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TRANSFORMATION OF ORDINARY SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS

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My lecture concerns ordinary second-order linear differential equations in the real domain. It is based on notions and methods of classic analysis. The results are of global character and admit of numerous applications. The essential parts of the lecture I had the honour of giving at the "Institut H. Poincaré" in Paris, in 1961.

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The foundations of the theory of transformations of ordinary second-order linear differential equations date back to the first half of the last century. They were laid by the German mathematician E. E. Kummer in a Latin treatise "*De generali quadam equatione diferentiali tertii ordinis*", published in 1834 in the programme of a secondary school at Legnica (Liegnitz), Poland, and later, in 1887 in Crelle's Journal (vol. 100). In this treatise Kummer formulates the following problem:

For two linear differential equations of the second order

$$\text{(q)} \qquad y'' + p(t)\, y' + q(t)\, y = 0\,,$$

$$\text{(Q)} \qquad \ddot{Y} + P(T)\, \dot{Y} + Q(T)\, Y = 0$$

we have to determine two functions w, X of one of the independent variables, e.g. t, such that for every solution Y of (Q) the function

$$y(t) = w(t)\, Y[X(t)]$$

is a solution of the differential equation (q).

From the above problem Kummer develops a theory followed by some interesting applications. His means consist in processes of derivations and simple algebraic operations. The problems of existence are not investigated. The essential element is a certain third-order nonlinear differential equation, as might well be gathered from the title of the treatise.

Kummer's theory was taken up by several followers but, at first, without any considerable success, owing probably to the general evolution of mathematics in the second half of the last century. The distinction itself between notions and methods according to the analytic or non-analytic character of differential equations, quite plain to us, needed some time to be established. An interesting proof of it is given by J. Hadamard who wrote the following about the beginnings of differential equations in the real domain: "Au temps de mes études, la méthode de Cauchy-Lipschitz (celle de Picard n'avait pas été créée) ne nous avait même pas été signalée. Lorsqu'un hasard mit quelques-uns d'entre nous en présence de l'exposé de Lipschitz, nous nous y intéressâmes comme à une démonstration nouvelle, mais sans nous rendre compte qu'il y avait là un résultat différent de celui que nous connaissions" (Encyclopédie Française I, 1937, 1·76–8). In the course of years extensive theories of the equivalence of differential systems and systems of differential equations were formed, especially E. Cartan's theory of the equivalence of Pfaff's systems, but the problem of revising Kummer's investigations and their further development still remained to be settled.

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The following theory is based on the assumption that the differential equations (q), (Q) are of the form:

$$\text{(q)} \qquad y'' = q(t)\, y\,,$$

$$\text{(Q)} \qquad \ddot{Y} = Q(T)\, Y,$$

where q, Q stand for continuous functions inside open intervals $j = (a, b)$, $J = (A, B)$, either bounded or not.

With regard to the theory of transformations it is appropriate to divide differential equations (q) according to the number of the zeros of their integrals.

A differential equation (q) is said to be of type (m) – m standing for an integer – if it has integrals with m zeros inside the interval j, but not with $m + 1$ zeros.

If $m = 1$, then there are no conjugated values inside the interval j.

If $m \geqq 2$, then there are two privileged numbers a_1, b_{-1} inside the interval j. The number a_1 is the greatest number inside the interval j to which there are no numbers conjugated on the left; the number b_{-1} is the smallest number inside the interval j to which there are no numbers conjugated on the right. With regard to the reciprocal position of the numbers a_1, b_{-1} we divide differential equations (q) into two kinds i. e. *general* and *special*, according to whether the numbers a_1, b_{-1} are or are not conjugated.

Apart from differential equations (q) of the finite type (m) I have just spoken about, there are others, of the infinite type, characterized by the property that their integrals have an infinite number of zeros inside the interval j. The latter are divided into three classes according to whether only the left end or only the right one or both ends of the interval j are accumulation points of the zeros of the integrals of the corresponding differential equation.

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After these preliminary remarks I shall proceed to the actual subject. Let me start with an outline of a "King's route" leading to the core of the theory in question. In order to eliminate from the very beginning some exceptional cases, I shall assume that differential equations (q), (Q) are either of finite types with conjugated numbers or of infinite types, so that in both cases each of their integrals has at least one zero inside the intervals j, J.

Let **r**, **R** be the linear spaces formed by the integrals of the differential equations (q), (Q). In each of them we choose an arbitrary basis: $u, v \in \mathbf{r}$, $U, V \in \mathbf{R}$, i. e. an ordered pair of linearly independent integrals u, v of the differential equation (q), and U, V of the differential equation (Q). According to our assumption the integral u has at least one zero in the interval j and analogously the integral U has one in the interval J; let us consider some of the zeros $t_0 \in j$ of the integral u and some $T_0 \in J$ of the integral U.

Let α, A be phases of the bases u, v; U, V such that $\alpha(t_0) = 0$, $A(T_0) = 0$. Note that the phases α, A are continuous functions inside the intervals j, J in which they satisfy – with the exception of the zeros of the integrals v, V – the relations $\operatorname{tg} \alpha = u/v$, $\operatorname{tg} A = U/V$ and vanish in t_0, T_0.

Let us consider the equation

(1) $$\alpha(t) = A(T).$$

It is evidently satisfied for $t_0 \in j$, $T_0 \in J$ because for these values both sides of the equation vanish. As each phase of α, A increases or decreases, there exists just one function $T = X(t)$ defined in a vicinity $i \subset j$ of t_0 having in t_0 the value T_0 and inside the interval i identically satisfying equation (1); i stands for the widest interval with the mentioned property. Similarly there exists just one function $t = x(T)$ defined in a vicinity $I \subset J$ of T_0 having in T_0 the value t_0 and inside the interval I identically satisfying equation (1); I stands for the widest interval with the mentioned property. It is evident that the intervals i, I are in relations: $I = X(i)$, $i = x(I)$ as well as that the functions X, x are reciprocally inverse and expressed by the formulae:

$$X(t) = A^{-1}[\alpha(t)], \quad x(T) = \alpha^{-1}[A(T)].$$

It can further be proved that the functions X, x belong to the class C_3 and satisfy third-order nonlinear differential equations:

(Qq) $$-\{X, t\} + Q(X)\, X'^2 = q(t),$$

(qQ) $$-\{x, T\} + q(x)\, \dot{x}^2 = Q(T),$$

where the symbol $\{\ \}$ stands for the Schwarz derivative.

Note that the equations (Qq), (qQ) may be replaced by the single relation:

$$\frac{1}{4}\left[\frac{1}{X'(t)}\right]'' + Q(X)\, X'(t) = \frac{1}{4}\left[\frac{1}{\dot{x}(T)}\right]^{\cdot\cdot} + q(x)\, \dot{x}(T),$$

where t, T are associated in the sense that $t = x(T) \in i$, $T = X(t) \in I$.

Now let us return to the above bases u, v and U, V in the spaces **r**, **R**.

By means of these bases we define the linear correspondence between the spaces **r**, **R** so that we reciprocally coordinate every two integrals $y \in \mathbf{r}$, $Y \in \mathbf{R}$ having – with respect to the bases – the same constant coordinates c_1, c_2: $y = c_1u + c_2v$, $Y = c_1U + c_2V$. It can be proved that every two thus coordinated integrals can be conveniently normalized if multiplied by certain constants k, K ($\neq 0$), dependent only on the choice of the bases and on the choice of the zeros t_0, T_0; if we then denote the normalized integrals ky, KY again by y, Y, we get the following transformation formulae:

$$y(t) = \frac{Y[X(t)]}{\sqrt{|X'(t)|}}, \quad Y(T) = \frac{y[x(T)]}{\sqrt{|\dot{x}(T)|}}.$$

They can by replaced by the unique symmetric relation

$$\sqrt[4]{(|X'(t)|)}\, y(t) = \sqrt[4]{(|\dot{x}(T)|)}\, Y(T),$$

which holds for every two associated numbers $t = x(T) \in i$, $T = X(t) \in I$.

It remains to determine the position of the intervals i ($\subset j$), I ($\subset J$), inside which, as we know, the functions X, x are defined. Let me just mention the fact that the intervals i, I need not but may coincide with the intervals j, J, but always, in a sense, reach the end-points of the intervals j, J.

We have arrived at the following result:

To every linear correspondence between the spaces $\mathbf{r}$, $\mathbf{R}$ there exist reciprocally inverse functions $T = X(t)$, $t = x(T)$ defined inside certain widest intervals $i \subset j$, $I \subset J$ by which every two corresponding and suitably normalized integrals $y \in \mathbf{r}$, $Y \in \mathbf{R}$ are reciprocally transformed in the sense of the above transformation formulae. The functions X, x are defined by the equation $\alpha(t) = A(T)$, where α, A are suitable phases of the differential equations (q), (Q) and satisfy the third-order nonlinear equations (Qq), (qQ). The intervals i, I always, in a sense, reach the end-points of the intervals j, J and in special cases coincide with them.

Let us add that to every solution w ($\neq 0$), X of Kummer's transformation problem there exists a linear correspondence between the spaces $\mathbf{r}$, $\mathbf{R}$ which determines the functions X, x in the described way by means of suitable phases α, A of the differential equations (q), (Q).

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The above reflections have led us to the core of the theory of transformations of second-order linear differential equations. We are now facing the problems concerning primarily differential equations (Qq), (qQ): the problem of existence and uniqueness of the solutions, of the significance of the solutions for the considered transformations, of the relations between the solutions, and further problems that I shall mention later. Note that the equations (Qq), (qQ) essentially represent the third-order nonlinear differential equation at which Kummer arrived in his original treatise in 1834.

Let us now turn to the results with respect to the problems I have just spoken about. The starting point is the theorem concerning the existence and uniqueness of solutions of the equation (Qq):

Let $t_0 \in j$, $X_0 \in J$, X_0' $(\neq 0)$, X_0'' be arbitrary numbers. There exists a unique solution $X(t)$ of the differential equation (Qq) *defined inside a certain interval i $(\subset j)$ which satisfies Cauchy's initial conditions*

$$X(t_0) = X_0\,, \quad X'(t_0) = X_0'\,, \quad X''(t_0) = X_0''$$

and which is the widest in the sense that every solution of the differential equation (Qq) *satisfying the same initial conditions forms a part of $X(t)$. The solution $X(t)$ and its inverse function $x(T)$ are stated by the formulae*

$$X(t) = A^{-1}[\alpha(t)]\,, \quad x(T) = \alpha^{-1}[A(T)]\,;$$

α, A stand for suitable phases of (q), (Q) *vanishing in numbers t_0, X_0.*

For the significance of the solutions of (Qq), (qQ) for transformations of (q), (Q) the following results are valid:

For every integral $Y(T)$ of (Q) *and every solution $X(t)$ of* (Qq) *the function*

$$y(t) = \frac{Y[X(t)]}{\sqrt{|X'(t)|}} \tag{2}$$

is a solution of (q). *Analogously*

$$Y(T) = \frac{y[x(T)]}{\sqrt{|\dot{x}(T)|}}\,, \tag{2'}$$

where $x(T)$ stands for the function inverse to $X(t)$. Analogous results hold for integrals $y(t)$ of (q) *and the solutions $x(T)$ of* (qQ) *as well.*

In this connection I might finally describe the relations between the solutions of the differential equations (Qq), (qQ) and their special cases (qq), (QQ):

$$-\{X, t\} + q(X)\,X'^2 = q(t)\,, \tag{qq}$$

$$-\{x, T\} + Q(x)\,\dot{x}^2 = Q(T)\,. \tag{QQ}$$

It holds first of all that for every solution $X(t)$ of (Qq) the corresponding inverse function $x(T)$ is a solution of (qQ); analogously for every solution $x(T)$ of (qQ) the inverse function $X(t)$ is the solution of (Qq).

Now let G_{11}, G_{12}, G_{21}, G_{22} stand for sets of solutions of (qq), (qQ), (Qq), (QQ) and consider arbitrary solutions $x_{\alpha\beta} \in G_{\alpha\beta}$, $y_{\gamma\delta} \in G_{\gamma\delta}$; $\alpha, \beta, \gamma, \delta = 1, 2$. It can be proved that the composed functions $x_{\alpha\beta}y_{\beta\delta}$, as far as they exist, belong to the set $G_{\alpha\delta}$ so that they are the solutions of the corresponding differential equations. In certain cases the sets G_{11}, G_{22} together with binary operations defined by composing functions, form groups having a common unit element, i.e. the function t. At the same time the sets G_{12}, G_{21} consist of reciprocally inverse elements in the sense that between the sets G_{12}, G_{21} there exists a one-one correspondence in which every two corresponding elements $x_{12} \in G_{12}$, $x_{21}' \in G_{21}$, if composed, give the unit: $x_{12}x_{21}' = x_{21}'x_{12} = t$.

Generalizing these results we arrive at an abstract algebraic structure of the following form: It consists of non-empty sets G_{11}, G_{12}, G_{21}, G_{22} between which there are binary relations (multiplications) having the following properties:

i) For every two elements $x_{\alpha\beta} \in G_{\alpha\beta}$, $y_{\gamma\delta} \in G_{\gamma\delta}$ there exists a product $x_{\alpha\beta}y_{\gamma\delta}$ if $\beta = \gamma$, and then the product belongs to the set $G_{\alpha\delta}$;

ii) the sets G_{11}, G_{22} are groups having a common unit 1;

iii) between the sets G_{12}, G_{21} there is a one-one correspondence such that the product of any two corresponding elements $x_{12} \in G_{12}$, $x'_{21} \in G_{21}$ is the unit element 1: $x_{12}x'_{21} = x'_{21}x_{12} = 1$.

I believe that the study of these abstract structures by methods of modern algebra would lead to valuable results. As we have seen, models of these structures have already been found.

6

The further complex of problems to which the preceding investigations lead concern the so called complete transformations.

We have seen that the reciprocally inverse functions defined by the equation $\alpha(t) = A(T)$ and representing the solutions of differential equations (Qq), (qQ) exist in certain vicinities i, I of the initial values $t_0 \in j$, $T_0 \in J$, but the vicinities i, I do not necessarily coincide with the intervals j, J. That means that the transformations of the integrals y, Y by functions x, X according to formulae (2), (2′) do not always concern the whole integrals y, Y but generally only their parts inside the intervals i $(\subset j)$, I $(\subset J)$. Let us call *complete* such a solution X of the equation (Qq) that exists inside the whole interval j and whose values cover the whole interval J. If X is a complete solution of the differential equation (Qq), then the corresponding inverse function x is a complete solution of the differential equation (qQ) and vice versa. The same term "complete" will be used for transformations of integrals y, Y of differential equations (q), (Q) determined by complete solutions x, X of differential equations (qQ), (Qq) according to formulae (2), (2′). By complete transformations the integrals y, Y reciprocally transform each other in their entire extent.

The basic problems of the theory of complete solutions of the differential equation (Qq) concern the existence and generality of complete solutions and the analysis of the structure of the set of these solutions. From vast material I shall confine myself merely to essential matters in the case of the differential equations (q), (Q) being of finite types with conjugated points.

The main results are as follows:

Differential equations (q), (Q) *of finite types with conjugated points admit of complete reciprocal transformations of their integrals if and only if they are of the same type and kind, i. e. both at the same time general or special.*

If these conditions are satisfied, then there exist complete solutions $X(t)$ *of the differential equation* (Qq) *increasing as well as decreasing which at an assumed* *number $t_0 \in j$ reach a given value $T_0 \in J$; this value may be chosen arbitrarily within certain limits. If the differential equations* (q), (Q) *are general, then there exist altogether ∞^1 complete increasing and ∞^1 complete decreasing solutions of the differential equation* (Qq). *If the differential equations* (q), (Q) *are special, then there exist altogether ∞^2 complete increasing and ∞^2 complete decreasing solutions of the differential equation* (Qq).

The set of complete solutions of the differential equation (Qq) *consists of two classes of which one, $\mathscr{M}$, is formed by complete increasing solutions whereas the other, $\bar{\mathscr{M}}$, is formed by decreasing ones. If the differential equations* (q), (Q) *are general, then there exist in the classes $\mathscr{M}$, $\bar{\mathscr{M}}$ countable subsets dense in $\mathscr{M}$, $\bar{\mathscr{M}}$. If all the integrals of the differential equation* (Q) *are bounded, then every complete solution $X \in \mathscr{M}$ or $X \in \bar{\mathscr{M}}$ may be uniformly, with arbitrary precision, approximated by elements of the countable subsets.*

7

The preceding theory may be applied to transformations of a given second-order differential equation (q) into itself. I shall now proceed to a survey of transformations of the differential equation (q) into itself, assuming – this time – the differential equation of infinite type whose integrals have an infinite number of zeros towards both ends of the corresponding interval j. The interpretation will be clearer if we assume that $j = (-\infty, \infty)$. Thus I shall be able to mention an extensive theory of the so called dispersions which is closely connected to our subject.

Let q be a continuous function inside the interval $j = (-\infty, \infty)$ and the integrals of (q) have an infinite number of zeros in the directions towards both ends of the interval j.

Moreover I shall assume – with regard to the theory of dispersions – that the values of the function q in the interval j are always negative.

Let $t \in j$ be an arbitrary number. Consider an arbitrary integral u of the differential equation (q) vanishing in t and another arbitrary integral v whose derivative v' vanishes in t.

Let $n = 1, 2, \ldots$ and denote:

$\varphi_n(t)$ or $\varphi_{-n}(t)$ the n-th zero of the integral u lying behind the zero t or before it;
$\psi_n(t)$ or $\psi_{-n}(t)$ the n-th zero of the function v' lying behind the zero t or before it;
$\chi_n(t)$ or $\chi_{-n}(t)$ the n-th zero of the function u' lying behind the number t or before it;
$\omega_n(t)$ or $\omega_{-n}(t)$ the n-th zero of the integral v lying behind the number t or before it.

Moreover let $\varphi_0(t) = \psi_0(t) = \chi_0(t) = \omega_0(t) = t$.

Let $\nu = 0, \pm 1, \pm 2, \ldots$ The function $\varphi_\nu, \psi_\nu, \chi_\nu, \omega_\nu$ is called the ν-th *central* (c.) *dispersion of the first, second, third, fourth kind* of the differential equation (q). Particularly $\varphi_1, \psi_1, \chi_1, \omega_1$ are so called *basic c. dispersions* of the corresponding kinds. The definition of these functions does not depend on a special choice of the integrals u, v. The term "dispersion" is justified by the fact that the corresponding functions express – in the above sense – the dispersion of the zeros of the integrals and their derivatives of the differential equation (q). The attribute "central" is connected – at least in case of c. dispersions of the first kind – with further properties of the functions in question, as we shall see later.

The central dispersions we have just defined admit of detailed analysis and have interesting properties. I shall mention only some of them.

First of all, the dispersions of the first kind form an infinite cyclic group $\mathfrak{C}$ which is generated by the basic c. dispersion φ_1 and whose unit is the function $\varphi_0(t) = t$. The multiplication in the group $\mathfrak{C}$ is defined by forming composed functions. The central dispersions of the first kind with even indices form a subgroup of $\mathfrak{C}$. Analogous results are valid for c. dispersions of the second kind.

It further holds that every c. dispersion of any kind has – at every point – a continuous derivative of the first order that may be expressed in simple and elegant formulae. Let me just note the formulae for the derivative $\varphi'(t)$ or $\chi'(t)$ of an arbitrary c. dispersion of the first kind or third kind:

$$(3_1) \qquad \varphi'(t) = \frac{u^2[\varphi(t)]}{u^2(t)} \quad \text{for} \quad u(t) \neq 0\,;$$

$$\varphi'(t) = \frac{u'^2(t)}{u'^2[\varphi(t)]} \quad \text{for} \quad u(t) = 0\,;$$

$$(3_2) \qquad \chi'(t) = -\frac{1}{q[\chi(t)]}\,\frac{u'^2[\chi(t)]}{u^2(t)} \quad \text{for} \quad u(t) \neq 0\,;$$

$$\chi'(t) = -\frac{1}{q[\chi(t)]}\,\frac{u'^2(t)}{u^2[\chi(t)]} \quad \text{for} \quad u(t) = 0\,,$$

where u stands for an arbitrary integral of the differential equation (q). From the above formulae we see – in particular – that every c. dispersion of the first kind belongs to the class C_3.

Well, let us return to the transformation of the differential equation (q) into itself. The connection between the theory of transformations and the theory of dispersions is given by the above formula (3_1), which expresses the derivative $\varphi_\nu'(t)$ of the c. dispersion of the first kind with the index ν in the form of

$$\varphi_\nu'(t) = \frac{u^2[\varphi_\nu(t)]}{u^2(t)} \quad (u(t) \neq 0)\,.$$

Hence the relation:

$$(4) \qquad (-1)^\nu\, u(t) = \frac{u[\varphi_\nu(t)]}{\sqrt{\varphi_\nu'(t)}}\,,$$

which holds for $t \in (-\infty, \infty)$. We see that every integral u of the differential equation (q) is completely transformed by any arbitrary c. dispersion of the first kind into itself, in the sense of formula (4). Thus we find a special solution of Kumer's original problem in the particular case of transformation of the differential equation (q) into itself, a solution given by the functions $w(t) = 1/\sqrt{\varphi_\nu'(t)}$, $X(t) = \varphi_\nu(t)$. Hence we conclude that every c. dispersion of the first kind satisfies the above third-order nonlinear differential equation (qq):

$$-\{X, t\} + q(X)\, X'^2 = q(t)\,.$$

As for the set of solutions X of the differential equation (qq) defined inside the interval $(-\infty, \infty)$ the following results can be proved:

All the solutions X form a continuous group $\mathfrak{G}$ dependent on three parameters; the multiplication of $\mathfrak{G}$ is defined by forming composed functions and the unit element of $\mathfrak{G}$ is the function $\varphi_0(t) = t$. All the increasing solutions X form, in the group $\mathfrak{G}$, a normal subgroup $\mathfrak{P}$ of index 2, whereas the decreasing solutions form, in $\mathfrak{G}$, a coset modulo $\mathfrak{P}$. There holds: $\mathfrak{G} \subset \mathfrak{P} \subset \mathfrak{C} \subset \mathfrak{S}$ where $\mathfrak{C}$ stands for the cyclic group of c. dispersions of the first kind and $\mathfrak{S}$ stands for its subgroup formed by the c. dispersion of the first kind with even indices. The group $\mathfrak{S}$ is a normal subgroup of $\mathfrak{G}$, the group $\mathfrak{G}/\mathfrak{S}$ is isomorphic to the group of all unimodular second-order matrices, and the group $\mathfrak{C}$ forms the centre of the group $\mathfrak{P}$. This property of the group $\mathfrak{C}$ is the reason for the attribute "central" in the term c. dispersions.

8

The theory of transformations of second-order linear differential equations I have hitherto spoken about may be extended to transformations of derivatives (first order) of integrals of one of the differential equations (q), (Q) into integrals or their derivatives of the other.

Suppose that the functions q, Q inside the intervals j, J do not vanish and belong to the class C_2.

Let

$$q_1(t) = q(t) + \sqrt{(|q(t)|)}\, [1/\sqrt{|q(t)|}]''\,,$$
$$Q_1(T) = Q(T) + \sqrt{(|Q(T)|)}\, [1/\sqrt{|Q(T)|}]^{\cdot\cdot}\,.$$

It can be proved that the above theory of transformations, if applied to differential equations (q_1), (Q) and (q_1), (Q_1), represents a theory of transformations of integrals and derivatives of integrals of (Q) into derivatives of integrals of (q). The corresponding transformation formulae are of the form:

$$u'(t) = \sqrt{(|q(t)|)}\, \frac{U[X_1(t)]}{\sqrt{|X_1'(t)|}}\,,$$

$$u'(t) = \sqrt{\left\{\frac{|q(t)|}{|Q[X_2(t)]|}\right\}}\, \frac{U'[X_2(t)]}{\sqrt{|X_2'(t)|}}\,,$$

where X_1 and X_2 stand for solutions of third-order nonlinear differential equations (Qq_1) and (Q_1q_1).

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The theory of the transformations of second-order linear differential equations admits of numerous applications. It made it possible to formulate and solve a series of problems of the following kind: Certain properties of the integrals given, all differential equations (q) whose integrals have the given properties are to be determined.

Let us note some of the problems that have already been solved:

One has determined: all differential equations (q) of a given finite or infinite type and kind; all oscillatory differential equations (q) whose integrals have zeros in equal distances or whose zeros form convex sequences (G. Szegö's problem); all differential equations with a given basic c. dispersion of the first kind; all differential equations (q) with coinciding basic c. dispersions of the first and second kind, as well as all differential equations (q) with coinciding basic c. dispersions of the third and fourth kind.

The theory in question has furthermore been successfully applied in the study of the sets of differential equations (q) having the same basic c. dispersion of the first kind, in solving second-order boundary-value problems, in extending Floquet's theory if the periodicity of the functions is not assumed, and in solving a number of problems of various kind in the theory of linear differential equations of higher orders. Some problems of the above theory of transformations have been formulated in the complex domain if the function q is analytic, eventually entire, and have led to valuable results.

About all this a good many interesting, perhaps even unexpected facts could be given but due to lack of space I cannot go into details.

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