Olga A. Oleinik

Energetic estimates analogous to the Saint-Venant principle and their applications

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ENERGETIC ESTIMATES ANALOGOUS TO THE SAINT-VENANT PRINCIPLE AND THEIR APPLICATIONS

O.A. Oleinik, Moscow

In 1855 Saint-Venant [1] formulated a principle which is of an exceptional importance in the theory of elasticity as well as in its applications in the construction mechanics. During the last hundred years numerous studies have been devoted to the Saint-Venant principle and to the clarification of conditions of its applicability. A strict mathematical formulation of the Saint-Venant principle together with its justification for cylindrical bodies was given by Toupin [2] in 1965 and for arbitrary twodimensional bodies by Knowles [3]. A survey of investigations concerning this problem is found in Gurtin's paper [4].

The Saint-Venant principle can be expressed in the form of an a priori energetic estimate of the solution of the system of equations of the elasticity theory. It was found that estimates of this type can be established for wide classes of partial differential equations and systems. Theorems of the Phragmen-Lindelöf type, existence and uniqueness theorems for solutions of boundary value problems in both bounded and unbounded domains in the class of functions with unbounded energy integrals, theorem on the behavior of solutions in the neighborhood of non-regular points of the boundary (in the neighborhood of angles, ribs etc.) and in the neighborhood of infinity can be obtained as consequences of the energetic estimates which express the Saint-Venant principle. A number of such results was obtained in [5] - [11].

As the simplest example let us consider the Saint-Venant principle for the Laplace equation in a domain Ω of a special shape.

<u>Theorem 1.</u> Let a bounded domain Ω from the class C^1 coincide for $|x_n| < T$ with a cylinder $\{x\colon x' \in \Omega', -T < x_n < T\}$ where $x = (x_1, \dots, x_n)$, $x' = (x_1, \dots, x_{n-1})$, $T = \text{const.}$, Ω' is a domain in the space $R^{n-1}_{x'}$. Assume that $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$,

$$(1)\quad \Delta u = f \text{ in } \Omega, \quad \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = \psi, \quad \Delta u \equiv \sum_{j=1}^{n} u_{x_j x_j},$$

where $f \equiv 0$ in Ω_T, $\psi \equiv 0$ on $\partial\Omega \cap \partial\Omega_T$ and, moreover,

$$(2)\quad \int\limits_{\Omega^+} f dx - \int\limits_{\partial\Omega \cap \partial\Omega^+} \psi ds = 0, \quad \int\limits_{\Omega^-} f dx - \int\limits_{\partial\Omega \cap \partial\Omega^-} \psi ds = 0,$$

where $\Omega_\tau = \Omega \cap \{x: |x_n| < \tau\}$, $\tau = \text{const} > 0$, $\tau \leqq T$, $\Omega^+ = \Omega \cap \{x: x_n > T\}$, $\Omega^- = \{x: x_n < -T\}$, $\partial\Omega$ is the boundary of Ω, ν is the direction of the unit outer normal to $\partial\Omega$. Then

(3) $$\int_{\Omega_{\tau_0}} |\nabla u|^2\, dx \leqq \exp\{-2\lambda^{\frac{1}{2}}(\tau_1 - \tau_0)\} \int_{\Omega_{\tau_1}} |\nabla u|^2\, dx ,$$

where $|\nabla u|^2 \equiv \sum_{j=1}^{n} u_{x_j}^2$, τ_0, $\tau_1 = \text{const} > 0$, $\tau_0 < \tau_1 \leqq T$,

(4) $$\lambda = \inf_{v \in M} \left\{ \int_{\Omega'} \sum_{j=1}^{n-1} v_{x_j}^2\, dx' \left[\int_{\Omega'} v^2 dx'\right]^{-1} \right\},$$

M is the family of all functions $v(x')$ continuously differentiable on $\bar{\Omega}'$ which satisfy the condition

(5) $$\int_{\Omega'} v(x')dx' = 0 .$$

Proof. With regard to (1) we obtain for $-T < a < T$

$$\int_{\Omega^+} f dx = \int_{\Omega \cap \{x: x_n < a\}} \Delta u dx = \int_{\partial\Omega \cap \partial\Omega^+} \psi ds + \int_{x_n = a} u_{x_n} dx'.$$

This together with the conditions (2) implies

(6) $$\int_{x_n = a} u_{x_n} dx' = 0 , \quad -T < a < T .$$

Let $S_\tau^+ = \{x: x_n = \tau\}$, $S_\tau^- = \{x: x_n = -\tau\}$, $S_\tau = S_\tau^+ \cup S_\tau^-$. Accor-ding to the Green formula, we have for arbitrary positive $\tau \leqq T$

$$0 = \int_{\Omega_\tau} u \Delta u dx = -\int_{\Omega_\tau} |\nabla u|^2 dx + \int_{S_\tau^+} u u_{x_n} dx' - \int_{S_\tau^-} u u_{x_n} dx' .$$

Taking into account the relation (6), we conclude that for any constants C_τ^+ and C_τ^-

$$\int_{\Omega_\tau} |\nabla u|^2 dx = \int_{S_\tau^+} (u + C_\tau^+) u_{x_n} dx' - \int_{S_\tau^-} (u + C_\tau^-) u_{x_n} dx' .$$

Consequently

$$(7)\qquad \int_{\Omega_\tau} |\nabla u|^2 dx \leqq \Big[\int_{S_\tau^+} (u+C_\tau^+)^2 dx'\Big]^{\frac{1}{2}} \Big[\int_{S_\tau^+} u_{x_n}^2 dx'\Big]^{\frac{1}{2}} + $$
$$+ \Big[\int_{S_\tau^-} (u+C_\tau^-)^2 dx'\Big]^{\frac{1}{2}} \Big[\int_{S_\tau^-} (u_{x_n})^2 dx'\Big]^{\frac{1}{2}} .$$

Let us choose the constants C_τ^+ and C_τ^- so that

$$\int_{S_\tau^+} (u + C_\tau^+) dx' = 0 , \qquad \int_{S_\tau^-} (u + C_\tau^-) dx' = 0 .$$

Then the inequality (7) and the relation (8) imply that

$$(8)\qquad \int_{\Omega_\tau} |\nabla u|^2 dx \leqq \lambda^{-\frac{1}{2}} \Big[\int_{S_\tau^+} \sum_{j=1}^{n-1} u_{x_j}^2 dx'\Big]^{\frac{1}{2}} \Big[\int_{S_\tau^+} u_{x_n}^2 dx'\Big]^{\frac{1}{2}} +$$
$$+ \lambda^{-\frac{1}{2}} \Big[\int_{S_\tau^-} \sum_{j=1}^{n-1} u_{x_j}^2 dx'\Big]^{\frac{1}{2}} \Big[\int_{S_\tau^-} u_{x_n}^2 dx'\Big]^{\frac{1}{2}} \leqq$$
$$\leqq \frac{1}{2} \lambda^{-\frac{1}{2}} \int_{S_\tau} \Big(\sum_{j=1}^{n-1} u_{x_j}^2 + u_{x_n}^2\Big) dx' = \frac{1}{2} \lambda^{-\frac{1}{2}} \int_{S_\tau} |\nabla u|^2 dx' .$$

Set $F(\tau) = \int_{\Omega_\tau} |\nabla u|^2 dx$, $0 \leqq \tau \leqq T$. We obtain from (8) that

$$F(\tau) \leqq \frac{1}{2} \lambda^{-\frac{1}{2}} \frac{dF}{d\tau} .$$

Multiplying this inequality by $\exp\{-2\lambda^{\frac{1}{2}}\tau\}\cdot 2\lambda^{\frac{1}{2}}$ and integrating from τ_0 to τ_1 we obtain the inequality (3). The theorem is proved.

The conditions (2) for a membrane correspond in the Saint-Venant principle for an elastic body to the condition that the forces acting at the ends are statically equivalent to zero. The number λ defined by the conditions (4), (5) equals to the first non-zero eigenvalue of the Neumann boundary value problem

$$(9) \qquad \sum_{j=1}^{n-1} v_{x_j x_j} + \lambda v = 0 \quad \text{in} \quad \Omega', \quad \frac{\partial v}{\partial \nu}\Big|_{\partial\Omega'} = 0 \,.$$

It is easily seen that $\lambda = \pi^2/l^2$ for $n = 2$, where l is the length of the interval S^+_τ. The following theorem of the Phragmen--Lindelöf type (a uniqueness theorem) for the solution of the Neumann problem in an infinite cylinder Ω is a consequence of the estimate (3).

Theorem 2. Let $\Omega = \{x\colon x' \in \Omega', \ -\infty < x_n < +\infty\}$, $\Delta u = 0$ in Ω, $\frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0$, $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ and let for a constant b

$$\int_{x_n = b} u_{x_n}\, dx' = 0 .$$

Then $u \equiv$ const in Ω provided there is a sequence $R_j \to \infty$ with

$$(10) \qquad \int_{\Omega_{R_j}} |\nabla u|^2 dx \leq \varepsilon(R_j) \exp\left\{2\lambda^{\frac{1}{2}} R_j\right\},$$

where $\varepsilon(R_j) \to 0$ for $R_j \to \infty$.

This theorem is an immediate consequence of Theorem 1.

The constant $2\lambda^{\frac{1}{2}}$ which appears in the exponential function in the inequalities (3) and (10) is the best possible, i.e. it cannot be replaced by a greater constant. This is demonstrated by the following example. Let $v(x')$ be a non-trivial solution of the problem (9) corresponding to the first non-zero eigenvalue λ. Then

$$\int_{\Omega'} v(x')\, dx' = 0 .$$

Put $u(x) = v(x') \exp\{\lambda^{\frac{1}{2}} x_n\}$. The function $u(x)$ satisfies all assumptions of Theorem 2 except the condition (10). The inequality (10) holds for $u(x)$ provided $\varepsilon(R_j) = \text{const} > 0$ and hence $u \equiv$ $\equiv$ const. Indeed, for any $R > 0$ we have

$$\int_{\Omega_R} |\nabla u|^2 dx = C_1 \int_{-R}^{R} \exp\{2\lambda^{\frac{1}{2}} x_n\} dx_n \leq C_2 \exp\{2\lambda^{\frac{1}{2}} R\}, \quad C_1, C_2 = \text{const}.$$

Analogously to the proof of Theorem 1 we can prove an estimate of the type (3) and a Phragmen-Lindelöf theorem for solutions of the Dirichlet problem for the Laplace equation. The following assertion holds.

Theorem 3. Let Ω be the domain defined in Theorem 1, $\Delta u = f$ in Ω, $u|_{\partial\Omega} = \psi$ with $f \equiv 0$ in Ω_T, $\psi \equiv 0$ on $\partial\Omega \cap \partial\Omega_T$, $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$. Then $u(x)$ satisfies the inequality (3) with

$$(11) \qquad \lambda = \inf_{v \in M_1} \left\{ \int_{\Omega'} \sum_{j=1}^{n-1} v_{x_j}^2 \, dx' \left[\int_{\Omega'} v^2 dx' \right]^{-1} \right\},$$

M_1 being the family of all functions $v(x')$ continuously differentiable in $\bar{\Omega}'$ and such that $v|_{\partial\Omega'} = 0$.

Let us notice that no conditions are put on f in $\Omega \setminus \Omega_T$ and on ψ in $\partial\Omega \setminus \partial\Omega_T$ in Theorem 3 in contradistinction to Theorem 1.

Theorem 4. Let $\Omega = \{x: x' \in \Omega', -\infty < x_n < +\infty\}$, $\Delta u = 0$ in Ω, $u|_{\partial\Omega} = 0$, $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$, $\partial\Omega \in C^1$. Then $u \equiv 0$ in Ω provided there is a sequence $R_j \to \infty$ satisfying the inequality (10) with λ defined by the relation (11) and $\mathcal{E}(R_j) \to 0$ for $R_j \to \infty$.

Similarly as in the case of the Neumann problem the constant $2\lambda^{\frac{1}{2}}$ in the inequality (10) is the best possible which is demonstrated by the example of the solution $u(x) = w(x') \exp\{\lambda^{\frac{1}{2}} x_n\}$ of the equation $\Delta u = 0$ where $w(x')$ is the eigenfunction corresponding to the first eigenvalue of the Dirichlet problem

$$\sum_{j=1}^{n-1} w_{x_j x_j} + \lambda w = 0 \quad \text{in } \Omega', \quad w|_{\partial\Omega'} = 0.$$

Theorem 5. Let Ω be a subset of the half-space $x_n > 0$ and let $S_\tau = \Omega \cap \{x: x_n = \tau\}$ be a domain in the plane $x_n = \tau$. Let $\Delta u = f$ in Ω, $u|_{\partial\Omega} = \psi$ with $f \equiv 0$ and $\psi \equiv 0$ for $x_n \leqq T$, $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$. Then

$$(12) \qquad \int\limits_{\Omega_{\tau_0}} |\nabla u|^2 dx \leqq \exp\left\{-\int\limits_{\tau_0}^{\tau_1} 2\lambda^{\frac{1}{2}}(\tau)d\tau\right\} \int\limits_{\Omega_{\tau_1}} |\nabla u|^2 dx ,$$

where

$$\Omega_\tau = \Omega \cap \{x: x_n < \tau\}, \qquad \tau_0 < \tau_1 \leqq T ,$$

$$\lambda(\tau) = \inf_{v\in N_\tau} \left\{ \int\limits_{S_\tau} \sum_{j=1}^{n-1} v_{x_j}^2 dx' \left[\int\limits_{S_\tau} v^2 dx' \right]^{-1} \right\},$$

N_τ is the family of all functions $v(x')$ continuously differentiable in $\bar{S}_\tau$ satisfying $v = 0$ on $\partial\Omega \cap \bar{S}_\tau$.

In this way the exponential factor occuring in the inequality (12) analogous to the Saint-Venant principle, can have an arbitrary character of decrease depending on the metric properties of the domain. Generalizations of Theorems 1 to 5 to the case of elliptic and parabolic equations of the second order in domains Ω of general shapes are given in [5] - [7] .

Let us now consider the Saint-Venant principle for the biharmonic equation which results from plane problems of the linear elasticity theory. Let Ω be a bounded domain in the plane (x_1, x_2) from the class C^1 such that $\Omega \subset \{x: x_2 > 0\}$ and the intersection of the domain Ω with the straight line $x_2 = \tau$ is a set S_τ which consists of a finite number of intervals. Let $l(\tau)$ equal the length of the largest interval from S_τ , $\tau = $ $= \mathrm{const} > 0$. In the domain Ω let us consider the equation

$$(13) \qquad \Delta\Delta u = f , \qquad \Delta\Delta u = u_{x_1x_1x_1x_1} + 2u_{x_1x_1x_2x_2} + u_{x_2x_2x_2x_2} ,$$

with boundary conditions

$$(14) \qquad u|_{\partial\Omega} = \psi_1 , \qquad \frac{\partial u}{\partial\nu}\Big|_{\partial\Omega} = \psi_2 ,$$

assuming that $f \equiv 0$ in Ω_T , $\psi_1 \equiv 0$, $\psi_2 \equiv 0$ on $\partial\Omega \cap$ $\cap \partial\Omega_T$ where $\Omega_\tau = \Omega \cap \{x: x_2 < \tau\}$, ν is the direction of the outer normal to $\partial\Omega$, $T = \mathrm{const} > 0$. In the domain Ω we obtain an estimate for $u(x)$ which expresses the Saint-Venant principle for a two-dimensional elastic body. Special cases of this estimate are established in [3], [12] in a different way.

<u>Theorem 6.</u> Let $u(x)$ be a solution of the problem (13), (14)

in a domain Ω, $f \equiv 0$ in Ω_T, $\psi_1 \equiv \psi_2 \equiv 0$ on $\partial\Omega \cap \partial\Omega_T$, $u \in C^4(\Omega) \cap C^3(\bar{\Omega})$. Then an estimate

$$(15) \qquad \int_{\Omega_{\tau_0}} E(x)dx \leqq \left[\phi(\tau_0, \tau_1)\right]^{-1} \int_{\Omega_{\tau_1}} E(x)dx$$

holds, where $E(x) = u^2_{x_1x_1} + 2u^2_{x_1x_2} + u^2_{x_2x_2}$, $\tau_0 < \tau_1 \leqq T$, the function $\phi(x_2, \tau_1)$ satisfies the identity

$$(16) \qquad \phi_{x_2x_2} - \mu(x_2)\phi = 0$$

for $\tau_0 \leqq x_2 \leqq \tau_1$ and the initial conditions

$$(17) \qquad \phi(\tau_1, \tau_1) = 1, \quad \phi_{x_2}(\tau_1, \tau_1) = 0,$$

where $\mu(\tau)$ is an arbitrary continuous function satisfying

$$(18) \quad 0 < \mu(\tau) \leqq \lambda(\tau) = \inf_{v \in N} \left\{ \int_{S_\tau} E dx_1 \left[\left| \int_{S_\tau} (v^2_{x_2} - vv_{x_2x_2} + v^2_{x_1}) dx_1 \right| \right]^{-1} \right\},$$

N is the family of functions $v(x_1,x_2)$ twice continuously differentiable in a neighborhood of $\bar{S}_\tau$ and such that $v = 0$, $v_{x_1} = 0$, $v_{x_2} = 0$ at the endpoints of the intervals from S_τ.

Proof. Integrating by parts we obtain

$$0 = \int_{\Omega_{\tau_1}} \phi u \Delta\Delta u dx = \int_{\Omega_{\tau_1}} E\phi dx - \int_{\Omega_{\tau_1}} (u^2_{x_2} - uu_{x_2x_2} + u^2_{x_1}) \phi_{x_2x_2} dx +$$

$$+ \int_{S_{\tau_1}} (u_{x_2x_2x_2}u - u_{x_2x_2}u_{x_2} - u_{x_1x_2}u_{x_1}) \phi dx_1 +$$

$$+ \int_{S_{\tau_1}} (u^2_{x_2} - uu_{x_2x_2} + u^2_{x_1}) \phi_{x_2} dx_1 .$$

This implies

$$(19) \qquad \int_{\Omega_{\tau_1}} E\phi dx = \int_{\Omega_{\tau_1}} (u^2_{x_2} - uu_{x_2x_2} + u^2_{x_1}) \phi_{x_2x_2} dx -$$

$$- \int_{S_{\tau_1}} (u_{x_2x_2x_2}u - u_{x_2x_2}u_{x_2} - u_{x_1x_2}u_{x_1})\phi\, dx_1 -$$

$$- \int_{S_{\tau_1}} (u_{x_2}^2 - uu_{x_2x_2} + u_{x_1}^2)\phi_{x_2} dx_1 .$$

Taking here $\phi \equiv 1$ we conclude

$$\text{(20)} \qquad \int_{\Omega_{\tau_1}} E(x)dx = - \int_{S_{\tau_1}} (u_{x_2x_2x_2}u - u_{x_2x_2}u_{x_2} - u_{x_1x_2}u_{x_1})dx_1 .$$

Let us introduce a function $\phi = \phi(x_2, \tau_1)$ defined for $\tau_0 \leqq \leqq x_2 \leqq \tau_1$ by the equation (16) and initial conditions (17) and continued linearly for $0 \leqq x_2 \leqq \tau_0$ so that for $x_2 = \tau_0$ the function ϕ is continuous and has a continuous derivative ϕ_{x_2}. Taking into account (18) we obtain from (19), (20)

$$\int_{\Omega_{\tau_1}} E\phi\, dx \leqq \int_{\Omega_{\tau_1} \setminus \Omega_{\tau_0}} E\mu^{-1}(x_2)\, \phi_{x_2x_2} dx + \int_{\Omega_{\tau_1}} Edx .$$

Hence with regard to (16)

$$\text{(21)} \qquad \int_{\Omega_{\tau_0}} E(x)\, \phi(x_2, \tau_1)dx \leqq \int_{\Omega_{\tau_1}} E(x)dx .$$

Let us now study the function $\phi(x_2, \tau_1)$. We shall show that $\phi > 0$, $\phi_{x_2} < 0$ for $0 \leqq x_2 < \tau_1$. Integrating the equation

$$\phi_{x_2x_2} - \mu(x_2)\phi = 0$$

from x_2 to τ_1 we obtain

$$\text{(22)} \qquad \phi_{x_2}(x_2, \tau_1) = - \int_{x_2}^{\tau_1} \mu(x_2)\, \phi(x_2, \tau_1)dx_2 , \qquad \tau_0 \leqq x_2 \leqq \tau_1 .$$

If the inequality $\phi(x_2, \tau_1) > 0$ is not valid for $\tau_0 \leqq x_2 \leqq \tau_1$ then there exists a point $x_2 = \alpha$ such that $\phi(\alpha, \tau_1) = 0$, $\phi(x_2, \tau_1) > 0$ for $x_2 > \alpha$. Obviously $\phi_{x_2}(\alpha, \tau_1) \geqq 0$. On the other hand, the relation (22) implies that $\phi_{x_2}(\alpha, \tau_1) < 0$.

The contradiction just obtained proves that $\Phi(x_2, \tau_1) > 0$, $x_2 \leqq \tau_1$. The identity (22) implies that $\Phi_{x_2}(x_2, \tau_1) < 0$ for $\tau_0 \leqq x_2 < \tau_1$. Consequently, the inequality (21) implies the estimate (15). The theorem is proved.

Let us now assume that $\lambda(x_2) \geqq \mu = \text{const} > 0$. Then for $\tau_0 \leqq x_2 \leqq \tau_1$

$$\Phi(x_2, \tau_1) = \frac{1}{2}\left[\exp\{\mu^{\frac{1}{2}}(\tau_1 - x_2)\} + \exp\{-\mu^{\frac{1}{2}}(\tau_1 - x_2)\}\right].$$

In this case the estimate (15) implies the inequality

$$(23) \quad \int_{\Omega_{\tau_0}} E(x)dx \leqq 2\exp\left\{-\mu^{\frac{1}{2}}(\tau_1 - \tau_0)\right\} \int_{\Omega_{\tau_1}} E(x)dx .$$

Let us now estimate μ. It is known that if $v(x_1, x_2)$ is such that $v = 0$, $v_{x_1} = 0$, $v_{x_2} = 0$ at the endpoints of the intervals from S_τ then

$$\int_{S_\tau} v_{x_2}^2 dx_1 \leqq \lambda_1^{-1}(\tau) \int_{S_\tau} v_{x_2 x_1}^2 dx_1 , \quad \lambda_1 = \frac{\pi^2}{l^2(\tau)} ,$$

$$\int_{S_\tau} v^2 dx_1 \leqq \lambda_2^{-1}(\tau) \int_{S_\tau} v_{x_1 x_1}^2 dx_1 , \quad \lambda_2 = \frac{(4.73)^4}{l^4(\tau)} ,$$

$$\int_{S_\tau} v_{x_1}^2 dx_1 \leqq \lambda_3^{-1}(\tau) \int_{S_\tau} v_{x_1 x_1}^2 dx_1 , \quad \lambda_3 = \frac{4\pi^2}{l^2(\tau)} .$$

Here $l(\tau)$ is the length of the largest interval from S_τ. Let $l = \sup_{0 \leqq \tau \leqq T} l(\tau)$. With regard to the above inequalities we obtain

$$\left|\int_{S_\tau} (v_{x_2}^2 - v v_{x_2 x_2} + v_{x_1}^2) dx_1\right| \leqq \frac{1}{2}\lambda_1^{-1} \int_{S_\tau} 2 v_{x_1 x_2} dx_1 +$$

$$+ \lambda_3^{-1} \int_{S_\tau} v_{x_1 x_1}^2 dx_1 + \frac{\theta}{2} \int_{S_\tau} v_{x_2 x_2}^2 dx_1 + \frac{\lambda_2^{-1}}{2\theta} \int_{S_\tau} v_{x_1 x_1}^2 dx_1 ,$$

where $\theta = \text{const} > 0$. Let us choose θ so that $\frac{1}{2}\theta = \lambda_3^{-1} + \frac{1}{2}(\theta\lambda_2)^{-1}$. By an easy computation we have $\frac{1}{2}\lambda_1^{-1} > \lambda_3^{-1} + \frac{1}{2}(\theta\lambda_2)^{-1} = \frac{\theta}{2}$. Therefore $\lambda(\tau) \geqq 2\lambda_1(\tau) \geqq 2\frac{\pi^2}{l^2} = \mu$. The estimate (23) implies the inequality

$$\int_{\Omega_{\tau_0}} E(x)dx \leq 2 \exp\left\{-2 \frac{\pi}{1\sqrt{2}}(\tau_1 - \tau_0)\right\} \int_{\Omega_{\tau_1}} E(x)dx .$$

This estimate is better than the corresponding ones obtained in [2], [12]. The following theorem is analogous to the Phragmen-Lindelöf theorem for the biharmonic equation.

Theorem 7. Let $\Omega \subset \{x\colon x_2 > 0\}$, let the set $S_\tau = \Omega \cap \cap \{x\colon x_2 = \tau\}$ be nonempty for all $\tau > 0$, $f \equiv 0$ in Ω, $\psi_1 \equiv 0$, $\psi_2 \equiv 0$ on $\partial\Omega$. Let $u(x)$ be a solution of the problem (13), (14) and $u \in C^4(\Omega) \cap C^3(\bar{\Omega})$. Then $u \equiv 0$ in Ω provided there is a sequence of numbers $R_j \to \infty$ for $j \to \infty$ and a constant $d > 0$ such that

$$(24) \qquad \int_{\Omega_{R_j}} E(x)dx \leq \varepsilon(R_j)\left[\phi(d,R_j)\right]^{-1},$$

where $\varepsilon(R_j) \to \infty$ for $R_j \to \infty$.

Proof. By virtue of Theorem 6 and the condition (24) we have

$$\int_{\Omega_d} E(x)dx \leq \left[\phi(d,R_j)\right]^{-1} \int_{\Omega_{R_j}} E(x)dx \leq \varepsilon(R_j)$$

for any R_j. Hence

$$\int_{\Omega_d} E(x)dx = 0$$

and consequently, $u \equiv 0$ in Ω_d since $u = 0$, $u_{x_1} = 0$, $u_{x_2} = 0$ on $\partial\Omega$. It is known that a solution of the equation $\Delta\Delta u = 0$ is an analytic function in Ω. Hence $u \equiv 0$ in Ω.

Given an unbounded domain Ω such that $\lambda(\tau) \geq \mu = \text{const} > > 0$, the condition (24) can be written in the form

$$(25) \qquad \int_{\Omega_{R_j}} E(x)dx \leq \varepsilon(R_j) \exp\left\{\mu^{\frac{1}{2}} R_j\right\}.$$

The problem whether the constant $\mu^{\frac{1}{2}}$ in the condition (25) is the best possible remains open. Theorems analogous to Theorem 6 and 7 can be established in the same way also for more complicated domains Ω, in particular, for the case of a domain Ω which has several branches which stretch to infinity along various directions. Such domains are studied for elliptic equations of the second order in [5], [6]. The method used here for investigating the problems (13), (14) was former used in [10] to study the behavior of solutions of the system of equations of the elasticity theory at non-regular points of the boundary. Analogous results may be established also for solutions of the problem (13), (14). In particular, the following theorem holds.

<u>Theorem 8.</u> Let a bounded domain Ω belong to the halfplane $\{x: x_2 > 0\}$, $\sigma = \bar{\Omega} \cap \{x: x_2 = 0\}$ being nonempty. Let $u(x)$ be a solution of the problem (13), (14), $u \in H_2(\Omega) \cap C^4(\Omega) \cap \cap C^3(\bar{\Omega} \setminus \sigma)$ and let the curve $\partial\Omega \setminus \sigma$ belong to the class C^1, $f \equiv 0$, $\psi_1 \equiv 0$, $\psi_2 \equiv 0$ in a certain neighborhood of the set σ. Then

$$\int_{\Omega} E(x)\, \phi\, (x_2) dx < \infty \quad ,$$

where $\phi(x_2)$ satisfies the equation $\phi_{x_2x_2} - \frac{1}{2}\mu(x_2)\phi = 0$ and the initial conditions $\phi(\alpha) = 1$, $\phi_{x_2}(\alpha) = 0$, $0 < x_2 \leqq \alpha$ where α is a constant, the function $\mu(x_2)$ is defined by the relation (18) and by the assumption $\mu(x_2) \to \infty$ for $x_2 \to 0$.

It is possible to establish estimates for the function $\phi(x_2)$ which characterize the growth of $\phi(x_2)$ for $x_2 \to 0$ in dependence on the geometric properties of the domain Ω in a neighborhood of the set σ.

Let us remark that estimates analogous to the Saint-Venant principle for solutions of the Dirichlet problem for the system of equations of the elasticity theory are established in [10] while for the mixed problem they are given in [11]. Inequalities analogous to the Saint-Venant principle as well as theorems of Phragmen-Lindelöf type which are their consequences, hold under certain conditions for solutions of general boundary value problems for both elliptic and parabolic equations. These estimates are given in [13] - [15]. In these papers an approach is used which is connected with a study of analytic continuation of solutions in a domain of variation of one of the independent variables of some specially constructed auxiliary systems.

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Author's address: Moskovskiĭ universitet, Mechaniko-matematičeskiĭ fakultet, Moskva B-234, USSR