Pitcher [P1]–[P4] for the purposes of estimating of parameters of stochastic process.

Let T_ε, $\varepsilon \geqq 0$, be measurable transformations of the probability space $(\Omega, \mathscr{B}, P)$ such that maps $\varepsilon \to f \circ T_\varepsilon$ into $L^1(P)$ are differentiable for sufficiently large algebra $\mathscr{E}$ of functions f. Define v by $\partial_v(f \circ T_0) = \partial f \circ T_\varepsilon / \partial\varepsilon|_{\varepsilon=0}$. Assume that $Q_\varepsilon = P \circ T_\varepsilon^{-1} = g_\varepsilon Q_0$ and $\varepsilon \mapsto g_\varepsilon$ is differentiable at zero as a map to $L^1(Q_0)$. Then P is differentiable along v and $\varrho_v(P) = -\partial g_\varepsilon \circ T_0/\partial\varepsilon|_{\varepsilon=0}$.

2.11. Example. Let w_t be a standard Wiener process on $[0, 1]$ and u be an adapted process with $|u| \in \bigcap L^p$. Define ξ^ε by

$$d\xi_t^\varepsilon = \sigma(t, \xi_t^\varepsilon)\, dw_t + [b(t, \xi_t^\varepsilon) + \varepsilon\sigma^2(t, \xi_t^\varepsilon)\, u_t]\, dt\,, \qquad \xi_0^\varepsilon = x\,,$$

where σ, b satisfy some usual conditions. Transformations T_ε are given by $T_\varepsilon(\omega)(t) = \xi_t^\varepsilon(\omega)$. By the Girsanov theorem measures $Q_\varepsilon = P \circ T_\varepsilon^{-1}$ are equivalent and g_ε are given by the formula

$$g_\varepsilon = \exp\left[\varepsilon \int_0^1 u_t\, d\xi_t^0 - 2^{-1}\varepsilon \int_0^1 u_t(2b(t, \xi_t^0) + \varepsilon\sigma^2(t, \xi_t^0)\, u_t)\, dt\right].$$

So $\partial g_\varepsilon/\partial\varepsilon|_{\varepsilon=0}$ exists and is equal to $\int_0^1 u_t\, d\xi_t^0 - \int_0^1 u_t b(t, \xi_t^0)\, dt$. Hence

$$\varrho_v = \int_0^1 \sigma(t, \xi_t^0)\, u_t\, dw_t\,.$$

In the case $\sigma = 1$, $b = 0$ we have the following expression for v:

$$v(\omega)(t) = \int_0^t u_s\, ds\,.$$

Recently the Malliavin calculus has been developed for processes with jumps (see references in [BS], in particularly, papers by J. Bismut, K. Bichteler, J. Gravereaux, J. Jacod, R. Leandre) and this also leads to vector fields of differentiability. In [CP] there is a special construction of a vector field on a standard Poisson space such that the corresponding logarithmic derivative coincides with a stochastic integral as in the example above. It would be interesting to compare this construction with that of [Sm] suggested for more general spaces of configurations.

The described method of constructing of vector fields of differentiability is especially convenient if one deals with groups. In [AM], [Kh2], [M2], [M3], [MM1], [MM2], [Sha] there are interesting examples of this kind for loop-groups and groups of diffeomorphisms. Similar results can be derived from constructions in [Sh2], [Sh3], [Ho3] for infinite dimensional rotation group and infinite dimensional torus. There are also interesting relations between stochastic integration and differentiability along vector fields (which appear also in the example above). In particular, one of the possible constructions [N] of a stochastic integral is to define it as a logarithmic derivative along a suitable vector field (possibly, operator-valued). Some information concerning these connections and further references can be found in [NZ], [N], [Nu]. We return to this question in the next section.

2.12. Remark. It is natural to ask how to determine densities g_ε or transformations T_ε knowing ϱ_v (like in Proposition 2.3 and Example 2.4). Up to now the best result in this direction is due to A. B. Cruzeiro [Cr] and is as follows. Let γ be a symmetric gaussian measure on a LCS X, $v\colon X \to H = H(\gamma)$ be a vector field, satisfying the conditions i) $v \in W^\infty(X, H)$, ii) for all λ

$$\exp(\lambda\|v\|) + \exp(\lambda\|\nabla V\|) + \exp(\lambda|\delta v|) \in L^1(\gamma)\,.$$

Then there exists a family of transformations U_t such that

$$U_t(x) = x + \int_0^t v(U_s x)\,\mathrm{d}s \qquad \text{for all } t \text{ and } \gamma\text{-a.e. } x\,,$$

$\mu \circ U_t^{-1}$ has a density k_t with respect to μ and denoting δv by ϱ_v

$$k_t = \exp\left(\left(\int_0 \varrho_v(U_{-s}x)\,\mathrm{d}s\right)\right). \tag{2.6}$$

In [DS] the formula (2.6) was announced in a more general case of a Banach manifold with a family of transformations U_t generating the field v and with a measure μ differentiable along v provided $\varrho_v(U_{-t}x)$ is continuous in t. But in fact the proof (analogous to that of [B2]) consists in differentiating in t the integral $\int f(U_t x)\, k_t \mu$ and applying the integration by parts formula for $\partial_v k_t$. Unlike Remark 2.4 both operations claim in general additional conditions (cf [B2]). One of the possibilities here is to claim integrability of $C\exp(\sup_{t\in[0,1]} |\varrho_v(U_{-t}x)|$. In [AM], [M3], [MM1], [MM2] there are generalizations of the Cruzeiro's theorem to the case of some infinite dimensional groups.

Various applications of infinite dimensional integration by parts formulae can be found in [AKR], [AR], [Bo7], [BS], [CZ], [K].

3. Logarithmic derivatives, subspaces of differentiability and their applications

In this section we discuss the following topics: integrability and differentiability properties of logarithmic derivatives, subspaces of differentiability and some applications of these objects.

A. Properties of logarithmic derivatives.

In [U] A. V. Uglanov suggested the following useful lemma.

3.1. Lemma [U]. *There exists a constant C such that for each nonnegative twice differentiable function φ: $R^1 \to R^1$ with absolutely continuous φ'' the following holds:*

$$J(\varphi) = \int (\varphi')^2 \varphi^{-1}\,\mathrm{d}t \leqq C\left(\int [|\varphi'| + |\varphi''| + |\varphi'''|]\,\mathrm{d}t\right), \qquad 0/0 = 0\,.$$

3.2. Corollary [U]. *If a nonnegative measure μ on a LCS X is 3 times differentiable along v then $\varrho_v(\mu) \in L^2(\mu)$.*

In particular, if a stable measure μ is differentiable along v then it is infinitely differentiable (see [Bo3]) and hence $\varrho_v(\mu) \in L^2(\mu)$ which gives an affirmative answer to Problem 6 in [BS]. In [Kr] E. Krugova found exact connection between the order of differentiability of a measure and integrability of its logarithmic derivative. We formulate her result in 3.4.

An easy example [U] ($\varphi(t) = |t|\,|\log|t||^{-1}$ in $(-\delta, \delta)$) shows that $J(\varphi)$ can't be estimated by means of $\|\varphi'\|_1$ and $\|\varphi''\|_1$.

3.3. Remark. *Notice that 3.2 is trivial for measures with bounded supports. Indeed, if the support of φ belongs to the segment of the length I and φ'' is absolutely continuous, then*

$$\int (\varphi')^2 \varphi^{-1} \leqq 3^{1/2} I \sup|\varphi''| \leqq 2I\|\varphi^{(3)}\|_1\,.$$

Proof. Fix $\delta > 0$ and put $\psi = \varphi + \delta$, $f = \psi^{1/2}$. Then

$$\int(\varphi'')^2 = \int(\psi'')^2 f^2 + 4\int(f')^4 + 8\int ff''(f')^{\,2} =$$

$$= 4\int(f'')^2 f^2 + 4\int(f')^4 + 8/3\int f((f')^{\,3})' =$$

$$= 4\int(f'')^2 f^2 + 4/3\int(f')^4 = 4\int(f'')^2 f^2 + 3^{-1}\int(\psi')^4\psi^{-2}\,.$$

Hence $\int(\varphi')^4 \varphi^{-2} \leqq 3\int(\varphi'')^2$ and we obtain the estimate above.

3.4. Proposition [Kr]. *Let μ be nonnegative. If $d_h^2\mu$ exists then $\varrho_h(\mu) \in L^{2-\varepsilon}(\mu)$ for all $\varepsilon > 0$. In addition,*

$$\|\varrho_h(\mu)\|_{2-\varepsilon} \leqq (1 + \varepsilon^{-1}) \|d_h\mu\| + (1 - \varepsilon)\, \varepsilon^{-1} \|d_h^2\mu\| \tag{3.1}$$

If $d_h^3\mu$ exists then $\varrho_h(\mu) \in L^{3-\varepsilon}(\mu)$ for all $\varepsilon > 0$ and for some constant $c(\varepsilon)$

$$\|\varrho_h(\mu)\|_{3-\varepsilon} \leqq c(\varepsilon)\,(\|d_h\mu\| + \|d_h^2\mu\| + \|d_h^3\mu\|) \tag{3.1'}$$

3.5. Remark. It would be interesting to find conditions ensuring inclusions $\varrho_h \in L_p$ for all $p > 1$. Even analyticity is not enough for this as the following trivial example (suggested by A. Popov) shows: $\varphi(t) = t^2 \exp(-t^2)$, functions $|\varphi'(t)|^p\, \varphi(t)^{1-p}$ are not integrable at the origin for all $p \geqq 3$.

3.6. Proposition. *Assume that μ_n are differentiable along h_n, h_n converge weakly to h, $\sup_n \|\varrho_{h_n}(\mu_n)\|_p \leqq C$ for some $p > 1$. If μ_n converge weakly to a measure λ (in fact weak convergence of finite dimensional projections suffices) then λ is differentiable along h and $\|\varrho_h(\mu)\|_p \leqq C$.*

Proof. On the linear space L spanned by $f = \exp(ig)$, $g \in X^*$ define a functional $F(f) = -\int \partial_h f \lambda$. Note that

$$F(f) = -ig(h) \int \exp(ig)\, \lambda = \lim -ig(h_n) \int \exp(ig)\, \mu_n =$$

$$= \lim \int \exp(ig)\, d_{h_n}\mu_n = \lim \int \exp(ig)\, \varrho_{h_n}(\mu_n)\, \mu_n = \lim \int f \varrho_{h_n}(\mu_n)\, \mu_n .$$

The same is true for all $f \in L$. By Holder inequality and weak convergence of finite dimensional projections of μ_n we have:

$$|F(f)| \leqq C \limsup \left(\int |f|^q \mu_n\right)^{1/q} = C\left(\int |f|^q \lambda\right)^{1/q} .$$

Therefore there exists $g \in L^p(\lambda)$ with $F(f) = \int fg\lambda$ and $\|g\|_p \leqq C$. This implies differentiability of λ and the equality $\varrho_h(\lambda) = g \in L^p(\lambda)$.

3.7. Proposition [U]. *Let measures μ and λ on a LCS X be n times differentiable along vectors from a finite dimensional linear space L and $\mu \ll \lambda$. Then there exists a density $F = \mu/\lambda$ that is λ-a.e. n times differentiable along directions in L.*

Proof. In order to simplify notations consider the case $n = 1$, $L = R^1 h$. Choose a complementary hyperplane Y and differentiable conditional measures μ^y and λ^y with absolutely continuous densities f^y, g^y on lines $y + R^1 h$. It is easy to see that $\mu \ll \nu \times l$, $\lambda \ll \nu \times l$, where ν is the projection of $|\lambda|$ on Y and l is the standard Lebesgue measure on R^1. Thus $\mu = f(\nu \times l)$, $\lambda = g(\nu \times l)$, $d_h\mu = \varphi(\nu \times l)$, $d_h\lambda = \psi(\nu \times l)$. Then ν-a.e. $f(y + th) = f^y(t) = \int_{-\infty}^t \varphi(y + sh)\, ds$, $g(y + th) = g^y(t) = \int_{-\infty}^t \psi(y + sh)\, ds$. It suffices to notice that $F = f/g$.

3.8. Corollary. *If μ is twice differentiable along h, then $\varrho_h(\mu)$ admits a.e. h-differentiable version.*

Not always F has a continuous version even if μ, λ are smooth: take the standard Gaussian measure μ on the line and $\lambda = t\mu$ or $\lambda = t^2\mu$.

3.9. Remark [U]. If $\lambda \geqq 0$ and $d_h^3\lambda$ exists then $\varrho_h(\lambda)$ admits a version ϱ that satisfies the equality $d_h^2\lambda = \varrho\, d_h\lambda + (\partial_h\varrho)\,\lambda$ in the sense that $\varrho \in L^1(d_h\lambda)$, $\partial_h\varrho \in$ - $L^1(\lambda)$. But if λ is only twice differentiable or signed this does not hold as easy examples show.

B. Subspaces of differentiability.

Subspaces of differentiability of measures were introduced in [Bo1]. The subspace of differentiability $D(\mu)$ of a measure μ in a LCS X is the linear space of all vectors h of Fomin differentiability equipped with the norm $h \mapsto \|d_h\mu\|$. The space $D_C(\mu)$ of all vectors of Skorohod differentiability is defined in the same way. As shown in [Bo1], [Bo3] $D(\mu)$, $D_C(\mu)$ are Banach and compactly embedded into X. $D(\mu)$ is closed in $D_C(\mu)$ and isomorphic to a closed linear subspace of $L^1(\mu)$ ([Bo3]), while $D_C(\mu)$ is isomorphic to a dual space ([Kh2], [BS]). All l^p, $1 \leqq p \leqq 2$, are isomorphic to some subspaces of differentiability [Bo5]. In particular, $D(\mu)$ need not be Hilbertized. However, Albeverio - Høegh-Krohn subspace $H(\mu) = \{h \in D(\mu) : \varrho_h(\mu) \in L^2(\mu)\}$ with the norm from L^2 is Hilbert. In a similar way, the space $H_p(\mu) = h \in D(\mu) : \varrho_h(\mu) \in L^p(\mu)\}$ equipped with the natural L^p--norm is complete.

Further information can be found in [BS], [H1], [Sh1], [Sh4].

3.10. Proposition. *Let μ be a measure on a LCS X and F be a barrelled space (for example, Fréchet space) continuously embedded in X. If $F \subseteqq F(\mu)$ then the map $R : v \mapsto \varrho_v(\mu)$, $F \to L^1(\mu)$, is continuous. If $F \subseteqq H(\mu)$, then the same is true for $R : F \to L^2(\mu)$.*

Proof. By the closed graph theorem the natural inclusion $F \to D(\mu)$ is continuous, since $D(\mu)$ is Banach and continuously embedded in X. On the other hand the map $v \mapsto \varrho_v(\mu)$, $D(\mu) \to L^1(\mu)$, is an isometry. These arguments are valid also in the second case.

Under some additional assumptions this assertion was proved in [AHK] by more complicated arguments.

3.11. Theorem. *Let μ be a nonzero measure on a LCS X. Then there exists a Hilbert space H compactly embedded in X with the following properties: 1) $D(\mu) \subset H$, 2) for each n one can find a probability measure ν_n on X with a compact support and n times Fréchet differentiable along H. If μ is separable then H is also separable.*

Proof. According to [Bo7, Theorem 1] there exists a probability measure ν on X with a compact support and 4 times differentiable along directions from

$D = D(\mu)$. Note that $D \subset H(\nu) \subset D(\nu)$. This follows from Corollary 3.2 but in this particular case can be checked directly as noted in Remark 3.4. Denote $H(\nu)$ by H and set $\nu_n = \nu * \ldots * \nu$, where the convolution is taken $n + 1$ times. Then ν_n have compact supports and are n times Fréchet differentiable along H. Indeed, for all $v_1, \ldots, v_n \in H$ the derivative $\mathrm{d}_{v_1} \ldots \mathrm{d}_{v_n} \nu_n = \mathrm{d}_{v_1} \nu * \ldots * \mathrm{d}_{v_n} \nu * \nu$ exists. Since $\|v\|_{D(\nu)} \leqq C\|v\|_H$, we get the following estimate for the map $T: a \to (\nu_n)_a$:

$$\|T^{(n)}(a) - T^{(n)}(b)\| \leqq \sup\{\|\mathrm{d}_{v_1}\nu * \ldots * \mathrm{d}_{v_n}\nu * \nu_a - \mathrm{d}_{v_1}\nu * \ldots * \mathrm{d}_{v_n}\nu * \nu_b\|,$$
$$\|v_i\|_H \leqq 1\} \leqq C^n \|\mathrm{d}_{a-b}\nu\| \leqq C^{n+1}\|a - b\|_H .$$

This estimate implies the n-fold Fréchet differentiability of T. Note that if μ is separable, then the measure ν constructed in [Bo7, Theorem 1] is separable too, which gives separability of H.

This theorem gives an affirmative answer to the question posed by V. Bentkus some years ago and mentioned in [BS, Problem 8-e]. As shown in [Bo5], [BS] not always one can find a Gauss measure γ with $D(\mu) \subset D(\gamma)$. Also, $D(\mu)$ need not be isomorphic to a Hilbert space. We don't know whether ν_n can be chosen infinitely differentiable or analytic along H.

Recall that a probability measure μ on a LCS X is called stable of the order $\alpha \in (0, 2]$ if for each n there exists $a_n \in X$ such that the distribution of the random vector $(X_1 + \ldots + X_n)/n^{1/\alpha} - a_n$ coincides with μ provided X_i are independent with distribution μ. Gaussian measures are stable of the order 2.

Combining Theorem 1 from [Bo3] and Uglanov's Corollary 3.2 we get the following result for stable measures.

3.12. Theorem. *Let μ be a measure on a LCS X, stable of some order α. Then the subspaces of continuity $C(\mu)$ and differentiability $D(\mu)$ coincide with the Hilbert space $H(\mu)$, μ is infinitely differentiable along $H(\mu)$ and, if $\alpha \geqq 1$, analytic along $H(\mu)$. If $\alpha \geqq 1$ or μ is not completely asymmetric in the sense of [Bo4], then $H(\mu) = Q(\mu) \approx \{h: \mu_{th} \sim \mu\ \forall t\}$.*

3.13. Remark. Let μ be a stable measure. 1°. We don't know whether $H(\mu)$ is always separable. 2°. If $\varrho_a(\mu)$ is in all $L^p(\mu)$? 3°. Is it true that μ and μ_a are singular for all vectors a not belonging to $C(\mu)$? 4°. For stable product-measures there is an estimate [BS]:

$$\|\mu_a - \mu\| \geqq 2(1 - \vartheta(\alpha)/\|\mathrm{d}_a\mu\|^{\alpha/4}) .$$

If an analogous estimate holds in a general case? Such estimate would imply the positive answer to the previous question.

C. Mappings and equations connected with logarithmic derivatives and some applications.

3.14. Remark. By the Pettis theorem the vector measure $\mathrm{d}\mu: \mathscr{B}(X) \to D(\mu)^*$ defined by the formula $\mathrm{d}\mu(A)(h) = \mathrm{d}_h\mu(A)$ has a bounded semivariation (see

[Bo3]), but as observed in [Bo3, Proposition 5] this measure has unbounded total variation if $D(\mu)$ is infinite dimensional and admits an equivalent Hilbert norm. Recently M. Khafisov has proved that the same holds for all $D(\mu)$ with $\dim D(\mu) = \infty$. If $D(\mu)$ is Hilbert this gives a vector measure in X, since we can identify $D(\mu)$ with its dual. Such measure usually is of bounded variation and this is exploited below.

Now we shall discuss the notion of a vector logarithmic derivative of a measure μ on a LCS X. This notion turned to be very usefull in applications to quantum field theory, stochastic quantization and infinite dimensional diffusions (see [AHK], [AKR], [AR], [CZ], [K]). There exist several constructions of this object.

I. In the first construction assume that μ is differentiable along directions from some Hilbert space E continuously embedded to X. We identify E and E^*. Then the measure $d\mu$ defined above takes values in E and consequently in X. If this X-valued measure possesses the Radon - Nikodym density Λ with respect to μ then Λ is called the logarithmic derivative of μ along E. According to [Bo3] such Λ exists if the embedding $E \to X$ is absolutely summing (which is always the case for Hilbert X).

II. In the second construction assume that H is a densely embedded Hilbert space in X. Identifying again H with H^* we get the triple $X^* \subset H^* = H \subset X$. Assume also that $X^* \subset D(\mu)$. If there exists a Borel map $\beta\colon X \to X$ with

$$ {}_{X^*}(k, \beta(x))_X = l_k(\mu)\,(x) \qquad \text{a.s. for all} \quad k \in X^* \subset X\,, $$

then we say that β is the logarithmic H-derivative.

Note that if $E = H$ and both maps exist then they coincide.

The following two problems arise in the connection with this construction. First, whether logarithmic derivatives completely determine measures (up to a multiplication by a constant) and, second, how to find a measure with a given logarithmic derivative. The latter problem posed in [ASF] admits two different formulations: a) knowing that β is a logarithmic derivative of some μ to reconstruct μ; b) find conditions on β (necessary and sufficient or sufficient) for existence of a measure for which β serves as logarithmic derivative. The problem a) was partially solved in [AR], where it was proved that μ can be reconstructed as an invariant measure for the diffusion with the drift β. The partial solution to b) was suggested in [K]. We study these problems in a separate paper.

3.15. Remark. In [H1] and [Sh4] there was discussed a natural question: which random variables ξ can be obtained as logarithmic derivatives of probability measures? Certainly, conditions $0 < E|\xi| < \infty$ and $E\xi = 0$ are necessary. It was proved in [H1] and [SH4] that, conversely, for each random variable ξ with $0 < E|\xi| < \infty$ and $E\xi = 0$ there exists a probability differentiable measure μ on the line such that the distribution of $\varrho_1(\mu)$ on (R^1, μ) equals to the distribution of ξ.

3.16. Example. *For each square integrable centered random variable ξ on the classical Wiener space there exists an adapted square integrable process η such that $\xi = \int_0^1 \eta_t \, \mathrm{d}w_t$ and thus ξ equals to ϱ_v for the field v derined by $v(\omega)(t) = \int_0^t \eta_s(\omega) \, \mathrm{d}s$.*

Proof. This follows from Example 2.11 in view of the well-known fact that ξ can be represented as a stochastic integral.

3.17. Example. *Let γ be a centered Gaussian measure on a LCS X. Then for each $\xi \in L^2(\gamma)$ with $\int \xi\gamma = 0$ there exists a vector field $v\colon X \to H$ such that γ is differentiable along v and $\varrho_v(\gamma) = \xi$ a.e. In addition, v can be chosen in $W^{2,1}(X, H)$ with $\|v\|_{2,1} = \|\xi\|_2$.*

Proof. One can reduce this example to the previous one, but the direct checking is also possible. Indeed, for simple fields of the form $v(x) = f(x)\,h$ with smooth cylindrical f and $h \in H$ one has: $\mathrm{d}_v\gamma = [\partial_h f + f\varrho_h]\,\gamma$. Hence

$$\|\varrho_v\|_2^2 = \int [(\partial_h f)^2 + f^2\varrho_h^2]\,\gamma + \int \partial_h(f^2)\,\varrho_h\gamma = \int [(\partial_h f)^2 - (\partial_h\varrho_h)\,f^2]\,\gamma =$$
$$= \int [(\partial_h f)^2 + \|h\|_H^2 f^2]\,\gamma.$$

Extending this correspondence by linearity, we get an isometry from $W^{2,1}(X, H)$ to $L^2(\gamma)$ and it suffices to check that the range is dense in the orthogonal complement to constants. Moreover, it is sufficient to consider only one-dimensional case, since a dense subspace is spanned by functions $g(x) = g_1(l_h(x))\,g_2(x)$, where g_1 is centered and g_2 is independent with l_h. Direct calculation shows that for each $\varphi \in D(R)$ with $\int \varphi\sigma = 0$, σ being standard gaussian, $v\colon x \mapsto \exp(t^2)\int_{-\infty}^x \varphi(t)\,\sigma(\mathrm{d}t)$ belongs to $D(R)$ and satisfies $v' - xv = \varphi$.

3.18. Remark. In recent years there has been a considerable progress in the study of infinite dimensional Dirichlet forms, in particular, the classical Dirichlet forms of the type $H(f, f) = \int (\nabla f, \nabla f)\,\mu$ with $(\nabla f, \nabla f) = \sum (\partial_{h_i} f)^2$. It turned out that closability of H is connected with continuity of μ along h_i (see [AR], [AKR]), while differentiability of μ is important for essential selfadjointness of the corresponding operator (see [Ko]).

3.19. Example. Theorem 5 in [K] asserts that the Dirichlet operator corresponding to a measure μ on $\mathcal{S}(R^n)'$ is the Schrodinger representation of some Araki hamiltonian provided μ possesses continuous and square integrable logarithmic derivatives for vectors in $\mathcal{S}(R^n)$. This assertion is a trivial reformulation of the fact that such condition implies Albeverio - Høegh-Krohn differentiability and hence follows directly from the characterization of Albeverio - Høegh-Krohn differentiability obtained in [Bo5]. In particular, an additional condition of continuity imposed in [K] can be omitted. In a similar way some results in [AKR], [AR] can be obtained as corollaries of results in [Bo2], [Bo3], [Bo5] on relationships between various differential properties of measures.

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