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Luděk Zajíček

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Supergeneric Results and Gateâux Differentiability of Convex and Lipschitz Functions on Small Sets

L. ZAJÍČEK

Praha*)

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We recall some recent results on Gateâux differentiability of convex functions on small convex sets and show that some of them are consequences of supergeneric results on differentiability of convex functions defined on the whole space. We show that supergeneric theorems imply also some results on differentiability of convex functions (and on singlevaluedness of monotone operators and metric projections) on small sets which can be of the first category in itself. A result concerning (relative) differentiability of a locally Lipschitz convex function defined on a closed convex set C with nonempty interior at boundary points of C is proved. A simple observation concerning differentiability of Lipschitz functions on small sets is also presented.

CONTENTS

1. Introduction
2. Support points of convex and arbitrary sets, related notions and lemmas
3. Supergeneric results
4. Differentiability of convex functions on small convex sets
5. Differentiability of Lipschitz functions on small convex sets
6. Differentiability of convex functions at boundary points of convex bodies
7. Concrete consequences of supergeneric results

1. Introduction

My lecture on the 25th Winter School in Abstract Analysis was entitled "Are supergeneric results interesting?". I was seeking for interesting non-trivial consequences of supergeneric results in the Banach space theory, in the formulation of

*) Department of Mathematical Analysis, Charles University, 186 00 Praha 8, Sokolovská 83, Czech Republic

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which the "supergeneric notions" (like σ-porosity) are not used. Now I realized that the applications concerning the Banach-Steinhaus theorem were trivial and the others concerning Fréchet differentiability should be further developed.

Therefore I consider here only questions concerning Gateâux differentiability of convex functions on real Banach spaces and closely related questions concerning singlevaluedness of monotone operators and metric projections. In these cases the supergeneric results (which are recalled in Section 3) say that exceptional sets are "σ-cone supported" (which, in separable spaces, is equivalent with the statement that these exceptional sets can be covered by countably many "Lipschitz hypersurfaces").

The main aims, except the search for interesting concrete consequences of supergeneric results (cf. Section 7), are the following:

(i) To comment (in Section 4) recent results (from [V], [R], [VV], [N], [BFK], [WC]) concerning Gateâux differentiability of convex functions (and singlevaluedness of monotone operators) on small convex sets. In particular, we show that the just mentioned results in separable spaces can be easily deduced from the 1978 supergeneric result of [Z2]. Also some (but not all) these results in non-separable spaces can be easily obtained from recent results of [Z4] and [H].

(ii) To make a simple observation (in Section 5) that the works of Aronszajn [A] and Phelps [P1] easily imply results on Gateâux differentiability of Lipschitz functions on small convex subsets of a separable Banach space (Theorem 2). Thus we obtain a further (measure theoretical) proof of the result oin generic Gateâux differentiability of locally Lipschitz convex functions on small convex subsets of a separable space (cf. Remark 6, (b)).

(iii) To consider also Gateâux differentiability of functions (and singlevaluedness of monotone operators and metric projections) on some convex sets which are of the first category in itself (in Sections 4 and 5). For example, we obtain that:

a) If F is an arbitrary Lipschitz function on $C[0,1]$, then there exists a monotone real analytic function f on $[0,1]$ such that F is Gateâux differentiable at f (cf. Proposition 2).

b) If T is a monotone multivalued operator $T: C[0,1] \to (C[0,1])^*$, then there exists a monotone real analytic functions g on $[0,1]$ such that T is not multivalued at g (cf. Proposition 4).

Some results of this type are related to interesting Stegall's result [S1] on differentiability of the composition of a Gateâux differentiable with a convex function (cf. Remark 5, (c)).

(iv) As an easy consequence of properties of δ-convex mapping between Banach spaces [VZ], we obtain results on Gateâux differentiability (w.r.t. C) of a locally Lipschitz convex function defined on a closed convex set C with nonempty interior at boundary points of C.

In the following Section 2 we recall definitions and facts concerning support points and prove lemmas which we need for our applications of supergeneric results.

2. Support points of convex and arbitrary sets, related notions and lemmas

Let X be a real Banach space. By a ball we mean an open ball in X. The ball with center c and radius r is denoted by $B(c, r)$. If $0 \neq v \in X$ and $0 < c < \|v\|$, we define (the cone)

$$A(v, c) = \{x \in X : x = \lambda v + w, \lambda > 0, \|w\| < c\lambda\} = \bigcup_{\lambda > 0} \lambda B(v, c).$$

Let $C \subset X$ be a convex set and $x \in C$. A geometrical form of the Hahn-Banach theorem (applied to $\bar{C}$) easily implies that the following properties are equivalent.
(i) There exists a cone $A(v, c)$ such that $C \cap (x + A(v, c)) = \emptyset$.
(ii) There exists a functional $f \in X^*$ such that $f(x) = \sup\{f(t) : t \in C\}$.

If these conditions hold, then x is said to be a support point of C; in the opposite case it is called a non-support point of C. In the literature interesting results on the sets $S(C)$ and $N(C)$ of all support and non-support points of C, respectively, can be found. The following facts are almost obvious.

(F1) $S(\bar{C}) \cap C = S(C)$.

(F2) $N(\bar{C}) \cap C = N(C)$.

(F3) $N(C) = \operatorname{int} C$, if $\operatorname{int} C \neq \emptyset$.

(F4) If $x \in N(C)$ and $y \in C$, then $[x, y) \subset N(C)$; in particular $N(C)$ is a convex set.

(F5) If $N(C) \neq \emptyset$, then $N(C)$ is dense in C.

Basic deeper results read as follows.

(F6) ([FK]) If X is separable and $C \subset X$ is a closed convex nonempty set which is contained in no closed hyperplane, then $N(C) \neq \emptyset$.

(F7) ([P2]) If $C \subset X$ is a closed convex set, then $N(C)$ is a G_δ set.

One part of the well-known Biship-Phelps theorem (cf. [P3]) reads as follows.

(F8) If $C \subset X$ is a nonempty closed convex set, then $S(C)$ is dense in the boundary bdry C.

Let now C be an arbitrary (possibly non-closed) convex subset of X. Then it is easy to see that the following statements hold.

(F9) By (F2) and (F7) $N(C)$ is a (relative) G_δ subset of C.

(F10) If $N(C) \neq \emptyset$, then by (F5) and (F9) $N(C)$ is a dense G_δ (and therefore residual) subset of C.

(F11) The following properties are equivalent:

(i) C is the second category in itself.
(ii) C is of the second category in $\bar{C}$.
(iii) C is the Baire space in the relative topology.

(F12) If X is separable, C is of the second category in itself and is contained in no clased hyperplane, then $N(C)$ is a (relative) dense G_δ subset of C.

Remark 1.

(i) The original proof of (F6) is very short. A different (topological) proof of (F6) can be immediately obtained via Theorem 2.8. from [BFK]. On the other hand, using (F6) and (F7), we can obtain almost immediately (via (F11)) the mentioned theorem from [BFK].

(ii) The fact (F6) does not hold in general non-separable spaces; if C is the positive cone in $l^2(\Gamma)$, where Γ is uncountable, then $N(C) = \emptyset$ (cf [R], Proposition 1, (h)).

Let now M be an arbitrary subset of a Banach space X and $x \in M$. Following [Z4], we say that M is cone-supported at x if there exists a cone $A(v, c)$ and $r > 0$ such that $M \cap (x + A(v, c)) \cap B(x, r) = \emptyset$. A subset of X is said to be cone supported if it is cone supported at all its points. A set is called σ-cone supported if it can be written as a union of countably many cone supported sets. Clearly, if A is convex, then $x \in S(A)$ iff A is cone supported at x. Therefore we can say that $x \in A$ is a support (or non-support) point of an arbitrary set A if A is (is not, respectively) cone supported at x. Thus we have defined $N(A)$ and $S(A)$ for an arbitrary set $A \subset X$.

Remark 2. The statements (F1) and (F2) clearly hold also for an arbitrary C.

In $\mathbb{R}^2$ the notion of a σ-cone supported set was used by W. H. Young [Y] and H. Blumberg [B] under the names "ensemble ridée" and "sparse set", respectively; they used a different but equivalent definition. In separable Banach spaces σ-cone supported sets were used in [Z1] and [Z2]. In fact, if X is separable, then a more transparent definition of σ-cone supported sets can be used.

Lemma 1. *Let X be a separable Banach space. Then a set $M \subset X$ is σ-cone supported iff it can be covered by countably many Lipschitz hypersurfaces.*

(It is e.g. an easy consequence of Lemma 1 from [Z1], cf. also [Z6].) Here we use the following definition.

Definition 1. Let X be a Banach space and let $0 \neq v \in X$. We shall say that $M \subset X$ is a Lipschitz hypersurface associated with v if there exists a topological complement Z of $sp\{v\}$ and a Lipschitz function $f : Z \to \mathbb{R}$ such that

$$M = \{z + f(z)\, v : z \in Z\}.$$

If f is a difference of two Lipschitz convex functions on Z, we say that M is a δ-convex hypersurface.

Remark 3. Note that each σ-cone supported subset of a separable Banach space is a null set in the Aronszajn sense and thus also null for each non-degenerate Gaussian measure (cf. [Z6], [Z1]).

Later on, we will need the following elementary lemmas concerning support points and σ-cone supported sets.

Lemma 2. *Let X be a Banach space, let $C \subset X$ be an arbitrary set and let $A \subset X$ be a σ-cone supported set. Then the following statements hold.*

(i) If $N(C)$ is of the second category in C then $A \cap C$ is not residual in C (i.e. $C \setminus A$ is of the second category in C).

(ii) If $N(C)$ is residual in C then $A \cap C$ is of the first category in C.

Proof. Let $A = \bigcup_{n=1}^{\infty} A_n$, where each A_n is cone supported. If either $N(C)$ is of the second category in C and $A \cap C$ is residual in C or $N(C)$ is residual in C and $A \cap C$ is of the second category in C, then there clearly exists an index n such that $N(C) \cap A_n$ is of the second category in C. Therefore there exists an open set G such that $G \cap C \neq \emptyset$ and $N(C) \cap A_n \cap G$ is dense in $G \cap C$. Choose a point $x \in N(C) \cap A_n \cap G$. Since $x \in S(A_n)$, we have $x \in S(N(C) \cap A_n \cap G)$ and consequently (by Remark 2) $x \in S(G \cap C)$ which clearly contradicts to $x \in N(C)$.

Lemma 3. *Let X, Y be Banach spaces; let an open $G \subset X$, an arbitrary $M \subset Y$ and an $a \in G$ be given. Let $f : G \to Y$ be a continuous mapping which has a Gateâux derivative L at a such that $\overline{L(X)} = Y$. Then $f(a) \in S(M)$ implies $a \in S(f^{-1}(M))$.*

Proof. We can and will suppose that f is defined and continuous on whole X. Let $b := f(a) \in S(M) \subset M$. Then we can clearly find a ball $B(v, r) \subset Y$ such that $M \cap (b + tB(v, r)) = \emptyset$ for each $t \in (0, 1)$. Further we can easily find a ball $B(u, e) \subset X$ such that $L(B(u, e)) \subset B(v, \frac{r}{2})$. Thus we have

$$\lim_{t \to 0} \frac{f(a + tw) - f(a)}{t} = L(w) \in B\left(v, \frac{r}{2}\right)$$

for each $w \in B(u, e)$. Consequently for each such w we can find a natural number n such that

$$\frac{f(a + tw) - f(a)}{t} \in B\left(v, \frac{r}{2}\right) \qquad \text{whenever} \quad 0 < t < \frac{1}{n} \tag{1}$$

Since X is complete, we can find a natural n, a ball $B(z, p) \subset B(u, e)$ and a dense subset D of $B(z, p)$ such that (1) holds for each $w \in D$. Thus, for each $0 < t < \frac{1}{n}$, we have

$$f(a + tB(z, p)) = f(\overline{a + tD}) \subset \overline{f(a + tD)} \subset \overline{b + tB\left(v, \frac{r}{2}\right)} \subset b + tB(v, r) \subset Y \setminus M .$$

Consequently

$$\bigcup_{0 < t < \frac{1}{n}} (a + tB(z, p)) \cap f^{-1}(M) = \emptyset .$$

Thus obviously $a \in S(f^{-1}(M))$.

Lemma 4. *Let X, Y be Banach spaces, $G \subset Y$ be an open set and let $g: G \to X$ be a continuous mapping. Let $D \subset G$ be a set which is not σ-cone supported and, for each $a \in D, g$ has a Gateâux derivative L_a at a such that $\overline{L_a(Y)} = X$. then $g(D)$ is not σ-cone supported.*

Proof. Suppose on the contrary that $g(D) = \bigcup_{n=1}^{\infty} M_n$ and each M_n is cone supported. Then $D = \bigcup_{n=1}^{\infty} (f^{-1}(M_n) \cap D)$. If $a \in f^{-1}(M_n) \cap D$, then $f(a) \in M_n = S(M_n)$ and by Lemma 3 we have $a \in S(f^{-1}(M_n))$ and consequently $a \in S(f^{-1}(M_n) \cap D)$. Thus each set $f^{-1}(M_n) \cap D$ is cone supported which yields a contradiction.

Notation. *Let C be a subset of a Banach space X. We shall say that $M \subset C$ is u-dense (or c-dense) in C if $M \cap G$ is uncountable (has cardinality at least continuum, respectively) whenever $G \subset X$ is open and $C \cap G \neq \emptyset$.*

Lemma 5. *Let X, Y be Banach spaces, $G \subset Y$ be an open set and let $g: G \to X$ be a continuous mapping. Let $D \subset G$ be a nonempty set which is a Baire subspace of X and such $N(D)$ is residual in D. Let, for each $a \in D, g$ has a Gateâux derivative L_a at a such that $\overline{L_a(Y)} = X$ and let $S \subset X$ be a σ-cone supported set. Denote $C := g(D)$. Then $C \setminus S$ is u-dense in C. If Y is separable and D is an analytic set then $C \setminus S$ is even c-dense in C.*

Proof. Choose an open ball $B \subset X$ such that $B \cap C \neq \emptyset$ and put $D^* = D \cap g^{-1}(B) = g^{-1}(B \cap c)$. Lemma 2 implies that D^* is not σ-cone supported and thus $B \cap C$ is not σ-cone supported by Lemma 4. Thus $(B \cap C) \setminus S$ is not σ-cone supported and consequently it is uncountable. If Y (and therefore also X) is separable then we can suppose (by Lemma 1) that S is an F_σ set. If D is analytic then $(B \cap C) \setminus S$ is analytic as well and we can use the Hausdorff-Aleksandrov theorem to infer that it is of cardinality continuum.

3. Supergeneric results

A result which asserts that an exceptional set (e.g. the set of all non-differentiability points of a convex continuous function on a Banach space) is a first category set is called by some authors a generic result. Some generic results were improved: it was shown that some exceptional sets belong to a family Φ which is a proper subfamily of the family of all first category sets. In many of these "supergeneric results" Φ is the family of all σ-porous sets or a family defined in a similar way. There are many results of this type in the Real Analysis (cf. [Z5]).

In this section, we will cite several supergeneric results in infinite-dimensional Banach space which say that exceptional sets are σ-cone supported. The first three theorems concern separable Banach spaces; thus we can and will use the characterization of σ-cone supported sets from Lemma 1.

Theorem A. *Let X be a separable Banach space and let $T: X \to X^*$ be a monotone operator. Then the set of all points at which T is multivalued can be covered by countably many Lipschitz hypersurfaces.*

This theorem was proved in [Z2]. For slightly sharper results see [Vý].

Theorem A can be clearly applied to differentiability of continuous convex functions since the subdifferential ∂f is a monotone operator. In fact, in this case a sharper (the best possible) result holds [Z3] (cf. also [Vý] for the case of proper convex functions).

Theorem B. *Let X be a separable Banach space and let f be a continuous convex function on X. Then the set of all Gateâux non-differentiability points of f can be covered by countably many Lipschitz δ-convex hypersurfaces.*

To formulate results which concern the abstract approximation theory we need a notation.

Definition 2. Let X be a Banach space and let $F \subset X$ be a closed set. Let P_F be the metric projection on F (the nearest point mapping), $P_F(x) = \{y \in F : \|x - y\| = dist(F, x)\}$. Then we put

$$A(F) = \{x \in X : card(P_F(x)) \geq 2\}.$$

The set $A(F)$ was investigated in a number of articles. Sometimes the name "ambiguous locus of P_F" is used for $A(F)$ (cf. [DM]). Note that in the case of a nonempty compact set F we have that the metric projection P_F is singlevalued at all points $x \notin A(F)$.

The following result was proved in [Z1] and [Z6].

Theorem C. *Let X be a separable strictly convex (i.e. rotund) Banach space and let F be a closed subset of X. Then $A(F)$ can be covered by countably many Lipschitz hypersurfaces. If X is even a separable Hilbert space, then $A(F)$ can be covered by countably many Lipschitz δ-convex hypersurfaces.*

Theorem A was generalized to some non-separable Banach spaces (namely to Asplund spaces and to spaces with a strictly convex dual space) in [Z4]. Recently a further generalization (Theorem D below) was published in [H]. Following C. Stegall we shall say that a Banach space Y is a (GSG)-space if there exist an Asplund space Z and a continuous linear mapping $L: Z \to Y$ such that $Y = \overline{L(Z)}$. Note that each separable and each reflexive space is a (GSG)-space.

Theorem D. *Let X be a closed subspace of a (GSG)-space and let $T: X \to X^*$ be a monotone operator. Then the set of all points at which T is multivalued is σ-cone supported.*

The following non-separable analogue of the first part of Theorem C was proved in [M].

Theorem E. *Let X be a strictly convex Banach space with a uniformly Fréchet differentiable norm and let F be a closed of X. Then the set $A(F)$ is σ-cone supported.*

Finally I will mention 3 related problems which are, as far as I know, still open. The first one is the old well-known Stechkin's problem [Sn].

Problem 1. *Let X be a strictly convex Banach space and $F \subset X$ be a closed set. Is $A(F)$ necessarily a first category set?*

The second problem was formulated in [Z2].

Problem 2. *Can the exceptional set of Theorem A be covered by countably many Lipschitz δ-convex hypersurfaces?*

Note that L. Veselý [Vý] proved the positive answer in the case $X = \mathbb{R}^2$.

Problem 3. *Can Theorem D be generalized to the case when X belongs to Stegall's class (S) or even to the case of an arbitrary weak Asplund space X?*

Note that each subspace of a (GSG) space is an (S)-space (cf. [S1]) but the converse implication does not hold. However, as far as I know, each "concrete classical" (S)-space is a subspace of a (GSG)-space.

4. Differentiability of convex functions on small convex sets

The question of Gateâux differentiability of convex functions on small convex sets was investigated in a series of articles [V], [R], [VV], [N], [BFK], [WC].

Let $C \subset X$ be a convex set and let f be a function on C. We say that f is Gateâux differentiable at a point $a \in C$ (w.r.t. C) if there exists $L \in X^*$ such that, for every $c \in C$,

$$\lim_{t \to 0+} \frac{f(a + t(c-a)) - f(a)}{t} = L(c-a).$$

It is easy to see that if C is not contained in a closed hyperplane then the Gateâux derivative L, if it exists, is determined uniquely. Namely, in this case clearly $\overline{sp}(C-a) = X$, where $C - a := \{c - a : c \in C\}$.

We can ask whether each continuous convex function f on C is Gateâux differentiable at all points of a dense (u-dense, c-dense) subset of C or of a residual (i.e. "generic") subset of C, for different types of convex sets C. Of course, in the generic differentiability problem, only the case when C is of the second category in itself is interesting.

In all articles cited above the question of generic differentiability was considered and it was supposed that f is locally Lipschitz on C. In fact, if f is only supposed to be continuous and convex, then an example in [R] shows that the "generic Gateâux differentiability problem" can have negative answer even when C is a closed convex subset of l^2 with $N(C) \neq \emptyset$ (in this example even $\partial f(x) = \emptyset$ for each $x \in N(C)$). The same example is a counterexample also for the "dense differentiability problem".

In both "generic" and "dense" differentiability problems we can suppose that f is Lipschitz on C (since both residual an dense sets are "locally determined"). In this case f is a restriction of a convex Lipschitz function defined on the whole X. It can be proved using an infimal convolution (cf. [BFK], [WC]), but we can also use a more elementary argument. In fact, the geometrical form of the Hahn-Banach theorem easily implies that $\partial f(x) \neq \emptyset$ for each $x \in C$. Thus we can choose $g_x \in \partial f(x)$ for each $x \in C$ and obtain an extension as $\tilde{f}(t) := \sup\{f(x) + g_x(t - x) : x \in C\}$. Thus, using also (F11), we easily obtain the following (slightly non-formally formulated) fact.

(F13) In both "generic" and "dense" differentiability problem for locally Lipschitz convex functions f defined on an arbitrary convex subset C of a Banach space X, we can suppose without any loss of generality that f is a Lipschitz and convex function defined on the whole X. In the "generic differentiability problem" we can also suppose that C is closed.

In [V] and [R] it is supposed that C is a closed set, f is convex on C and locally Lipschitz on $N(C) \neq \emptyset$. Because of the fact (F4) above f is clearly Gateâux differentiable at a point $a \in N(C)$ w.r.t. C iff it is Gateâux differentiable w.r.t. $N(C)$. Thus we see that also in this case the generic problem reduces equivalently to the case of a convex Lipschitz function on a closed convex set. Note also the following easy fact.

(F14) Let X be a Banach space, C be a convex set, let f be a convex and Lipschitz function on C and let $a \in N(C)$ be given. Then the following statements are equivalent:

(i) f is Gateâux differentiable at a w.r.t. C.

(ii) The subdifferential $\partial f(a)$ is a singleton.

(iii) A Lipschitz convex extension of f over X is Gateâux differentiable at a.

(iv) Every Lipschitz convex extension of f over X is Gateâux differentiable at a.

Thus we obtain that the positive answer to the "generic differentiability problem" for closed sets with $N(C) \neq \emptyset$ is equivalent to a supergeneric result. Namely, the following fact clearly holds.

(F15) Let X be a Banach space. Then the following assertions are equivalent.

(i) If C is a closed convex set with $N(C) \neq \emptyset$ and if f is a (locally) Lipschitz convex function on C, then f is Gateâux differentiable (w.r.t. C) at all points of a residual subset of C.

(ii) Let f be a continuous convex function on X and let S be the set of all non-differentiability points of f. Then $S \cap C$ is of the first category in C for each closed convex set C with $N(C) \neq \emptyset$.

The following result was essentially proved in [V]. We will show that it is an easy consequence of Theorem A.

Theorem F. *Let X be a separable Banach space, let $C \subset X$ be convex set which is of the second category in itself and let f be a locally Lipschitz convex function on C. Then f is Gateâux differentiable (w.r.t. C) at all points of a residual subset of C.*

Proof. Considering f only on the closed affine hull on C and using (F6) and (F13), we can suppose that C is closed, $N(C) \neq \emptyset$ and f is convex and Lipschitz on X. Let A be the set of all points $x \in X$ at which f is not Gateâux differentiable. By Theorem A (applied to $T := \partial f$) or by Theorem B we have that A is σ-cone supported. By (F10) and Lemma 2 we obtain that $C \cap A$ is of the first category in C and the assertion of the theorem follows. For an alternative "measure teoretical" proof see Remark 6, (b).

Theorem F was generalized to (non-separable) (S)-spaces independently in [VV], [N] and [BFK]:

Theorem G. *Let X be a Banach space which belongs to Stegall's class (S), let $C \subset X$ be a convex set which is of the second category in itself and let f be a locally Lipschitz convex function on C. Then f is Gateâux differentiable (w.r.t. C) at all points of a residual subset of C.*

Note that Theorem G was proved in the special case when $N(C) \neq \emptyset$ already in [R]. (In this case it is an immediate consequence of Theorem H below which deals with general monotone operators.) It seems that Theorem G cannot be simply deduced from this special case (cf. Remark 1, (ii)). In this special case and under the (slightly) more strict assumption that X is a subspace of a (GSG)-space, Theorem G is an easy consequence of the "supergeneric" Theorem D (cf. the text after Theorem H). The following natural problem (formulated in [R] in the case $N(C) \neq \emptyset$) seems to be open.

Problem 4. *Can Theorem G be generalized to the case when X is a weak Asplund space?*

Remark 4.

(i) Each (S)-space is a weak Asplund space. It is an open problem whether there two types of spaces coincide.

(ii) In [WC] a claim can be found that Problem 4 has the positive answer in the case $N(C) \neq \emptyset$ (cf. also the review MR 94j:46046). But I do not understand the argument and think that Problem 4 is still open also in this special case.

The following theorem is a corollary of Theorem 2.6. (see Remarks 2.7., (2), (4)) of [BFK]; cf. also [VV], Corollary of Theorem B.

Theorem H. *Let X be a Banach (S)-space an let $C \subset X$ be a convex set which is of the second category in itself and such that $N(C) \neq \emptyset$. Let $T : X \to X^*$ be* *a monotone operator such that $D(T) \cap C$ is residual in C. Then T is single-valued at each point of a residual subset of C.*

Of course, in the (slightly less general) case when X is a subspace of a (GSG)-space, the assertion of Theorem H immediately follows from Theorem D and Lemma 2. Moreover, in this special case we can omit the assumption concerning $D(T)$ and assert that T is not multivalued at each point of a residual subset of C.

Using supergeneric results and lemmas on σ-cone supported sets we obtain easily the following theorem concerning the existence of "regular points" in (convex) sets which are typically of the first category in itself.

Theorem 1. *Let X, Y be Banach spaces and let $g : Y \to X$ be a continuous linear mapping such that $\overline{g(Y)} = X$. Let $D \subset Y$ be a (not necessarily convex) set which is a Baire subspace of Y such that $N(D)$ is residual in D. Let $C := g(D)$. Then the following assertions hold.*

(i) If X is a subspace of a (GSG)-space and $f : X \to \mathbb{R}$ is a continuous convex function, then f is Gateâux differentiable at all points of a u-dense subset of C.

(ii) If X is a subspace of (GSG)-space and $T : X \to X^*$ is a monotone operator, then the set of points of C at which T is not multivalued is u-dense in C.

(iii) Let X be a strictly convex Banach space which is moreover separable or which has a uniformly Fréchet differentiable norm. Let F be a closed subset of X. Then the set $C \setminus A(F)$ is u-dense in C. If $Y = X$ and $g = id$ then $C \setminus A(F)$ is a residual subset of $C = D$.

If Y is separable and D is analytic then we can write in (i)–(iii) "c-dense" instead of "u-dense".

Moreover, all assertions hold under a more general assumption concerning g. Namely, we can suppose that $g : G \subset Y \to X$ satisfies the assumptions of Lemma 5.

Proof. Of course, (i) is an immediate consequence of (ii) (we put $T := \partial f$). To prove (ii), it is sufficient to use Theorem D and Lemma 5. To prove (iii), we use Theorem C, Theorem E. Lemma 5 and Lemma 2.

For some concrete consequences of Theorem 1 see Section 7. I know such interesting consequences only in the case when D and C are convex and g is linear. But also in this most interesting case the assertions of Theorem 1 are probably new. However, as we shall see in the next remark, in some special cases at least the "dense" versions of these assertions are easy consequences of known results.

Remark 5.

(a) If $X = Y$, $g = id$ and $D = C$ is convex then Theorem G and Theorem H show that the assertions (i), (ii) hold also under the more general assumption that X is an (S)-space.

(b) Using Theorem G and Lemma 7 below, it is easy to obtain that (i) holds also in the case, when g is linear (i.e. satisfies the basic assumption of the theorem), Y is an (S)-space and X is arbitrary. If moreover $D = Y$, it is sufficient to suppose that Y is a weak Asplund space.

(c) At least a weaker version of the assertion (i) (with "dense" instread of "u-dense") holds also in the case when $D = Y$ and X is an (S)-space. In fact, it follows from Lemma 7 below and following Stegall's result (Theorem 2, (i) of [S1]):

Let G be an open subset of a Banach space Y. Let X be an (S)-space and let $g : G \to X$ be continuous and Gateâux differentiable on a dense G_δ subset oif G. Let $f : X \to \mathbb{R}$ be a continuous convex function. Then $f \circ g$ is Gateâux differentiable on a dense G_δ subset of G.

(d) The preceding observations strongly suggest that (i) (and perhaps also (ii)) can be generalized to the case of an (S)-space X.

(e) If Y and X are separable, X is strictly convex, D is closed convex and g is also locally Lipschitz, then we can obtain at least the "dense" version of (iii) by an alternative way. Namely, we can apply (as in, e.g., [Z6]) Proposition 3 below to the (Lipschitz) distance function $f(x) := dist(x, F)$. If g is moreover linear then (iii) is a consequence of Theorem 2 below.

5. Differentiability of Lipschitz functions on small convex sets

Let X be a separable Banach space. A set $M \subset X$ is said to be Gaussian null if $\mu(M) = 0$ whenever μ is a non-degenerate (i.e. such that $\mu(G) \neq 0$ for each nonempty open set $G \subset X$) Gaussian measure on X. R. R. Phelps, using the result of [A], proved in [P1] that each (locally) Lipschitz function on X is Gateâux differentiable at all points except a Gaussian null set. In this section we present some easy consequences of this result and of the following lemma from [P1].

Lemma P. *Let X be a separable Banach space. Let (w_n) be a sequence in X such that $w_n \to 0$ and $\overline{sp}\{w_n\} = X$. Then the set $\overline{conv}(\{w_n\} \cup \{-w_n\})$ is not Gaussian null.*

An alsmost immediate consequence of this lemma is the following fact.

Lemma 6. *Let X be a separable Banach space and let C be a closed convex subset of X which is contained in no closed hyperplane of X. Then $N(C)$ is not Gaussian null. Moreover, if Y is a Banach space and $L : X \to Y$ is a linear continuous mapping such that $\overline{L(X)} = Y$, then $L(N(C))$ is not Gaussian null.*

Proof. Since a translate of a Gaussian null set is Gaussian null, we can suppose $0 \in C$. We can clearly find a sequence (v_n) in C such that $\|v_n\| \leq 1$ and $\overline{sp}\{w_n\} = X$. Now put $w_n = \frac{v_n}{4^n}$, $x = \sum_{n=1}^{\infty} w_n$ and

$$E = x + \overline{conv}(\{w_n\} \cup (-w_n\}) = \overline{conv}(\{x + w_n\} \cup \{x - w_n\}).$$

Since clearly both $x + w_n \in C$ and $x - w_n \in C$, we have $E \subset C$. Lemma P implies that E is not Gaussian null and thus C is not Gaussian null as well. Since $S(C)$ is Gaussian null by Remark 3, we can find a non-degenerate Gaussian measure μ on X such that $\mu(N(C)) > 0$. Then the image measure $\nu = L(\mu)$ is clearly a non-degenerate Gaussian measure on Y and $\nu(L(N(C))) > 0$.

Proposition 1. *Let X be a separable Banach space, let C be a closed convex subset of X which is contained in no closed hyperplane of X and let f be a (locally) Lipschitz function on X. Then f is Gateâux differentiable at all points of a c-dense subset of $N(C)$.*

Proof. Consider a closed ball B such that $\operatorname{int} B \cap C \neq \emptyset$. Lemma 6 clearly implies that $B \cap N(C)$ is not Gussian null. Since the (Borel) set N of all Gateâux non-differentiability points of f is Gaussian null we obtain that $(B \cap N(C)) \setminus N$ is not Gaussian null. Since it is Borel, we conclude that it has cardinality continuum which completes the proof.

Remark 6.

(a) If we consider relative differentiability (w.r.t. C) of f then we can clearly suppose that $C \subset X$ is a general nonempty closed convex set and conclude that each (locally) Lipschitz function f on C is Gateâux differentiable (w.r.t. C) at each point of a c-dense subset of C.

(b) Using the fact that the set of all Gateâux differentiability points of a continuous convex function on a separable Banach space is a G_δ set (cf. [P3]), we see that Proposition 1 easily implies Theorem F.

(c) I do not know whether we can writte "G_δ convex set C" instead of "closed convex set C" in Proposition 1.

(d) I also do not know whether there exist "relative differentiability" versions of Proposition 1 which deal with non-separable X and/or Fréchet differentiability.

Quite similarly as Proposition 1 (using the second part of Lemma 6) we obtain its generalization that deals with convex sets which can be of the first category in itself.

Theorem 2. *Let X, Y be a separable Banach spaces and let D be a closed convex subset of Y which is contained in no closed hyperplane of Y. Let $L : Y \to X$ be a linear continuous mapping such that $\overline{L(Y)} = X$. Let f be a (locally) Lipschitz function on X. Then f is Gateâux differentiable at all points of a c-dense subset of $C := L(N(D))$.*

Of course, if we are interested in the "relative" (w.r.t. $L(D)$) differentiability of f then we can clearly suppose that X is an arbitrary Banach space and omit the assuption $\overline{L(Y)} = X$.

We present here only one from many concrete consequences of Theorem 2.

Proposition 2. *Let F be a locally Lipschitz function on $C[0, 1]$. Then the set of all increasing real analytic functions g on $[0, 1]$ such that F is Gateâux differentiable at g has cardinality c.*

Proof. It is sufficient to apply Theorem 2 with $X = C[0, 1]$, $Y = l^2$, $D = \{(a_n) \in l^2 : a_n \geq 0\}$, $L((a_n)) = \sum_{n=1}^{\infty} 2^{-n} a_n x^{n-1}$ and $f = F$.

At least the "dense version" of Theorem 2 can be proved under weaker assumptions using the following lemma. I suppose that it is well-known; because of the lack of a reference I present the easy proof. If f is a mapping from a Banach space X to a Banach space Y and $x, v \in X$ are given then we consider the directional derivative

$$f'(x, v) = \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t}.$$

Lemma 7. *Let X, Y, Z be Banach spaces; $x, v \in X$. Let $g : X \to Y$ and $f : Y \to Z$ be mappings such that $g'(x, v)$ and $(f \circ g)'(x, v)$ exist and f is K-Lipschitz on a neighbourhood of $g(x)$. Then $f'(g(x), g'(x, v))$ exists and equals to $(f \circ g)'(x, v)$.*

If, moreover, g has at x a Gateâux derivative L and $f \circ g$ is Gateâux differentiable at x, then f is Gateâux differentiable at $g(x)$ w.r.t. $\overline{L(X)}$.

Proof. Since $g'(x, v)$ exists, we have
$g(x + tv) = g(x) + tg'(x, v) + \eta(t)$ with $\lim_{t \to 0} \frac{\eta(t)}{t} = 0$. Thus we can compute:

$$\lim_{t \to 0} \frac{f(g(x) + tg'(x, v)) - f(g(x))}{t} = \lim_{t \to 0} \frac{f(g(x + tv) - \eta(t)) - f(g(x))}{t} =$$

$$\lim_{t \to 0} \frac{f(g(x + tv)) - f(g(x))}{t} + \lim_{t \to 0} \frac{f(g(x + tv) - \eta(t)) - f(g(x + tv))}{t} =$$

$$(f \circ g)'(x, v) + \lim_{t \to 0} \frac{f(g(x + tv) - \eta(t)) - f(g(x + tv))}{t}.$$

Since, for sufficiently small $t \neq 0$,

$$\left\| \frac{f(g(x + tv) - \eta(t)) - f(g(x + tv))}{t} \right\| \leq \frac{K\|\eta(t)\|}{t},$$

the last limit is 0, which gives our assertion.

To prove the second part, consider $w_1, w_2 \in L(X)$. We can find $v_1, v_2 \in X$ such that $w_1 = g'(x, v_1)$, $w_2 = g'(x, v_2)$ and therefore $w_1 + w_2 = g'(x, v_1 + v_2)$. By the first part,

$$f'(g(x), w_1 + w_2) = (g \circ f)'(x, v_1 + v_2) = (g \circ f)'(x, v_1) + (g \circ f)'(x, v_2) = f'(g(x), w_1) + f'(g(x), w_2).$$

Thus the function $f'(g(x), .)$ is linear on $L(X)$. Since f is Lipschitz on a neighbourhood of $g(x)$, we can conclude (cf. e.g. [A]) that $f'(g(x), .)$ is continuous (even Lipschitz) and linear on $\overline{L(X)}$.

Proposition 3. *Let X, Y be separable Banach spaces and let D be a closed convex subset of Y which is contained in no closed hyperplane of Y. Let $G \subset Y$ be an open set containing D and let $g : G \to X$ be a Lipschitz mapping which has the Gateâux derivative L_a at each $a \in D$ such that $\overline{L_a(Y)} = X$. Let f be a (locally) Lipschitz function on X. Then f is Gateâux differentiable at all points of a dense subset of $C := L(N(D))$.*

Proof. It is sufficient to apply Proposition 1 to $f \circ g$ and then use Lemma 7.

6. Differentiability of convex functions at boundary points of convex bodies

In the following, we shall need the following well-known lemma. Its easy proof is omited.

Lemma 8. *Let f be a continuous convex function on a Banach space X, let $H \subset X$ be a closed hyperplane, $a \in H$ and let $S \subset X$ be a halfspace determined by H. If f is Gateâux differentiable at a w.r.t. H then f is Gateâux differentiable at a w.r.t. S.*

For the definitions of weak Asplund spaces and GDS spaces see [P3].

Theorem 3. *Let X be a weak Asplund space (or a GDS space), $C \subset X$ be a closed convex set with* $\operatorname{int} C \neq \emptyset$ *and let f be a locally Lipschitz convex function on C. Then f is Gateâux differentiable w.r.t. C at all points of a residual (dense, respectively) subset of the boundary* $\operatorname{bdry}(C) = S(C)$.

Proof. We can and will suppose that f is a Lipschitz convex function defined on X. Let $b \in \operatorname{bdry}(C)$ be given. Clearly there exists an open neighbourhood V of b such that $\operatorname{bdry}(C) \cap V$ is "a piece" of a convex hypersurface. More precisely, there exists a vector $0 \neq v \in X$, a topological complement Z of $sp\{v\}$, an open ball $B \subset Z$ and a continuous convex function $g : B \to \mathbb{R}$ such that $bd(C) \cap V = \{z + g(z)\, v : z \in B\}$. Using Lemmas 1.6., 1.7. and 1.5. of [VZ] we easily obtain that the mapping $h(z) := z + g(z)\, v$ is a δ-convex mapping on B. By Theorem 4.2. of [VZ] we obtain that the function $f \circ h$ is a locally δ-convex mapping, i.e. it can be locally represented in the form $p - q$, where p and q are continuous convex functions. Now note (cf. [P3], Theorem 4.24. and Proposition 6.8) that Z is also weak Asplund (GDS space, respectively). Thus we obtain that both h and $f \circ h$ are Gateâux differentiable at all points of a residual subset of B (a dense subset of B, respectively; in this case we can' use the fact that $g + p + q$ is densely differentiable on the domain of p and q). By Lemma 7 we obtain that, for each point z from a residual subset of B (a dense subset of B, respectively), the function f is Gateâux differentiable w.r.t. the tangent hyperplane H to C at the point $h(z)$; clearly H is the range of the Gateâux derivative $h'(z)$. Now it is sufficient to use Lemma 8 and the obvious fact that h is a homeomorphism.

Theorem 3 slightly suggests the following problem.

Problem 5. *For which Banach spaces X it is true that any Lipschitz convex function on any nonempty nowhere dense closed convex $C \subset X$ is Gateâux differentiable w.r.t. C at all points of a dense subset of $S(C)$?*

7. Concrete consequences of supergeneric results

In Real Analysis there is a long-standing tradition of detailed investigation of exceptional sets. From this point of view supergeneric results in Real Analysis are interesting. In Banach space theory such tradition does not exist and thus the following question should be considered.

Question. Are there some arguments which show that supergeneric results in Banach spaces are interesting?

The aim of the present section is to support the following answer.

Answer. In some cases supergeneric results in Banach spaces are interesting, since they have consequences which have the following attributes:

(i) The consequences are simply formulated.

(ii) The formulation of them, the classical notions are used only (in particular, no "supergeneric notion" is used).

(iii) These consequences cannot be proved by an essentially simpler way without supergeneric notions.

Several propositions which are consequences of supergeneric results (via Theorem 1) and have properties (i) and (ii) are formulated in this section. I conjecture that they satisfy also (iii) but, of course, I am not able to prove this claim. In any case, I know now no alternative proofs of Propositions 6 and 7. Propositions 4 and 5 can be proved also using measure theoretical arguments (via Lemma 6; see Remark 7, (iv) and Remark 8 below) which are not essentially simpler.

Proposition 4. *Let $T: C[0, 1] \to (C[0, 1])^*$ be a monotone operator. Then the set of all increasing real analytic functions f on $[0, 1]$ such that T is not multivalued at f has cardinality c.*

Proof. It is sufficient to apply Theorem 1, (ii) with $X = C[0, 1]$, $Y = l^2$, $D = \{(a_n) \in l^2 : a_n \geq 0\}$ and $g((a_n)) = \sum_{n=1}^{\infty} 2^{-n} a_n x^{n-1}$.

Remark 7.

(i) Of course, the above proposition cannot be obtained directly as a consequence of a generic result since even $C^1[0, 1]$ is a first category subset of $C[0, 1]$. From similar reasons the generic results are not sufficient also in the following propositions.

(ii) It is easy to prove that there exists a continuous convex function on $C[0, 1]$ such that $T := \partial f$ is multivalued at each polynomial $f \in C[0, 1]$.

(iii) If T is a subdifferential of a continuous convex function on $C[0, 1]$ then the assertion of Proposition 4 can be proved also via Remark 5, (b).

(iv) Proposition 4 is also a consequence of Lemma 6 and the fact that a monotone operator on a separable Banach space can be multivalued only on a Gaussian null set. This fact follows immediately from (supergeneric) Theorem A and Remark 3, but it can be also obtained (via [P1]) from a result on singlevaluedness of monotone operators which is proved in [A].

Proposition 5. *Let $p > 1$ and let $F \subset L^p[0, 1]$ be a closed nonempty set. Then the set of all increasing real analytic functions f on $[0, 1]$ which do not belong to $A(F)$ has cardinality c.*

Proof. It is sufficient to apply Theorem 1, (iii) with $X = L^p[0, 1]$, $Y = l^2$, $D = \{(a_n) \in l^2 : a_n \geq 0\}$ and $g((a_n)) = \sum_{n=1}^{\infty} 2^{-n} a_n x^{n-1}$.

Remark 8. Proposition 5 can be proved also by measure theoretical arguments via Remark 5, (e).

Proposition 6. *Let Γ be an uncountable set, $p > 1$ and let $T : l^p(\Gamma) \to (l^p(\Gamma))^*$ be a monotone operator. Then the set of all $x \in l^1(\Gamma)$ at which T is not multivaled is u-dense in $l^p(\Gamma)$.*

Proof. It is sufficient to apply Theorem 1, (ii) with $X = l^p(\Gamma)$, $Y = D = l^1(\Gamma)$ and $g = id$.

Remark 9.

(a) The space $l^1(\Gamma)$ is not weak Asplund space (cf. [DGZ], p. 7). Therefore it is not an (S)-space.

(b) If T is a subdifferential of a continuous convex function on $l^p(\Gamma)$ then the assertion of Proposition 6 can be deduced also from Stegall's theorem cited in Remark 5, (c).

Proposition 7. *Let Γ be an uncountable set, $p > 1$ and let F be a closed subset of $l^p(\Gamma)$. Then the set of all $x \in l^1(\Gamma)$ which do not belong to $A(F)$ is u-dense in $l^p(\Gamma)$.*

Proof. It is sufficient to apply Theorem 1, (iii) with $X = l^p(\Gamma)$, $Y = D = l^1(\Gamma)$ and $g = id$.

Remark 10. Since $g = id$ in the proofs of Propositions 6 and 7, we can use directly Lemma 3 and Theorems D and E and easily obtain that we can write "residual in $l^1(\Gamma)$" (or "c-dense in $l^p(\Gamma)$") instead of "u-dense in $l^p(\Gamma)$" in the assertions of these propositions.

The above propositions were obtained by applications of supergeneric results to some convex (non-σ-cone supported) sets which arise quite naturally. Of course, we can deduce from supergeneric results the existence of some "regular points" also in some "very small" non-convex sets. For example, using the theory of δ-convex mappings [VZ], it is not difficult to prove that Theorem B implies the following result.

Proposition 8. *Let X be a separable Banach space. Then there exists a C^1 hypersurface $M \subset X$ such that each continuous convex function on X is Gateâux differentiable at each point of a residual subset of M.*

Note that the proof of Theorem 5.4 from [Vý] easily implies the following analoguous fact.

Proposition 9. *Let X be a separable Banach space. Then there exists a Lipschitz hypersurface $F \subset X$ such that each monotone operator $T: X \to X^*$ is not multivalued at each point of a dense subset of F.*

Unfortunately, I cannot claim that sets like the hypersurfaces M and F from Propositions 8 and 9 appear in natural questions. Thus Propositions 8 and 9 do not support the positive answer to the above Question too much.

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