EQUADIFF 5

Ivo Vrkoč

Strongly maximal matrix functions in regions containing stable solutions

In: Michal Greguš (ed.): Equadiff 5, Proceedings of the Fifth Czechoslovak Conference on Differential Equations and Their Applications held in Bratislava, August 24-28, 1981. BSB B.G. Teubner Verlagsgesellschaft, Leipzig, 1982. Teubner-Texte zur Mathematik, Bd. 47. pp. 352--355.

Persistent URL: http://dml.cz/dmlcz/702320

STRONGLY MAXIMAL MATRIX FUNCTIONS IN REGIONS CONTAINING STABLE SOLUTIONS

Ivo Vrkoč
Prague, ČSSR

Ito stochastic equations

(1) $$dx = a(t,x)dt + B(t,x)dw$$

are considered where $w(t)$ is an n-dimensional Wiener process, $a(t,x)$ is an n-dimensional vector function, $B(t,x)$ is an $n*n$ matrix function.

Hypothesis (A). $B_{ij}(t,x)$, $a_i(t,x)$ are defined for $t \geq 0$, $x \in R^n$, are bounded, Lipschitz continuous in x and Hölder continuous in t.

Let D be a given bounded region and K a compact subset of D.

Hypothesis (B). The matrix function $H(t,x) = B(t,x)B^T(t,x)$ is uniformly positive definite on $(0,\infty) * S$ for every compact subset S of $\bar{D}-K$.

Define $P(B,x_0) = P\{\exists t : x(t;0,x_0) \notin D\}$, where $x(t;t_0,x_0)$ is the solution of (1), $x(t_0;t_0,x_0) = x_0$. We write $H_0(t,x) \geq H(t,x)$ (the diffusion generated by H_0 is greater than that generated by H) iff $H_0(t,x)-H(t,x)$ is positive semidefinite at every point of $\langle 0,\infty) * D$.

Definition (of stability). A compact set K is uniformly stable with respect to (1) iff for every neighbourhood U of K and every number $\varepsilon > 0$ there exists a neighbourhood U_ε of K such that

$$P\{\exists t : x(t;t_0,x_0) \notin U,\ t \geq t_0\} \leq \varepsilon \quad \text{for } x_0 \in U_\varepsilon .$$

Definition (of maximality). Let $a(t,x)$, $B_0(t,x)$, a bounded region D and a subset K be given fulfilling Hypotheses (A), (B). We say that the matrix function $B_0(t,x)$ is strongly maximal (with respect to $a(t,x),D,K$) if $P(B_0,x_0) \geq P(B,x_0)$ for every initial value $x_0 \in D$ and for every matrix function $B(t,x)$ fulfilling Hypotheses (A), (B) and $B(t,x) \leq B_0(t,x)$.

Motivation of the problem. Let a technical device be described by $\dot{x} = a(t,x)$. The influence of random perturbations on such a system can be sometimes described by (1), where $B(t,x)$ determines the intensity and distribution of the random perturbations. Frequently the probability that the parameter x leaves the region D is required to be small. If $a(t,x)$ and $B(t,x)$ are given precisely then this probability $P(B,x_0)$ can be calculated. But often only an upper bound $B_0(t,x)$ for $B(t,x)$ $(B(t,x) \leq B_0(t,x))$ is available. Certainly B_0 is a good upper bound only if $P(B,x_0) \leq$ $\leq P(B_0,x_0)$, i.e. if B_0 is strongly maximal.

A similar problem was studied in [1] - [3] but in these papers the probability $P(B,x_0)$ was considered on a finite time interval.

Before the results can be formulated, further assumptions are to be imposed on D and K.

Hypothesis (C). The region D is bounded and is of the type $C^{(3)}$, i.e. for every point $x^{(o)} \in \dot{D}$ (boundary of D) there exist a neighbourhood U of $x^{(o)}$, an index i and a function $x_i = h(x_1,\dots x_{i-1}, x_{i+1},\dots x_n)$ having the third continuous derivatives so that $D \cap U = \{x : x_i > h(x_1,\dots x_{i-1},x_{i+1},\dots,x_n) \cap U\}$. If $n=1$ then Hypothesis (C) is fulfilled for bounded intervals.

Hypothesis (C'). The set K is compact, can be expressed as $K=\overline{U}$ where U is a region, the boundary $\dot{U}$ consists of one component only and U fulfils Hypothesis (C).

Hypothesis (C''). The compact set K is a union of a finite number of disjoint sets K_i fulfilling Hypothesis (C').

Lemma. Let R be a symmetric matrix. There exist symmetric positive definite matrices $R^{(i)}$, $i=1,2$, such that $R = R^{(1)} - R^{(2)}$. The matrices $R^{(i)}$ are determined uniquely provided they have the same eigenvectors as R.

Further notation. Denote $r(x) = \text{dist}\,(x,K)$ for $x \in \overline{D-K}$. With regard to (C') there exist $\partial^2 r/(\partial x_i \partial x_j)(x)$ for $x \in \dot{K}$. We denote $R(x) = \{\partial^2 r/\partial x_i \partial x_j\}$ for $x \in \dot{K}$. Let $\nu(x)$ be the unit vector of the outward normal with respect to $D-K$.

Problem (P). Find a bounded solution $u(t,x)$ of

$$Lu = \partial u/\partial t + \sum_i a_i(t,x)\partial u/\partial x_i + \frac{1}{2}\sum_{i,j}(H_o)_{ij}(t,x)\partial^2 u/\partial x_i \partial x_j = 0$$

in the region $(0,\infty)\times(D-K)$,

fulfilling

$$u(t,x) = 1 \quad \text{for } x \in \dot{D},\ t \geq 0,$$

$$u(t,y) \rightarrow 0 \quad \text{for } y \rightarrow K \text{ uniformly with respect to } t.$$

We shall consider the Ito equation

(2) $$dx = a(t,x)dt + B_o(t,x)dw .$$

Theorem 1. Let the coefficients $a(t,x)$, $B_o(t,x)$ fulfil Hypotheses (A), (B), let the region D fulfil Hypotheses (C) and let the compact set K, $K \subset D$, be a union of two disjoint sets K_1, K_2 such that

1) $H_{ij}(t,x) \equiv 0$ for $t \geq 0$, $x \in K_1$ (if K_1 is nonempty),
2) K_2 fulfils Hypothesis (C'') and
$$\sum_i a_i(t,x)\nu_i(x) - \frac{1}{2}\sum_{i,j}(H_o)_{ij}(t,x)R^{(2)}_{ij}(x) \geq 0 \quad \text{for } t \geq 0,$$
$x \in K_2$ (if K_2 is nonempty),
3) K is uniformly stable with respect to (2),
4) every point of $D-K$ can be connected with $\dot{D}$ by a continuous curve lying in $D-K$.

Then $B_o(t,x)$ is strongly maximal if and only if the solution u

of the problem (P) is a convex function of x in $(0,\infty)\times(D-K)$.

The theorem gives conditions for the matrix function $B_0(t,x)$ to be strongly maximal. Notice that $B_0(t,x)$ need not be strongly maximal even if it is a constant matrix and even in the scalar case (see [1], [2]). The method of the proof uses modified results of [4], [5] on attainable and nonattainable sets and on degenerate partial differential equations of parabolic type.

Theorem 1 yields that a necessary condition for $B_0(t,x)$ to be strongly maximal is that the set K is convex. Using this fact as an assumption we obtain

Theorem 2. Let the coefficients $a(t,x)$, $B_0(t,x)$ fulfil Hypotheses (A), (B), let the region D fulfil Hypothesis (C) and let the compact set K be convex. Assume that K is uniformly stable with respect to (2) and that at least one of the following assumptions is fulfilled:

1) $H(t,x) = 0$ for $t \geq 0$, $x \in K$

2) K fulfils (C') .

Then the statement of Theorem 1 is valid.

Scalar case (n=1) . In this case $D = (x_1, x_2)$. We shall assume (without loss of generality) that $K = \{x_1\}$. In this case we obtain more explicit results.

Theorem 3. Let functions $a(t,x)$, $B(t,x)$ fulfil Hypotheses (A), (B). Assume that $a(t,x_1) = B(t,x_1) = 0$ and that the solution $x(t) = x_1$ is uniformly stable with respect to (1). Let the function $a(t,x)$ be a convex function of x in $(0,\infty)\times D$. The function $B_0(t,x)$ is strongly maximal if and only if $a(t,x_2) \leq 0$.

Theorem 3 can be derived from Theorem 2 and it is a starting point for deriving theorems involving no assumption on convexity of $a(t,x)$. Let $f'(x)$ be the derivative of f with respect to x .

Theorem 4. Let $a(t,x)$, $B(t,x)$ fulfil Hypotheses (A), (B), $D=(0,1)$, $a(t,0) = B(t,0) = 0$, let $x(t) = 0$ of (1) be uniformly stable, a' and $3''$ continuous, $a(t,1) < 0$. Denote $g = \sup \frac{1}{2}B^2(t,1)/(-a(t,1))$. Assume there exists a number $m \geq g$ such that

$$(a'(t,x) + B'(t,x) + B(t,x)B''(t,x))s^2 +$$
$$+ (2a(t,x) + 5B(t,x)B'(t,x))s + 6B^2(t,x) + a''(t,x)s^3 \geq 0$$

for all $t \geq 0$, $x \in (0,1)$, $s \in (m, m+2)$.

Then the function $B(t,x)$ is strongly maximal.

A very simple condition for strong maximality can be given in the autonomous scalar case, i.e. when $n=1$ and $a(t,x)$, $B(t,x)$ do not depend on t .

Theorem 5. Let $a(x)$, $B(x)$ be real, Lipschitz continuous functions, $D = (0,1)$, $a(0) = B(0) = 0$, $B(x) \neq 0$ for $x \in (0,1\rangle$. If the solution $x(t) = 0$ is stable with respect to (1) then $B(x)$ is strongly maximal if and only if $a(x) \leqq 0$ for $x \in (0,1)$.

Notice that the condition $a \leqq 0$ is neither necessary nor sufficient in the nonautonomous case. The condition of uniform stability of K can be given in terms of Lyapunov functions.

References

[1] I. Vrkoč: Some maximum principles for stochastic equations. Czech.Math.J., V.19(94), 1969, 569-604.

[2] I. Vrkoč: Some explicit conditions for maximal local diffusions in one-dimensional case. Czech.Math.J., V.21(96), 1971, 236-256.

[3] I. Vrkoč: Conditions for strong maximality of local diffusions in multi-dimensional case. Czech.Math.J., V.28(103), 1978, 200-244.

[4] A. Friedman: Stochastic differential equations and applications vol. 2. Academic Press New York, 1976.

[5] A. Friedman: Interaction between stochastic differential equations and partial differential equations. Lecture Notes in Control and Information Sci. Springer Verlag vol. 16, 1979, 156-171.