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CONVERGENCE PROPERTIES OF THE FINITE DIFFERENCE METHOD FOR SOLVING VARIATIONAL INEQUALITIES

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There are many works dedicated to the approximative solution of variational inequalities; we mention only the books /1/, /2/. Results concerning the order of convergence of the discretization methods for such problems are contained, for instance, in /3/, /4/, /5/, /6/. This contribution gives an information about some recent results of the author, applying ideas of the general theory of difference schemes /7/ to variational inequalities (/8/, /9/, /10/, /11/, /12/, /13/, /14/). Especially there is given a certain answer to a question stated by Lions /15/ concerning the convergence of the free boundaries.

1. Statement of the problem

Let us consider the following elliptic obstacle problem in the bounded domain $\Omega \subset R_n$ (with boundary $\Gamma = \partial\Omega$):
To find

$$u \in K = \{ v | v \in W_2^1(\Omega),\ v \geq \psi\ ,\ v|_\Gamma = \mu \} \tag{1}$$

fulfilling

$$a(u,\ v-u) \geq (f,\ v-u) \quad \text{for all} \quad v \in K, \tag{2}$$

where

$$a(v,\ w) = \int_\Omega \Big(\sum_{j=1}^{n} k(x) \frac{\partial v}{\partial x_j} \frac{\partial w}{\partial x_j} + qvw \Big) dx \tag{3}$$

and $(f,\ w)$ denotes the scalar product in $L_2(\Omega)$. In the case of sufficiently smooth ψ the restriction

$$\psi \equiv 0 \tag{4}$$

is possible without loss of generality. Further let be

$$k(x) \geq k_0 > 0,\ q(x) \geq 0,\ f(x) \leq 0\ (x \in \Omega);\ \mu(x) \geq 0\ (x \in \Gamma). \tag{5}$$

Under relatively weak additional assumptions on the smootheness of the data Γ, μ, k, q and f there exist an unique solution $u \in W_2^1(\Omega)$ of the problem (1)-(5), cf. /16/.

2. Two examples

1) Oil pressure in a journal bearing (cf. /17/, /12/).
With R the radius, B the length of the bearing, Ω the mantle surface of the axis (removed into the x_1x_2-plane), ν the frequency of the rotating shaft, η the viscosity of the oil and

$d=d(x_1, x_2; t)$ the oil film thickness at the time t one gets a problem (1)-(5) for the determination of the oil pressure u, where n=2, $\Omega = (0, 2\pi R)\times(0, B)$, $\psi = 0$ the cavitation pressure, μ the outside pressure, $k=d^3$, $q = 0$ and

$$f = -12\eta(\nu\pi R\frac{\partial d}{\partial x_1} + \frac{\partial d}{\partial t}).$$

2) Elastoplastic torsion of cylindric bars (cf. /18/).
If Ω is the cross-section of the bar, one has a problem (1)-(5) for the torsion function u, where n=2, $\psi(x)=\mathrm{dist}(x, \Gamma)$, $\mu=0$, $k=1$, $q=0$ and $f=-C$ with $C>0$ proportional to the torsion angle.

3. Finite difference approximation

Now we approximate the problem (1)-(5) in the case of the rectangular domain

$$\Omega = \{x \mid x = (x_1,\dots,x_n),\ 0<x_j<l_j,\ j=1,\dots,n\} \tag{6}$$

by means of the uniform set of grid points

$$\omega = \Omega \cap \mathcal{R}_h, \quad \gamma = \Gamma \cap \mathcal{R}_h, \quad \bar\omega = \omega \cup \gamma, \tag{7}$$

$$\mathcal{R}_h = \{x \mid x_j = i_j h_j;\ h_j = l_j N_j^{-1};\ i_j = 0, \pm1, \pm2, \dots;\ N_j > 0 \text{ integer}\}$$

and the difference expression

$$\Lambda y \equiv -\sum_{j=1}^{n}(a_j(x)y_{\bar x_j})_{x_j} + q(x)y(x) \quad (x\in\omega), \tag{8}$$

$$a_j(x) = k(x^{-0.5j}) = k(x_1,\dots,x_j-h_j/2,\dots,x_n)$$

by the following discrete problem (the discrete variational inequality we replace here by the equivalent form of difference inequalities):

$$\Lambda y \geq f(x),\ y(x) \geq 0,\ y(x)(\Lambda y - f(x)) = 0 \quad (x\in\omega); \qquad y(x) = \mu(x) \quad (x\in\gamma). \tag{9}$$

It holds:

1° Problem (9) has a unique solution y.

2° Problem (9) is stable with respect to f, i.e. a perturbation Δf of f leads to such perturbations Δy of y that

$$\|\Delta y\|_{(1)} \leq c\,\|\Delta f\|_{(-1)},$$

where $\|\cdot\|^{-}_{(1)}$ and $\|\cdot\|_{(-1)}$ denote the discrete W_2^1-norm and the dual norm, respectively.

3^o If the solution of (1)-(5) is regular in the sense that

a) the free boundary

$$\Gamma_o = \overline{\Omega^+} \cap \Omega^-, \quad \Omega^- = \{x | x \in \Omega, \ u(x)=0\} = \overline{\Omega^-}, \ \Omega^+ = \Omega \setminus \Omega^-$$

is a regular hypersurface and

b) we have

$$u \in C^{1,1}(\overline{\Omega}) \cap C^m(\overline{\Omega^+}) \quad \text{with} \quad m=2 \quad \text{or} \quad 3 ,$$

then for the error $z(x) = y(x) - u(x)$ $(x \in \omega)$ one has the qualitative error estimation

$$\| z \|_{(1)} = O(h^{m/2}). \tag{10}$$

4^o The estimation (10) is optimal, which is shown by a counterexample (see /13/).

5^o Defining the discrete free boundary

$$\gamma_o = \overline{\omega^+} \cap \omega^-, \quad \omega^- = \{x | x \in \omega , \ y(x)=0\} , \ \omega^+ = \omega \setminus \omega^-,$$

$$\overline{\omega^+} = \bigcup_{x \in \omega^+} S(x), \quad S(x) \text{ - stencil of } \Lambda \text{ in } x,$$

one has under the conditions of 3^o and $f(x) \leq -f_o < 0$ the following estimation of the Hausdorff distance of the two sets γ_o and Γ_o:

$$\text{dist}(\gamma_o, \Gamma_o) = \begin{cases} O(h^{m/4} (\ln \frac{1}{h})^{1/4}) , & n = 2 , \\ O(h^{(m-1)/4}) , & n = 3. \end{cases}$$

The discrete problem (9), especially in the case of the applications mentioned above, we have solved by a combination of penalty and iteration methods. In each iteration step then it is to be solved a linear difference scheme with strongly varying coefficients. For this sake it was developed a special modification (see /14/) of the Alternating Triangulation Method (ATM, see /19/).

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