Lars Inge Hedberg

Nonlinear potential theory and Sobolev spaces

In: Miroslav Krbec and Alois Kufner and Jiří Rákosník (eds.): Nonlinear Analysis, Function Spaces and Applications, Proceedings of the Spring School held in Litomyšl, 1986, Vol. 3. BSB B. G. Teubner Verlagsgesellschaft, Leipzig, 1986. Teubner Texte zur Mathematik, Band 93. pp. 5--30.

Persistent URL: http://dml.cz/dmlcz/702426

NONLINEAR POTENTIAL THEORY AND SOBOLEV SPACES

Lars Inge Hedberg
Linköping, Sweden

1. Introduction

We shall consider the Sobolev spaces $W^{m,p}(\mathbb{R}^N)$, where m is a positive integer, and $1 < p < \infty$. The elements of $W^{m,p}(\mathbb{R}^N)$ are L^p-functions f whose weak partial derivatives, denoted $D^\alpha f$, also belong to L^p for $|\alpha| \le m$. The norm $\|f\|_{W^{m,p}}$ is defined by

$$\|f\|^p_{W^{m,p}} = \sum_{0\le|\alpha|\le m} \int_{\mathbb{R}^N} |D^\alpha f|^p \, dx.$$

It is well known that there is a close connection between the space $W^{1,2}$ and the classical potential theory of Gauss, Frostman, H. Cartan etc. Two properties of $W^{1,2}$ play an important role here. One is its Hilbert space structure, and the other is its property of being closed under contractions, i.e. essentially that if $u \in W^{1,2}$ then $u^+ = \max(u,0) \in W^{1,2}$, and $\|u^+\|_{W^{1,2}} \le \|u\|_{W^{1,2}}$.
During the last two decades a theory of potentials and capacities has been developed, which is connected to $W^{m,p}$ in much the same way as the classical theory is connected to $W^{1,2}$. It is remarkable that a very large part of the classical theory has been carried over to this more general situation, in spite of the fact that $W^{m,p}$ neither is a Hilbert space (for $p \ne 2$), nor is closed under contractions (for $m \ne 1$).

This theory has increased our understanding of the $W^{m,p}$-spaces, and in view of the importance of these spaces in the theory of partial differential operators, there should be many applications.

The purpose of these lectures is to give an introduction, and a survey of parts of the theory. Then the interested reader should be able to find his way through the theory by means of the bibliography. Especially we want to draw attention to the recent treatise by V.G. Maz'ja (1985). The bibliography does not claim to be complete, but it is not limited to papers mentioned in the text.

2. Some basic results

For $mp > N$ the elements in $W^{m,p}(\mathbb{R}^N)$ can be represented as continuous functions by S.L. Sobolev's theorem. It is a rather natural idea to try to measure the lack of continuity when $mp \le N$ by means of a set function, (m,p) - capacity, $C_{m,p}$, which is associated to the norm of the space. (C. Loewner (1959), V.G. Maz'ja (1963).

Definition 1: Let $K \subset \mathbb{R}^N$ be compact. Then

$$C_{m,p}(K) = \inf \{\|\varphi\|^p_{W^{m,p}};\ \varphi \in C_0^\infty,\ \varphi(x) \ge 1 \text{ on } K\}.$$

We extend this definition to all sets in the following way.

Definition 2: Let $G \subset \mathbb{R}^N$ be open. Then

$$C_{m,p}(G) = \sup \{C_{m,p}(K);\ K \subset G,\ K \text{ compact}\}.$$

Let $E \subset \mathbb{R}^N$ be arbitrary. Then

$$C_{m,p}(E) = \inf \{C_{m,p}(G);\ G \supset E,\ G \text{ open}\}.$$

A capacity extended in this way to all sets is called an outer capacity.
A property that holds true for all x except those belonging to a set of zero (m,p)-capacity is said to be true (m,p)-quasieverywhere.

For $m = 1$ and $p = 2$ the extremal problem in Definition 1 immediately leads to a second order linear partial differential equation and to classical potential theory. For $p \ne 2$, however, the corresponding equations are non-linear, and very difficult to handle.

Because of this the theory of (m,p)-capacities was not developed very far. It was a breakthrough when around 1970 it was realized by several people (B.Fuglede (1968), N.G. Meyers (1970), V.G. Maz'ja and V.P. Havin (1970), (1972), Ju.G. Rešetnjak (1969)) that one can get much further by redefining (m,p)-capacity slightly.

The key to this observation is A.P. Calderón's theorem (1961) about representation of $W^{m,p}$ as spaces of Bessel potentials.

The Bessel kernel, G_m, is most easily defined through its Fourier transform,

$$\hat{G}_m(\xi) = (1 + |\xi|^2)^{-m/2}.$$

Then G has the following properties for $0 < m \le N$:

(a) $G_m > 0$, G_m is radial, decreasing, and continuous for $x \neq 0$.

(b) $G_m(x) = \frac{A}{|x|^{N-m}} + o(\frac{1}{|x|^{N-m}})$ as $x \to 0$ for $0 < m < N$.

(c) $G_N(x) = A \log \frac{1}{|x|} + o(1)$, as $x \to 0$.

(d) $G_m \in L^1$, and $G_m(x) = O(e^{-c|x|})$, as $x \to \infty$.
(See also E.M. Stein's (1970) book).

Let S denote the Schwartz class of C^∞ functions that tend rapidly to zero at infinity. Let $f \in S$. Then for any real number m, there is a uniquely defined function $g \in S$ such that $f = G_m * g$. (Here $*$ denotes convolution). In fact, $g = G_{-m} * f$. We introduce a norm,

$$\|f\|_{m,p} = \|g\|_p,$$

and we denote by $L^{m,p}(\mathbb{R}^N)$ the closure of S in the norm $\|\cdot\|_{m,p}$.

Equivalently, $L^{m,p}(\mathbb{R}^N)$ can be defined (for $m > 0$) as the space of L^p-functions f such that $f = G_m * g$, with $g \in L^p$.

Now according to Calderón's theorem (which is a consequence of the Calderón-Zygmund theory of singular integrals),

$$L^{m,p}(\mathbb{R}^N) = W^{m,p}(\mathbb{R}^N)$$

for $1 < p < \infty$ and all positive integers m, and there are constants A_1 and A_2 such that

$$A_1\|f\|_{m,p} \le \|f\|_{W^{m,p}} \le A_2\|f\|_{m,p}.$$

We now redefine (m,p)-capacity in the following way. Let K be compact, and set

$$\Omega_K = \{f \in S ;\ f \ge 1 \text{ on } K\},$$

so that Ω_K is a convex subset of S.

Definition 3: For a compact $K \subset \mathbb{R}^N$

$$C'_{m,p}(K) = \inf \{\|f\|^p_{m,p};\ f \in \Omega_K\} .$$

The definition of $C'_{m,p}$ is extended to arbitrary sets as in Definition 2.

Clearly there are constants A_1 and A_2 such that

$$A_1\, C'_{m,p}(E) \le C_{m,p}(E) \le A_2\, C'_{m,p}(E)$$

for all sets. This constant is not going to be important for us, so we shall from now on drop the distinguishing notation $C'_{m,p}$, and assume that $C_{m,p}$ is defined by Definition 3.

The following properties are obvious or easy to prove.

Proposition 1: $E_1 \subset E_2 \Rightarrow C_{m,p}(E_1) \le C_{m,p}(E_2)$.

Proposition 2: $C_{m,p}(\bigcup_{i=1}^{\infty} E_i) \le \sum_{i=1}^{\infty} C_{m,p}(E_i)$.

This apparently slight change in the definition of capacity has very important consequences for the extremal functions. In fact we can now easily prove the following theorem. ($\bar{\Omega}_K$ denotes the closure of Ω_K in $L^{m,p}$.)

Theorem 1: Let $K \subset \mathbb{R}^N$ be compact, $m > 0$, and $1 < p < \infty$. Then there is a unique element $f_K = G_m * g_K$ in $\bar{\Omega}_K$ such that $\|g_K\|^p_p = C_{m,p}(K)$. Moreover,

(a) *There is a positive Radon measure μ_K such that*

$$f_K = G_m * (G_m * \mu_K)^{q-1}, \quad \frac{1}{p} + \frac{1}{q} = 1,$$

and thus

$$C_{m,p}(K) = \int (G_m * \mu_K)^q dx = \int f_K \, d\mu_K;$$

(b) supp $\mu_K \subset K$;

(c) $f_K(x) \le 1$ *everywhere on* supp μ_K;

(d) $f_K(x) \ge 1$ (m,p)- *q.e. on* K;

(e) $\mu_K(K) = C_{m,p}(K)$;

(f) $C_{m,p}(K) = \sup \{ (\frac{\mu(K)}{\|G_m * \mu\|_q})^p;\ \mu \geq 0,\ \text{supp}\ \mu \subset K \}$;

(g) $C_{m,p}(K) = \sup \{\mu(K);\ \mu \geq 0,\ \text{supp}\mu \subset K,\ G_m*(G_m*\mu)^{q-1}(x) \leq 1$ *for all* $x \in \text{supp}\ \mu\}$.

For any Radon measure $\mu \geq 0$ the function $G_m*(G_m * \mu)^{q-1} = V^{\mu}_{m,p}$ is called a *nonlinear potential* of μ. We observe that for $p = q = 2$,

$$V^{\mu}_{m,2} = G_m*(G_m * \mu) = G_{2m} * \mu,$$

so then we have a classical potential. Then Theorem 1 is a well-known result in classical potential theory, although it is more often formulated for the Newton kernel $|x|^{2-N}$ or the M. Riesz kernel $R_m(x) = |x|^{m-N}$, $0 < m < N$, than for G_m. See e.g. the book by N.S. Landkof (1966).

Proof of Theorem 1: By the uniform convexity of L^p for $1 < p < \infty$ there is a uniquely determined extremal $f_K = G_m * g_K$ in the closure $\bar{\Omega}_K$ of Ω_K.

Let $\varphi = G_m * \psi$ be a non-negative function in S. Then $f_K + t\varphi \in \bar{\Omega}_K$ for all $t \geq 0$, so that

$$\int |g_K + t\psi|^p dx \geq \int |g_K|^p dx = C_{m,p}(K),\ t \geq 0.$$

Taking the derivative for $t = 0$ it follows that

$$\int |g_K|^{p-2}\ g_K\ \psi\ dx \geq 0$$

for all $\psi \in S$ such that $G_m * \psi \geq 0$.

Set $|g_K|^{p-2}\ g_K = h$. Then $h \in L^q$, and

$$|h|^q = |g_K|^p.$$

There is a distribution $T = G_{-m} * h$, belonging to $L^{-m,q}(R^N)$, such that $h = G_m * T$. It follows that

$$\int (G_m * T)\psi\ dx \geq 0.$$

But by the properties of convolutions of distributions this is the same as saying that

$$\langle T, G_m * \psi\rangle \geq 0,$$

i.e. $\langle T, \varphi\rangle \geq 0,$

$\langle\, ,\,\rangle$ denoting the action of a distribution on a test function.

This being true for all positive test functions φ, T is a positive measure, which we denote μ_K. Thus $h = G_m * \mu_K$, and $g_K = h^{q-1} = (G_m * \mu_K)^{q-1}$, which proves part (a) of the theorem. (The final equality follows from a change of order of integration.)

If the same reasoning is repeated with $\varphi \in S$ such that supp $\varphi \subset K^c$, so that $f_K + t\varphi \in \bar{\Omega}_K$ for all $t \in R$, we find that

$$\langle T, \varphi\rangle = \langle \mu_K, \varphi\rangle = 0$$

for all such φ, and thus

$$\text{supp } \mu_K \subset K,$$

which is (b).

We observe that $f_K = G_m * g_K$ is a lower semicontinuous function, so that the set $\{x;\ f_K(x) > 1\}$ is open. It follows that for all test functions φ with supp $\varphi \subset \{x;\ f_K(x) > 1\}$, we have $f_K + t\varphi \in \bar{\Omega}_K$ for all t with $|t|$ sufficiently small. Again we find that $\langle \mu_K, \varphi\rangle = 0$ for all such φ, so that

$$\text{supp } \mu_K \subset \{x;\ f_K(x) \leq 1\},$$

which is (c).

In order to prove (d) we consider a sequence $(f_n)_{n=1}^{\infty}$ in Ω_K such that $\|f_n\|^p_{m,p} \to \inf_{f\in\Omega_K} \|f\|^p_{m,p} = C_{m,p}(K)$. By the uniform convexity of L^p, $(f_n)_1^{\infty}$ is a Cauchy sequence in $L^{m,p}$. By the definition of capacity

$$C_{m,p}(\{x;\ |f_{n_1}(x) - f_{n_2}(x)| \geq \varepsilon\}) \leq \varepsilon^{-p} \|f_{n_1} - f_{n_2}\|^p_{m,p}.$$

By choosing a sufficiently sparse subsequence $(f_{n_i})_{i=1}^{\infty}$ one proves in a standard way, using Proposition 2, that $\lim_{i\to\infty} f_{n_i}(x) = f_K(x)$ (m,p)-q.e., (and uniformly outside an open set of arbitrarily small capacity). Thus $f_K(x) \geq 1$ (m,p)-q.e. on K.

Now let μ be a positive Radon measure with support in K, and let $f = G_m * g \in \Omega_K$. Then

$$\mu(K) \le \int f\, d\mu = \int (G_m * g)\, d\mu = \int (G_m * \mu) g\, dx \le \|G_m * \mu\|_q \, \|g\|_p .$$

It follows from Hölder's inequality that the same holds true for $f \in \bar{\Omega}_K$. In particular, $f = f_K$ gives

$$\frac{\mu(K)}{\|G_m * \mu\|_q} \le C_{m,p}(K)^{1/p},$$

and thus

$$\sup \{ (\frac{\mu(K)}{\|G_m * \mu\|_q})^p ;\ \mu \ge 0,\ \text{supp}\ \mu \subset K \} \le C_{m,p}(K).$$

On the other hand, choosing $\mu = \mu_K$, (c) gives that

$$\mu_K(K) \ge \int f_K\, d\mu_K$$

and also that

$$\int f_K\, d\mu_K = \int (G_m * g_K)\, d\mu_K = \int (G_m * \mu_K) g_K\, dx$$

$$= \int (G_m * \mu_K)^q\, dx = \int g_K^p\, dx = C_{m,p}(K) = \|G_m * \mu_K\|_q\, C_{m,p}(K)^{1/p},$$

i.e. we have equality in Hölder's inequality. It follows that

$$\mu_K(K) = C_{m,p}(K),$$

and $$(\frac{\mu_K(K)}{\|G_m * \mu_K\|_q})^p = C_{m,p}(K),$$

which proves (e) and (f).

Finally, in order to prove (g), we consider a positive measure μ with supp $\mu \subset K$ such that $V^{\mu}_{m,p}(x) \le 1$ on supp μ.

Then $\int V^{\mu}_{m,p}\, d\mu \le \mu(K)$.

But $\int V^{\mu}_{m,p}\, d\mu = \int G_m * (G_m * \mu)^{q-1}\, d\mu = \int (G_m * \mu)^q\, dx$,

so $\|G_m * \mu\|_q \le \mu(K)^{1/q}$.

On the other hand, by (f)

$$\mu(K) \le \|G_m * \mu\|_q \, C_{m,p}(K)^{1/p},$$

whence $\mu(K) \le \mu(K)^{1/q} \, C_{m,p}(K)^{1/p}$,

and $\mu(K) \le C_{m,p}(K)$.

By (c) and (e) the measure μ_K gives equality, and this proves (g).

Remark: This approach to Theorem 1 is due to Maz'ja and Havin (1970), (1972) and to Rešetnjak (1969). Another very elegant approach, due to B. Fuglede (1968), is to apply the Minimax Theorem to the bilinear functional

$$\Phi(\mu,g) = \int (G_m * \mu) g \, dx = \int (G_m * g) \, d\mu.$$

See N.G. Meyers (1970). The advantage is that this works equally well for more general kernels $k(x,y)$ instead of $G_m(x - y)$. Here k does not have to be symmetric, and x and y do not even have to belong to the same space. The theory has been applied in weighted L^p-spaces, and in Besov and other spaces by D.R. Adams (1985) and (1986).

3. Comparison theorems.

In order to give a more concrete idea of the properties of (m,p)-capacities we give some comparison theorems. The results should be compared to those for classical potentials given by L. Carleson (1967).

We first recall the definition of Hausdorff measure. Let $h(r)$ be increasing and continuous for $r \ge 0$, with $h(0) = 0$, and let $K \subset \mathbb{R}^N$ be compact. Then, for any ρ, $0 < \rho \le +\infty$, we set

$$\Lambda_h^{(\rho)}(K) = \inf \{ \sum_{i=1}^{\infty} h(r_i);\ K \subset \bigcup_{i=1}^{\infty} B(x_i, r_i),\ r_i \le \rho \} ,$$

i.e. the infimum is taken over all coverings of K by balls $B(x_i, r_i)$ with radius $\le \rho$. Clearly $\Lambda_h^{(\rho_1)}(K) \le \Lambda_h^{(\rho_2)}(K)$ if $\rho_2 \le \rho_1$, so $\lim_{\rho\to 0} \Lambda_h^{(\rho)}(K) = \Lambda_h(K) \le +\infty$ exists. This is the Hausdorff measure of K with respect to h. If $h(r) = r^\alpha$ we write $\Lambda_\alpha(K)$. The set function $\Lambda_h^{(\infty)}(K)$ is sometimes called Hausdorff content. One can prove that $\Lambda_h^{(\infty)} = 0$ if and only if $\Lambda_h(K) = 0$. See e.g. Carleson (1967).

The following proposition is easy to prove. See Meyers (1970).

Proposition 3: Let B_r *denote a ball in* $\mathbb{R}^N$ *with radius* r, $0 < r \le 1$. *Then there are constants* A_1 *and* A_2 *such that*

(a) $A_1 r^{N-mp} \le C_{m,p}(B_r) \le A_2 r^{N-mp}$, $mp < N$

(b) $A_1(\log \frac{2}{r})^{1-p} \le C_{m,p}(B_r) \le A_2(\log \frac{2}{r})^{1-p}$, $mp = N$.

The first statement in the following theorem is an immediate consequence of proposition 3. The theorem is proved in Meyers (1979) and in Maz'ja and Havin (1972).

Theorem 2: Let $K \subset \mathbb{R}^N$ *be compact, and suppose that* K *belongs to the unit ball. Let* $h(r) = r^{N-mp}$ *if* $mp < N$, $h(r) = (\log \frac{2}{r})^{1-p}$ *if* $mp = N$. *Then*

(a) *there is a constant* A *such that*
$C_{m,p}(K) \le A\, \Lambda_h^{(\infty)}(K)$;

(b) $C_{m,p}(K) = 0$ *if* $\Lambda_h(K) < \infty$.

In the converse direction we have the following deeper result.

Theorem 3: Let $K \subset \mathbb{R}^N$ *be compact, and let* $h(r)$ *be an increasing continuous function with* $h(0) = 0$. *Suppose that*

$$\int_0^1 \left(\frac{h(r)}{r^{N-mp}} \right)^{q-1} \frac{dr}{r} < \infty.$$

Then there is a constant A *such that*

$$\Lambda_h^{(\infty)}(K) \le A\, C_{m,p}(K),$$

and thus $\Lambda_h(K) = 0$ *if* $C_{m,p}(K) = 0$.

Theorems 2 and 3 have the following corollary.

Corollary: Let M_d *be a smooth* d- *dimensional manifold in* $\mathbb{R}^N$. *Then* $C_{m,p}(M_d) = 0$ *if and only if* $mp \le N - d$.

Theorem 3 was proved by Maz'ja and Havin (1972) by means of quite difficult estimates of nonlinear potentials (found independently by D.R. Adams in his Minnesota thesis). A somewhat easier proof can now be given by means of an important inequality of T. Wolff (L.I. Hedberg and T. Wolff (1983)).

Set $W^{\mu}_{m,p}(x) = \int_0^1 \left(\frac{\mu(B(x,r))}{r^{N-mp}} \right)^{q-1} \frac{dr}{r}$, if μ is a positive Radon measure, and $0 < mp \le N$.

Theorem 4: Let μ be a positive Radon measure. There are constants A_1 and A_2 such that

$$A_1 \int W^{\mu}_{m,p} d\mu \le \int V^{\mu}_{m,p} d\mu \le A_2 \int W^{\mu}_{m,p} d\mu.$$

What is remarkable here, and due to Wolff, is the right inequality. The left inequality follows from an easy pointwise estimate: $V^{\mu}_{m,p}(x) \ge A_1 W^{\mu}_{m,p}(x)$ (see e.g. Hedberg (1972a)). The converse to this is false, if $p \le 2 - \frac{m}{N}$. To see this it is enough to let μ be a point mass. The original proof of Wolff was quite complicated. Different simplifications were given, until D.R. Adams (1985) finally deduced the inequality from a known inequality of B. Muckenhoupt and R. Wheeden (1974). We sketch this proof.

If $\mu \ge 0$ we define a maximal function

$$M_{m;\rho}\mu(x) = \sup_{0<r\le\rho} \frac{\mu(B(x,r))}{r^{N-m}}, \quad 0 \le m < N, \ \rho > 0.$$

Then by the theorem of Muckenhoupt and Wheeden

$$\|G_m * \mu\|_q \le A_\rho \|M_{m;\rho}\mu\|_q, \quad \text{for } 1 < q < \infty.$$

(This is actually a modification, due to D.R. Adams, of their result).

Set $$J_{m,q}\mu(x) = \left[\int_0^{1/2} \left(\frac{\mu(B(x,r))}{r^{N-m}} \right)^q \frac{dr}{r} \right]^{1/q}, \quad 1 \le q < \infty;$$

$$J_{m,\infty}\mu(x) = M_{m;1/2}\mu(x).$$

Then, for any δ, $0 < \delta \le \frac{1}{4}$

$$J_{m,q}\mu(x) \ge \left[\int_\delta^{2\delta} \left(\frac{\mu(B(x,r))}{r^{N-m}} \right)^q \frac{dr}{r} \right]^{1/q} \ge \frac{\mu(B(x,\delta))}{(2\delta)^{N-m}} (\log 2)^{1/q},$$

so that

$$M_{m;1/4}\mu(x) \le A\ J_{m,q}\mu(x).$$

We recall that $\int V^{\mu}_{m,p} d\mu = \|G_m * \mu\|^q_q$, so all we have to show is that for some constant A

$$\|J_{m,q}\mu\|^q_q \le A \int W^{\mu}_{m,p} d\mu.$$

But $$\int_{\mathbb{R}^N} (J_{m,q}\mu)^q\, dx = \int_{\mathbb{R}^N} \left(\int_0^{1/2} \left(\frac{\mu(B(x,r))}{r^{N-m}} \right)^q \frac{dr}{r} \right) dx =$$

$$= \int_0^{1/2} \left(\int_{\mathbb{R}^N} \mu(B(x,r))^q\, dx \right) \frac{dr}{r^{(N-m)q+1}},$$

and $$\int_{\mathbb{R}^N} \mu(B(x,r))^q\, dx = \int_{\mathbb{R}^N} \mu(B(x,r))^{q-1} \left(\int_{|y-x|<r} d\mu(y) \right) dx$$

$$= \int_{\mathbb{R}^N} \left(\int_{|x-y|<r} \mu(B(x,r))^{q-1}\, dx \right) d\mu(y) \le A\, r^N \int_{\mathbb{R}^N} \mu(B(y,2r))^{q-1}\, d\mu(y).$$

The result follows.

Theorem 3 now follows from Theorem 4 by means of the well-known lemma of O. Frostman, which gives the existence of a measure μ supported on K such that $\mu(B_r) \le h(r)$ for every ball B_r, and $\Lambda^{(\infty)}_h(K) \le A\mu(K)$. See Carleson (1967) and Maz'ja - Havin (1972).

4. Thinness of sets.

One of the fundamental ideas in classical potential theory is the concept of a thin set, which is the generalization to arbitrary sets of the idea of an irregular set for the Dirichlet problem. In other words, if $\Omega \subset \mathbb{R}^N$ is a domain, then a boundary point x is irregular if and only if Ω^c is thin at x. The irregular boundary points were characterized in terms of capacities by Wiener (1924) (the Wiener Criterion). Brelot (1940) defined thin sets in general, and extended the Wiener Criterion. Much of this theory has been generalized to the present nonlinear setting. The theory generalizes in a non-trivial and sometimes unexpected way, and there are still some open problems.

We briefly recall the classical situation, expressed in our notation. See e.g. the books Helms (1969) and Landkof (1966).

Definition 4: Let $E \subset \mathbb{R}^N$ be an arbitrary set. Then E is thin at a point x_0 if there exists a positive Radon measure μ such that

$$V^{\mu}_{1,2}(x_0) < \liminf_{x\to x_0, x\in E\setminus\{x_0\}} V^{\mu}_{1,2}(x).$$

This is interpreted as meaning that E is thin at x_0 if $x_0 \notin \bar{E}$. Note that $V^{\mu}_{1,2}$ is lower semicontinuous, so that always $V^{\mu}_{1,2}(x_0) \le \liminf_{x\to x_0, x\in E\setminus\{x_0\}} V^{\mu}_{1,2}(x)$.

The set of points where E is thin is denoted e(E).

The main result characterizing thin sets is the following:

Theorem 5: *Let* $E \subset \mathbb{R}^N$, $N \ge 2$, *and let* $x_0 \in \bar{E}$. *The following statements are equivalent* .

(a) E *is* (1,2)- *thin at* x_0.

(b) *Let* G *be a neighbourhod of* x_0 *and let* μ *be the* (1,2) - *capacitary measure for* $E \cap G$.

If G *is small enough, then* $V^{\mu}_{1,2}(x_0) < 1$.

(c) $$\int_0^1 \frac{C_{1,2}(E \cap B(x_0,r))}{r^{N-2}} \frac{dr}{r} < \infty.$$

In the special case when $E = \Omega^c$ and Ω is a domain in $\mathbb{R}^N$, $N \ge 2$, we have Wiener's theorem.

Theorem 6: *A point* $x_0 \in \partial\Omega$ *is regular for the Dirichlet problem for the Laplace equation in* Ω *if and only if*

$$\int_0^1 \frac{C_{1,2}(\Omega^c \cap B(x_0,r))}{r^{N-2}} \frac{dr}{r} < \infty.$$

An important consequence of Theorem 5 is the so called Kellogg property.

Theorem 7: $C_{1,2}(E \cap e(E)) = 0$ *for all* E.

The earliest nonlinear generalization seems to be due to V.G Maz'ja (1970).

Theorem 8: A point $x_0 \in \partial\Omega$ *is regular for the Dirichlet problem for the equation* $\operatorname{div}(\operatorname{grad} u\, |\operatorname{grad} u|^{p-2}) = 0$, $1 < p \le N$, *in* Ω *(for solutions in* $W^{1,p}$*) if*

$$\int_0^1 \left(\frac{C_{1,p}(\Omega^c \cap B(x_0,r))}{r^{N-p}} \right)^{q-1} \frac{dr}{r} < \infty.$$

It was proved by R. Gariepy and W.P. Ziemer (1977) that the same result is true for much more general quasilinear elliptic equations of the type $\operatorname{div} A(x,u,\operatorname{grad} u) = B(x,u,\operatorname{grad} u)$. On the other hand, until quite recently it was a completely open problem whether the converse is true. The following theorem is contained in results proved by P. Lindqvist and O. Martio (1985). See also V.I. Skrypnik (1984) for necessary conditions.

Theorem 9: The condition in Theorem 8 is both necessary and sufficient for regularity if $N - 1 < p \le N$.

The natural generalization of Definition 4 would be by means of the following statement about $E \subset \mathbb{R}^N$ and $x_0 \in \mathbb{R}^N$:

(A) There exists a positive Radon measure μ such that

$$V^{\mu}_{m,p}(x_0) < \liminf_{x \to x_0,\, x \in E\setminus\{x_0\}} V^{\mu}_{m,p}(x).$$

Another possibility is the following:

(B) There exists a positive Radon measure μ such that $V^{\mu}_{m,p}$ is bounded and

$$V^{\mu}_{m,p}(x_0) < \liminf_{x \to x_0,\, x \in E\setminus\{x_0\}} V^{\mu}_{m,p}(x).$$

The natural generalization of the Wiener integral condition is the following:

$$\text{(C)} \qquad \int_0^1 \left(\frac{C_{m,p}(E \cap B(x_0,r))}{r^{N-mp}} \right)^{q-1} \frac{dr}{r} < \infty, \quad mp \le N.$$

Unfortunately (A), (B) and (C) are not equivalent in general. What is known is the following result (Adams-Meyers (1972), Hedberg (1972a)).

Theorem 10: (a) (A) $\Leftrightarrow$ (B) $\Leftrightarrow$ (C) *for* $2 - \frac{m}{N} < p$ $(\le \frac{N}{m})$;

(b) (B) ⇒ (C) *for* $1 < p \leq 2 - \frac{m}{N}$;

(c) (C) $\not\Rightarrow$ (B) *for* $1 < p < 2 - \frac{m}{N}$.

Because of this we have to choose, which property to take as definition of a thin set. It turns out that (C) is the best choice.

Definition 5: A set $E \subset \mathbb{R}^N$ is (m,p)-thin at $x_0 \in \mathbb{R}^N$ if $mp \leq N$ and

$$\int_0^1 \left(\frac{C_{m,p}(E \cap B(x_0,r))}{r^{N-mp}} \right)^{q-1} \frac{dr}{r} < \infty.$$

The set of points where E is (m,p)-thin is denoted $e_{m,p}(E)$.

A good reason for saying that this is the right choice of a definition is that the Kellogg property generalizes. (Hedberg-Wolff (1983)).

Theorem 11: $C_{m,p}(E \cap e_{m,p}(E)) = 0$ *for all* E.

Corollary: The set of irregular boundary points in Theorem 8 has zero (1,p)- capacity.

If E is a Borel set, Theorem 11 follows quite easily from Wolff's inequality (Theorem 4), which was in fact proved for this purpose. One needs the following lemma of Wolff.

Lemma: If there is a Borel set E without the Kellogg property, then for any $\varepsilon > 0$ *there is a compact* $F \subset E$ *such that* $C_{m,p}(F) > 0$, *and*

$$\int_0^1 \left(\frac{C_{m,p}(F \cap B(x,r))}{r^{N-mp}} \right)^{q-1} \frac{dr}{r} < \varepsilon \text{ for all } x \in F.$$

Now assume that the Kellogg property fails, and choose F by the lemma. Let μ_F be its capacitary measure. Then $V_{m,p}^{\mu_F}(x) \leq 1$ everywhere on supp μ_F by Theorem 1(c), so by (g) in the same theorem, $\mu_F(B(x,r))) = \mu_F(F \cap B(x,r)) \leq C_{m,p}(F \cap B(x,r))$. Thus

$$W^{\mu_F}_{m,p}(x) = \int_0^1 \left(\frac{\mu_F(B(x,r))}{r^{N-mp}} \right)^{q-1} \frac{dr}{r} \le \int_0^1 \left(\frac{C_{m,p}(F \cap B(x,r))}{r^{N-mp}} \right)^{q-1} \frac{dr}{r} < \varepsilon$$

for all $x \in F$. By Theorems 1 and 4

$$\int V^{\mu_F}_{m,p} d\mu_F \le A \int W^{\mu_F}_{m,p} d\mu_F \le A\varepsilon\ \mu_F(F) = A\varepsilon\ C_{m,p}(F).$$

But by Theorem 1(a) $\int V^{\mu_F}_{m,p} d\mu_F = C_{m,p}(F)$, which is a contradiction if $A\varepsilon < 1$. This finishes the proof.

The truth of the Kellogg property is a strong indication that Definition 5 is a good definition of (m,p)-thinness. Much more is true, however. Theorem 4 leads one to a third equivalent definition of (m,p)-capacity, using $W^{\mu}_{m,p}$ instead of $V^{\mu}_{m,p}$. In fact, by Theorem 1

$$C_{m,p}(K) = \sup \left\{ \left(\frac{\mu(K)}{(\int V^{\mu}_{m,p} d\mu)^{1/q}} \right)^p ;\ \mu \ge 0,\ \text{supp}\ \mu \subset K \right\}.$$

Thus, by Theorem 4, if we define

$$C''_{m,p}(K) = \sup \left\{ \left(\frac{\mu(K)}{(\int W^{\mu}_{m,p} d\mu)^{1/q}} \right)^p ;\ \mu \ge 0,\ \text{supp}\ \mu \subset K \right\},$$

then there are constants A_1 and A_2 such that

$$A_1\ C_{m,p}(K) \le C''_{m,p}(K) \le A_2\ C_{m,p}(K).$$

It is then natural to pursue this idea further, and prove an analogue of Theorem 1, with $W^{\mu}_{m,p}$ now playing the role of a nonlinear potential. This was carried out in Hedberg and Wolff (1983). Later it was observed by Adams (1985) that this can be made to fit into the general theory of Meyers (1970), so that the existence of extremals, dual definition of capacity, etc. follow automatically.

Among other things, it turns out that the potentials $W^{\mu}_{m,p}$ provide the problem to which the definition of an (m,p)-thin set gives the answer. In fact, Theorem 5 has the following extension (Hedberg-Wolff (1983)).

<u>*Theorem 12*</u> : *A set* $E \subset \mathbb{R}^N$ *is* (m,p)- *thin* , $m > 0$, $mp \le N$, *at a point* x_0 *if and only if there exists a positive Radon measure* μ *such that*

$$W^{\mu}_{m,p}(x_0) < \liminf_{x \to x_0,\, x \in E \setminus \{x_0\}} W^{\mu}_{m,p}(x).$$

This is more than a curiosity, because $W^{\mu}_{m,p}$ appears in a natural way as a nonlinear potential if one makes a capacity theory for Besov spaces analogous to the theory for Bessel potential spaces given here. In particular, Theorem 4 shows that the (m,p)-capacity associated to the Besov space $B_m^{p,p}$ is equivalent to $C_{m,p}$. See Adams (1985) and (1986).

A further consequence of the theory is the following generalization of the well-known Choquet property (Choquet (1959)) in classical potential theory. See Hedberg and Wolff (1983).

Theorem 13: For any $E \subset \mathbb{R}^N$ *and any* $\varepsilon > 0$ *there is an open* G *such that* $e_{m,p}(E) \subset G$ *and* $C_{m,p}(E \cap G) < \varepsilon$.

The Kellogg property, Theorem 11, is an immediate consequence.

5. Traces.

By the definition of $W^{m,p}(\mathbb{R}^N)$ its elements are (or can be identified with) elements in $L^p(\mathbb{R}^N)$, i.e. the elements are equivalence classes of functions defined and equal outside Lebesgue nullsets. However, in applications it is important to be able to give values to these elements on many sets of zero measure, for example on manifolds of dimension $k \le N - 1$.

For $mp > N$ there is the Sobolev imbedding theorem, which tells us that every element in $W^{m,p}(\mathbb{R}^N)$ contains exactly one continuous representative, so that elements in $W^{m,p}(\mathbb{R}^N)$ can be identified with continuous functions in a natural way. If $mp \le N$ this is no longer possible, however, but according to the well-known imbedding theorems of S.L. Sobolev and others, the elements of $W^{m,p}(\mathbb{R}^N)$ have traces on k-manifolds, if k is large enough, the traces being integrable functions with respect to k-dimensional measure. See e.g. the books by S.L. Sobolev (1950) and R.A. Adams (1975).

In the case of $W^{1,2}$, traces can be defined on arbitrary sets of positive (1,2)-capacity by means of the "precisely defined functions" of J.-L. Lions and J. Deny (1953). We shall now see how this theory generalizes to the more general $W^{m,p}$- and $L^{m,p}$-spaces.

Let $f \in L^{m,p}(\mathbb{R}^N)$ (or $W^{m,p}(\mathbb{R}^N)$ if m is an integer), and let $(\chi_n)_1^\infty$ be an approximate identity, i.e. $\chi_n(x) = n^N \chi(\frac{x}{n})$, where $\chi \in C_0^\infty(B(0,1))$, $\chi \ge 0$, and $\int \chi\, dx = 1$. Set $f_n = f * \chi_n$. Then $f_n \to f$ in $L^{m,p}$, $f_n(x) \to f(x)$ a.e., and by the definition of capacity

$$C_{m,p}(\{x; |f_{n_1}(x) - f_{n_2}(x)| \ge \varepsilon\}) \le A\varepsilon^{-p} \|f_{n_1} - f_{n_2}\|_{m,p}^p.$$

As in Theorem 1(d) one easily proves that a sufficiently sparse subsequence converges to a function $\tilde{f}(x)$ outside a set of zero (m,p)-capacity, and uniformly outside an open set of arbitrarily small (m,p)-capacity. (D.R. Adams (1972) even proved that $\lim_{n\to\infty} f_n(x) = \tilde{f}(x)$ (m,p)-q.e., without choosing any subsequence.) It follows that

(a) $\tilde{f}(x)$ is defined (m,p)-q.e.

(b) $\tilde{f}(x) = f(x)$ a.e. (or more precisely, the function $\tilde{f}$ is a representative of the element f of $L^{m,p}$).

(c) for every $\varepsilon > 0$ there is an open set G with $C_{m,p}(G) < \varepsilon$ such that $\tilde{f} \in C(G^c)$, i.e. $\tilde{f}|_{G^c}$ is continuous on G^c (in the induced topology on G^c).

Functions with the properties (a) and (c) are called (m,p)-quasicontinuous. What gives the notion interest is the following uniqueness theorem.

<u>*Theorem 14*</u>*: Let f_1 and f_2 be two (m,p)- quasicontinuous functions, such that $f_1(x) = f_2(x)$ a.e. Then $f_1(x) = f_2(x)$ (m,p) - q.e.*

<u>*Corollary*</u>*: Every element in $L^{m,p}(\mathbb{R}^N)$ has an (m,p)- quasicontinuous representative, which is uniquely determined up to sets of zero (m,p) - capacity.*

Theorem 14 was first proved in the classical case by H. Wallin (1963). See also Deny-Lions (1953) and Deny (1964). The extension to the nonlinear case is due to Maz'ja and Havin (1972). See also T. Sjödin (1975).

It is now clear how to define the trace on an arbitrary set E of an element f of $L^{m,p}$ or $W^{m,p}$.

<u>Definition 6</u>: The trace $f|_E$ of $f \in L^{m,p}$ (or $W^{m,p}$) is the restriction to E of any (m,p)-quasicontinuous representative of f. Thus, $f|_E$ is defined (m,p)-q.e. on E.

We shall now use the preceding definition of trace in order to study the continuity of Sobolev functions from another point of view. Suppose that $f \in W^{m,p}$, and that the traces of f and a certain number of its derivatives (understood in the sense of distributions) vanish on some set K. What can be said about f on a neighbourhood of K?

Clearly, what we can say depends on the capacity of K. For example, if $C_{m,p}(K) = 0$, then $f|_K = 0$ is a vacuous statement. More generally, $D^\alpha f$ belongs to $W^{m-|\alpha|,p}$, so $D^\alpha f|_K$ is defined $(m-|\alpha|,p)$-quasieverywhere. Thus, $D^\alpha f|_K = 0$, is a statement that does not give any information about the function, if $C_{m-|\alpha|,p}(K) = 0$.

In order to formulate the precise result it is convenient to define a "condenser capacity".

Definition 7: Let K be a compact subset of an open set G, let k be a positive integer, and $p \geq 1$. Then

$$C_{k,p}(K;G) = \inf \{ \sum_{|\alpha|=k} \int |D^{\alpha}\varphi|^p dx;\ \varphi \in C_0^{\infty}(G),\ \varphi \geq 1 \text{ on } K\}.$$

We also define a "relative capacity".

Definition 8: Let K be a compact subset of an open ball B_δ with radius δ. Then

$$c_{k,p}(K;B_\delta) = \delta^{kp-N}\, C_{k,p}(K;B_\delta).$$

It is easily seen that $c_{k,p}$ is homogeneous of degree 0, i.e., if $\delta K = \{x;\ x = \delta y,\ y \in K\}$, $B_\delta = B(0,\delta)$, and $\delta K \subset B_\delta$, then

$$c_{k,p}(\delta K;\ B_\delta) = c_{k,p}(K;\ B_1).$$

Moreover, it is easily seen that if $kp < N$ and $\delta \leq 1$, then there is $A > 0$ such that for any compact K

$$A^{-1} c_{k,p}(K \cap B_\delta; B_{2\delta}) \leq \frac{C_{k,p}(K\cap B_\delta)}{\delta^{N-kp}} \leq A\, c_{k,p}(K\cap B_\delta;\ B_{2\delta}),$$

and if $kp > N$, then

$$c_{k,p}(K\cap B_\delta;\ B_{2\delta}) \geq A > 0$$

unless $K \cap B_\delta = \emptyset$. ($B_{2\delta}$ is concentric to B_δ.)

The result is the following inequality of Poincaré type. (We denote by $\nabla^k f$ the vector $(D^\alpha f)_{|\alpha|=k}$.)

Theorem 15: Let $f \in W^{m,p}(\mathbb{R}^N)$ *for some integer* m *and* $1 < p < \infty$, *and let* K *be a compact set contained in some ball* B_δ *with radius* δ. *Suppose that* $f|_K = 0$ *and* $D^\alpha f|_K = 0$ *for all* α, $0 \leq |\alpha| \leq m-k$, *for some* k, $1 \leq k \leq m$. *Then*

$$\int_{B_\delta} |f|^p dx \leq \frac{A\delta^{mp}}{c_{1,p}(K;B_{2\delta})} \int_{B_\delta} |\nabla^m f|^p dx, \quad \text{if } k = 1,$$

and

$$\int_{B_\delta} |f|^p dx \le A\delta^{(m-k+1)p} \int_{B_\delta} |\nabla^{m-k+1} f|^p dx + \frac{A\delta^{mp}}{c_{k,p}(K;B_{2\delta})} \int_{B_\delta} |\nabla^m f|^p dx,$$

if $2 \le k \le m$.

Proof: By homogeneity we can assume that $\delta = 1$, and we let B_δ be the unit ball, which we denote by B. By a well-known theorem of Hestenes (1941) we can redefine f outside B so that $\int_{2B} |\nabla^j f|^p dx \le A \int_B |\nabla^j f|^p dx$.

Thus, it is enough to prove that

$$\int_B |f|^p dx \le \frac{A}{c_{1,p}(K,B)} \int_{2B} |\nabla^m f|^p dx \tag{1}$$

or

$$\int_B |f|^p dx \le A \int_B |\nabla^{m-k+1} f|^p dx + \frac{A}{c_{k,p}(K,2B)} \int_{2B} |\nabla^m f|^p dx. \tag{2}$$

First let $f \in C^\infty$, without any assumption about its zeroes. For any x and y we write the Taylor expansion of f about y as

$$f(x) = \sum_{|\alpha| \le m-1} \frac{1}{\alpha!} (x - y)^\alpha D^\alpha f(y) + R_y^{(m-1)} f(x), \tag{3}$$

where

$$R_y^{(m-1)} f(x) = \frac{1}{(m-1)!} \int_0^{|x-y|} t^{m-1} (\sigma \cdot \nabla)^m f(x - t\sigma) dt, \quad \sigma = \frac{x-y}{|x-y|}.$$

It is well-known, and not hard to show, that

$$\int_{|y-x| \le 1} |R_y^{(m-1)} f(x)| dy \le A \int_{|y-x| \le 1} \frac{|\nabla^m f(y)|}{|x-y|^{N-m}} dy$$

(see e.g. Hedberg (1981), p. 246).

Now let $x \in B$, and integrate (3) with respect to y over B. It follows that

$$|f(x) - \int_{|\alpha| \le m-1} \frac{1}{|B|} \int_B \frac{1}{\alpha!} (x - y)^\alpha D^\alpha f(y) dy| \le$$

$$\le A \int_{2B} \frac{|\nabla^m f(y)|}{|x-y|^{N-m}} dy. \tag{4}$$

The sum on the left is a polynomial $P_{m-1}(x)$ of degree $\le m-1$, whose constant term is $f_B = \frac{1}{|B|} \int_B f(y)\, dy$. It follows from (4) that if $x \in B$

$$|f(x) - f_B| \le A \sum_{i=1}^{m-1} \int_B |\nabla^i f(y)| dy + A \int_{2B} \frac{|\nabla^m f(y)|}{|x-y|^{N-m}} dy. \tag{5}$$

Let μ be a probability measure on K, and integrate $f(x) - f_B$ with respect to μ. We find if $mp < N$ (we omit the other cases), using the estimate for the Bessel kernel given earlier,

$$|\int f(x)d\mu(x) - f_B| \le A \sum_{i=1}^{m-1} \int_B |\nabla^i f(y)|dy + A \int_{2B} |\nabla^m f(y)| \left(\int_K \frac{d\mu(x)}{|x-y|^{N-m}} \right) \le$$

$$\le A \sum_{i=1}^{m-1} \int_B |\nabla^i f(y)|dy + A (\int_{2B} |\nabla^m f(y)|^p dy)^{1/p} \|G_m * \mu\|_q. \qquad (6)$$

By Theorem 1 we can choose μ so that

$$\|G_m * \mu\|_q \le \frac{2}{C_{m,p}(K)^{1/p}} \le \frac{A}{c_{m,p}(K,2B)^{1/p}}.$$

Now let $f \in W^{m,p}(\mathbb{R}^N)$, and assume that $f|_K = 0$. Let $\{f_n\}_1^\infty$ be a regularizing sequence of C^∞ functions. We can assume that $\{f_n(x)\}$ converges uniformly outside a set G with $C_{m,p}(G) < \frac{1}{2} C_{m,p}(K)$, so that $f_n(x) \to 0$ uniformly on $K\setminus G$, and $C_{m,p}(K\setminus G) \ge \frac{1}{2} C_{m,p}(K)$.

If μ is chosen so that supp $\mu \subset K\setminus G$ it follows that $\int f_n(x)d\mu(x) \to 0$, as $n \to \infty$.

Applying (6) to f_n it follows after a passage to the limit that

$$|f_B| \le A \sum_{i=1}^{m-1} \int_B |\nabla^i f(y)|dy + A (\int_{2B} |\nabla^m f(y)|^p dy)^{1/p} c_{m,p}(K;2B)^{-1/p}. \qquad (7)$$

Substituting this into (5) we find, again after a passage to the limit, that

$$|f(x)| \le A\left(\sum_{i=1}^{m-1} \int_B |\nabla^i f(y)|dy + (\int_{2B} |\nabla^m f(y)|^p dy)^{1/p} c_{m,p}(K;2B)^{-1/p} + \right.$$

$$\left. + \int_{2B} \frac{|\nabla^m f(y)|}{|x-y|^{N-m}} dy \right), \qquad (m,p)\text{-q.e. in } B. \qquad (8)$$

Integrating over B, and using the fact that $c_{m,p}(K;2B)$ is bounded above, we find

$$(\int_B |f(x)|^p dx)^{1/p} \le A \sum_{i=1}^{m-1} (\int_B |\nabla^i f(y)|^p dx)^{1/p} +$$

$$+ A(\int_{2B} |\nabla^m f(y)|^p dx)^{1/p} \cdot c_{m,p}(K;2B)^{-1/p}.$$

By the well-known inequality

$$(\int_B |\nabla^i f(y)|^p dy)^{1/p} \le A(\int_B |\nabla f(y)|^p dy)^{1/p} + A(\int_B |\nabla^m f(y)|^p dy)^{1/p}$$

we obtain

$$(\int_B |f(x)|^p dx)^{1/p} \le A(\int_B |\nabla f(x)|^p dx)^{1/p} +$$

$$+ A(\int_{2B} |\nabla^m f(x)|^p dx)^{1/p} \; c_{m,p}(K;2B)^{-1/p}. \tag{9}$$

If it is further assumed that $\nabla f|_K = 0$, we can apply (9) to the first derivatives of f in $W^{m-1,p}$, and find

$$(\int_B |\nabla f(x)|^p dx)^{1/p} \le A(\int_B |\nabla^2 f(x)|^p dx)^{1/p} + A(\int_{2B} |\nabla^m f(x)|^p dx)^{1/p} c_{m-1,p}(K;2B)^{-1/p}.$$

If $D^\alpha f|_K = 0$ for all α with $0 \le |\alpha| \le m - k$, we continue like this step by step. Substituting into (9) we obtain

$$(\int_B |f(x)|^p dx)^{1/p} \le A(\int_B |\nabla^{m-k+1} f(x)|^p dx)^{1/p} +$$

$$+ A(\int_{2B} |\nabla^m f(x)|^p dx)^{1/p} (c_{m,p}(K;2B)^{-1/p} + \ldots + c_{k,p}(K;2B)^{-1/p}) \le$$

$$\le A(\int_B |\nabla^{m-k+1} f(x)|^p dx)^{1/p} + A(\int_{2B} |\nabla^m f(x)|^p dx)^{1/p} \; c_{k,p}(K;2B)^{-1/p},$$

since $c_{k,p}(K;2B) \le c_{m,p}(K;2B)$ if $k \le m$.

If $k = 1$ we have obtained (1), otherwise (2).

Remark 1: It is easily seen that the theorem can be improved. For example, if $mp < N$, then by applying Sobolev's lemma to (8) we can replace the L^p-norm on the left hand side by the L^{p^*}-norm, $\frac{1}{p^*} = \frac{1}{p} - \frac{m}{N}$. (See Hedberg (1972) for a quick proof of the Sobolev lemma.)

Remark 2: Theorem 15 was proved in Hedberg (1981), Corollary 4.3, in a straightforward but more complicated way. Closely related inequalities had been proved much earlier by Maz'ja. The proof given above is close to the proof given in the book Maz'ja (1985), §10.1.3, but Maz'ja avoids duality, and so obtains a result that holds for $p \ge 1$. N.G. Meyers (1978) has given an abstract approach to this kind of inequalities.

We finish this exposition by briefly mentioning a related, but more difficult result.

Theorem 16: Let m *be a positive integer, let* $1 < p < \infty$, *and let* $f \in W^{m,p}(\mathbb{R}^N)$. *Let* $\Omega \subset \mathbb{R}^N$ *be an arbitrary open set. Then the following statements are equivalent:*

(1) $D^{\alpha}f|_{\Omega^c} = 0$ *for all* α, $0 \le |\alpha| \le m - 1$;

(2) $f \in W_0^{m,p}(\Omega)$;

(3) *for any* $\varepsilon > 0$ *and any compact* $K \subset \Omega$ *there is a function* $\eta \in C_0^{\infty}(\Omega)$ *such that* $\eta = 1$ on K, $0 \le \eta \le 1$, *and* $\|f - \eta f\|_{m,p} < \varepsilon$.

Here the implications (3) ⇒ (2) and (2) ⇒ (1) are obvious or easy, so the hard part is the implication (1) ⇒ (3). This was proved in increasing generality in Hedberg (1978), (1981), and Hedberg-Wolff (1983). We shall not discuss the proof here, suffice it to mention that it consists in an explicit construction of the multiplier η, and that the Kellogg property in Theorem 7, and the Poincaré inequality in Theorem 15 play a decisive role. In a special case, a simple proof was given in the author's (1980) expository paper (p. 96). This case is sufficient for many applications, and the proof indicates the role of the Poincaré inequality and of thinness.

BIBLIOGRAPHY

Adams, D.R. (1972). *Maximal operators and capacity.* Proc. Amer. Math. Soc., 34, 152-156.

Adams, D.R. (1973). *Traces of potentials, II.* Indiana Univ. Math. J., 22, 907-918.

Adams, D.R. (1974). *On the exceptional sets for spaces of potentials.* Pacific J. Math., 52, 1-5.

Adams, D.R. (1975). *A note on Riesz potentials.* Duke Math. J., 42, 765-778.

Adams, D.R. (1976). *On the existence of capacitary strong estimates in* $\mathbb{R}^n$. Ark. Mat., 14, 125-140.

Adams, D.R. (1978a). *Sets and functions of finite* L^p*-capacity.* Indiana Univ. Math. J., 27, 611-627.

Adams, D.R. (1978b). *Quasi-additivity and sets of finite* L^p*-capacity.* Pacific J. Math., 79, 283-291.

Adams, D.R. (1979). L^p*-capacity integrals with some applications.* Proc. Symp. Pure Math., 35, 359-367.

Adams, D.R. (1981a). *Lectures on* L^p*-potential theory.* Univ. of Umeå, Department of Math., Report no 2.

Adams, D.R. (1981b). *Capacity and the obstacle problem.* Appl. Math. Optim., 8, 39-57.

Adams, D.R. (1985). *Weighted nonlinear potential theory.* Preprint. Trans. Amer. Math. Soc., to appear.

Adams, D.R. (1986). *The classification problem for the capacities associated with the Besov and Triebel-Lizorkin spaces.* Preprint. Banach Center Publications, to appear.

Adams, D.R., and Hedberg, L.I. (1984). *Inclusion relations among fine topologies in non-linear potential theory.* Indiana Univ. Math. J., 33, 117-126.

Adams, D.R., and Lewis, J.L. (1985). *Fine and quasi connectedness in nonlinear potential theory.* Ann. Inst. Fourier (Grenoble), 35:1, 57-73.

Adams, D.R., and Meyers, N.G. (1972). *Thinness and Wiener criteria for non-linear potentials.* Indiana Univ. Math. J., 22, 169-197.

Adams, D.R., and Meyers, N.G. (1973). *Bessel potentials. Inclusion relations among classes of exceptional sets.* Indiana Univ. Math. J., 22, 873-905.

Adams, D.R., and Polking, J.C. (1973). *The equivalence of two definitions of capacity.* Proc. Amer. Math. Soc., 37, 529-534.

Adams, R.A. (1975). *Sobolev spaces.* Academic Press.

Bagby, T. (1972). *Quasi-topologies and rational approximation.* J. Functional Anal., 10, 259-268.

Bagby, T. (1984). *Approximation in the mean by solutions of elliptic equations.* Trans. Amer. Math. Soc., 281, 761-784.

Brelot, M. (1940). *Points irréguliers et transformations continues en theorie du potentiel.* J. Math. Pures Appl., 19, 319-337.

Brelot, M. (1944). *Sur les ensembles effilés.* Bull. Sci. Math., 68, 12-36.

Brezis, H., and Browder, F.E. (1982). *Some properties of higher order Sobolev spaces.* J. Math. Pures Appl., 61, 245-259.

Bagby, T., and Ziemer, W.P. (1974). *Pointwise differentiability and absolute continuity.* Trans. Amer. Math. Soc., 191, 129-148.

Calderón, A.P. (1961). *Lebesgue spaces of differentiable functions and distributions.* Proc. Symp. Pure Math., 4, 33-49.

Carleson, L. (1972). *Selected problems on exceptional sets.* Van Nostrand, Princeton, N.J.

Choquet, G. (1959). *Sur les points d'effilément d'un ensemble. Application à l'étude de la capacité.* Ann. Inst. Fourier (Grenoble), 9, 91-101.

Deny, J. (1964). *Théorie de la capacité dans les espaces fonctionnels,* Séminaire de théorie du potentiel (Brelot-Choquet-Deny), 9, no 1.

Deny, J., and Lions, J.-L. (1953). *Les espaces de type de Beppo Levi.* Ann. Inst. Fourier (Grenoble), 5, 305-370.

Fernström, C. (1977). *On the instability of capacity.* Ark. Mat., 15, 241-252.

Fernström, C., and Polking, J.C. (1978). *Bounded point evaluations and approximation in L^p by solutions of elliptic partial differential equations.* J. Functional Analysis, 28, 1-20.

Fuglede, B. (1968). *Applications du théorème minimax a l'étude de diverses capacités.* C.R. Acad. Sci. Paris, Sér. A, 266, 921-923.

Gariepy, R., and Ziemer, W.P. (1977). *A regularity condition at the boundary for solutions of quasilinear elliptic equations.* Arch. Rat. Mech. Anal., 67:1, 25-39.

Gol'dštejn, V.M., and Rešetnjak, Ju.G. (1983). *Introduction to the theory of functions with generalized derivatives and quasiconformal mappings* (Russian). Nauka, Moscow.

Hansson, K. (1978). *On a maximal imbedding theorem of Sobolev type and spectra of Schrödinger operators.* Thesis, University of Linköping.

Hansson, K. (1979). *Imbedding theorems of Sobolev type in potential theory.* Math. Scand., 45, 77-102.

Hedberg, L.I. (1972a). *Non-linear potentials and approximation in the mean by analytic functions.* Math. Z, 129, 299-319.

Hedberg, L.I. (1972b). *On certain convolution inequalities.* Proc. Amer. Math. Soc., 36, 505-510.

Hedberg, L.I. (1972c). *Bounded point evaluations and capacity.* J. Functional Analysis, 10, 269-280.

Hedberg, L.I. (1973). *Approximation in the mean by solutions of elliptic equations.* Duke Math. J., 40, 9-16.

Hedberg, L.I. (1978). *Two approximation problems in function spaces.* Ark. Mat., 16, 51-81.

Hedberg, L.I. (1980). *Spectral synthesis and stability in Sobolev spaces,* in *Euclidean Harmonic Analysis* (Proc., Univ. of Maryland, 1979). Lecture Notes in Math., 779, 73-103. Springer-Verlag.

Hedberg, L.I. (1981). *Spectral synthesis in Sobolev spaces, and uniqueness of solutions of the Dirichlet problem.* Acta Math., 147, 237-264.

Hedberg, L.I. (1983). *On the Dirichlet problem for higher order equations,* in *Conference on Harmonic Analysis in Honor of Antoni Zygmund* (Chicago 1981), 620-633. Wadsworth.

Hedberg, L.I. (1986). *Approximation in Sobolev spaces and nonlinear potential theory,* in *Nonlinear Functional Analysis and its Applications.* Proc. Symp. Pure Math., 45.

Hedberg, L.I., and Wolff, Th.H. (1983). *Thin sets in nonlinear potential theory.* Ann. Inst. Fourier (Grenoble), 33:4, 161-187.

Helms, L.L. (1969). *Introduction to potential theory.* Wiley, New York.

Hestenes, M.R. (1941). *Extension of the range of a differentiable function.* Duke Math. J., 8, 183-192.

Kolsrud, T. (1982). *A uniqueness theorem for higher order elliptic partial* *differential equations*. Math. Scand., 51, 323-332.

Landkof, N.S. (1966). *Foundations of modern potential theory,* Nauka, Moscow. (English translation, Springer-Verlag, 1972).

Lindqvist, P., and Martio, O. (1985). *Two theorems of N. Wiener for solutions of quasilinear elliptic equations*. Acta Math., 155, 153-171.

Loewner, C. (1959). *On the conformal capacity in space*. J. Math. Mech., 8, 411-414.

Maz'ja, V.G. (1963). *The Dirichlet problem for elliptic equations of arbitrary order in unbounded regions*. Dokl. Akad. Nauk SSSR, 150, 1221-1224.

Maz'ja, V.G. (1970). *On the continuity at a boundary point of solutions of quasilinear equations*. Vestnik Leningrad. Univ., 25:13, 42-55. Correction, Ibid., 27:1 (1972), 160.

Maz'ja, V.G. (1985). *Sobolev spaces*. Springer-Verlag. (Russian version, *Prostranstva S.L. Soboleva,* Izd. Leningrad. Univ., 1985).

Maz'ja, V.G., and Havin, V.P. (1970). *A nonlinear analogue of the Newtonian potential, and metric properties of* (p,1) - *capacity.* Dokl. Akad. Nauk SSSR, 194:4, 770-773.

Maz'ja, V.G., and Havin, V.P. (1972). *Nonlinear potential theory.* Uspehi Mat. Nauk, 27:6, 67-138.

Maz'ja, V.G., and Shaposhnikova, T.O. (1985). *Theory of multipliers in spaces of differentiable functions.* Pitman.

Meyers, N.G. (1970). *A theory of capacities for potentials of functions in Lebesgue classes.* Math. Scand., 6, 255-292.

Meyers, N.G. (1972). *Continuity of Bessel potentials.* Israel J. Math., 11, 271-283.

Meyers, N.G. (1974). *Taylor expansion of Bessel potentials.* Indiana Univ. Math. J., 23, 1043-1049.

Meyers, N.G. (1975). *Continuity properties of potentials.* Duke Math. J., 42, 157-166.

Meyers, N.G. (1978). *Integral inequalities of Poincaré and Wirtinger type.* Arch. Rat. Mech. Anal., 68, 113-120.

Muckenhoupt, B., and Wheeden, R. (1974). *Weighted norm inequalities for fractional integrals.* Trans. Amer. Math. Soc., 192, 261-274.

Polking, J.C. (1972). *Approximation in L^p by solutions of elliptic partial differential equations.* Amer. Math. J., 94, 1231-1244.

Rešetnjak, Ju.G. (1969). *On the concept of capacity in the theory of functions with generalized derivatives.* Sibirsk. Mat. Ž., 10, 1109-1138.

Rešetnjak, Ju.G. (1972). *On the boundary behaviour of functions with generalized derivatives.* Sibirsk. Mat. Ž., 13, 411-419.

Sjödin, T. (1975). *Bessel potentials and extension of continuous functions on compact sets.* Ark. Mat., 13, 263-271.

Skrypnik, I.V. (1984). *A criterion for regularity of a boundary point for quasilinear elliptic equations.* Dokl. Akad. Nauk SSSR, 274, 1040-1043.

Sobolev, S.L. (1950). *Applications of functional analysis in mathematical physics.* Izd. Leningrad. Univ. (English translation: Amer. Math. Soc., Providence, R.I., 1963).

Stein, E.M. (1970). *Singular integrals and integrability properties of functions.* Princeton University Press, Princeton, N.J.

Wallin, H. (1963). *Continuous functions and potential theory.* Ark. Mat., 5, 55-84.

Wallin, H. (1977). *Metrical characterization of conformal capacity zero.* J. Math. Anal. Appl., 58, 298-311.

Webb, J.R.L. (1980). *Boundary value problems for strongly nonlinear elliptic equations.* J. London Math. Soc. (2), 21, 123-132.

Wiener, N. (1924). *The Dirichlet problem.* J. Math. Phys. Mass.Inst. Tech., 3, 127-146.