MULTIPLICITY RESULTS FOR NONLINEAR ELLIPTIC EQUATIONS

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I. Introduction

We consider nonlinear elliptic equations of the form

$$(1.1)\qquad \begin{cases} -\Delta u(x) = f(u(x)) + h(x) \ , & x \in \Omega \subset \mathbb{R}^n \\ u(x) = 0 & , \ x \in \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded and smooth domain in $\mathbb{R}^n$, h is a given data function and $f : \mathbb{R} \to \mathbb{R}$ is a C^k-function with $k \geq 0$.

We are interested in the existence and multiplicity of solutions of equation (1.1) in dependence of the given data function h . First, note that if f does *not interact* with the spectrum of $-\Delta$, i.e. if, denoting by λ_i , $i \in \mathbb{N}$, the eigenvalues and by e_i , $i \in \mathbb{N}$, the corresponding eigenfunctions of

$$(1.2)\qquad \begin{cases} -\Delta v = \lambda v \ , \ \Omega \\ v = 0 \ , \partial\Omega \ , \end{cases}$$

we have $f'(t) \neq \lambda_i$, for all $i \in \mathbb{N}$ and all $t \in \mathbb{R}$, then equation (1.1) has a unique solution for every given $h \in L^2(\Omega)$. More interesting are the situations in which f does interact with the spectrum of $-\Delta$. In fact, we will find interesting solution structures as a result of such interactions.

In the study of these equations many tools of Nonlinear Analysis, such as Leray-Schauder degree, variational methods, Morse theory, etc., have been applied to obtain (lower) estimates on the number of solutions of equation (1.1) under various assumptions on f . Here we will concentrate on a particular class of nonlinearities, namely those which *cross asymptotically* eigenvalues of $-\Delta$, i.e. which satisfy (under the simplifying assumption $f \in C^1(\mathbb{R})$):

$$]f'(-\infty), f'(+\infty)[\cap \sigma(-\Delta) \neq \emptyset$$

where $\sigma(-\Delta) = \{\lambda_i, \ i \in \mathbb{N}\}$ denotes the spectrum of $-\Delta$. Nonlinearities of this type have been termed "jumping nonlinearities" by S. Fučik [13], who has introduced many useful concepts in the study of these equations.

The simplest case of asymptotic crossing is given if f interacts only with the first eigenvalue of $-\Delta$, i.e. if

$$(1.3)\qquad f'(-\infty) < \lambda_1 < f'(+\infty) < \lambda_2 \ .$$

This situation has been considered by A. Ambrosetti - G. Prodi in 1972 [2]. In their remarkable paper they showed that under the additional assumption

(1.4) $\qquad f''(t) > 0 \ , \quad \forall t \in \mathbb{R} \ ,$

the solution structure of equation (1.1) can be completely characterized.

<u>Theorem</u>. (Ambrosetti-Prodi [2]). Assume that f satisfies (1.3) and (1.4). Then there exists a hypersurface $N_1 \subset C^{0,\alpha}(\Omega)$, $0 < \alpha < 1$, such that $C^{0,\alpha} \setminus N_1$ consists of two unbounded components N_0 and N_2 and

if $h \in N_0$, then (1.1) has no solution
if $h \in N_2$, then (1.1) has 2 solutions
if $h \in N_1$, then (1.1) has 1 solution .

(see section III for remarks to the proof).

In subsequent papers by various authors it has been shown that the phenomenon of either zero or (at least) two solutions occurs always if f crosses the first eigenvalue (and if f satisfies some growth conditions), see [1,6, 10] . The most general result in this direction is contained in D.G. de Figueiredo - S. Solimini [11], who use variational methods to obtain the existence of two solutions.

Here we would like to discuss the case that f crosses the first two eigenvalues, i.e.

(1.5) $\qquad f'(-\infty) < \lambda_1 < \lambda_2 < f'(+\infty) < \lambda_3 \ ,$

and the case that f crosses a higher eigenvalue λ_k , $k \geq 2$, i.e.

(1.6) $\qquad \lambda_{k-1} < f'(-\infty) < \lambda_k < f'(+\infty) < \lambda_{k+1} \ .$

Also, we will give results for the Sturm-Liouville problem

$$(1.7) \qquad \begin{cases} -u'' = f(u) + h \ , & \text{in} \quad (0,\pi) \\ u(0) = u(\pi) = 0 \end{cases}$$

with f crossing an arbitrary (finite) number of eigenvalues.

The results which we present seem to indicate that any additional eigenvalue which is crossed by f produces two additional solutions (for suitable given h). In fact A. Lazer - P.J. McKenna conjectured in [18] that for f satisfying

(1.8) $\qquad f'(-\infty) < \lambda_1 < \ldots < \lambda_k < f'(+\infty) < \lambda_{k+1}$

equation (1.1) has for $h(x) = t \sin x + h_1(x)$ with t sufficiently negative at least $2k$ solutions. In [19] they proved this conjecture for the corresponding Sturm-Liouville problem (1.7).

The theorems and proofs concerning the situations (1.5), (1.6) and (1.7) indicate that the appearance of additional solutions under crossing of eigenvalues is a bifurcation phenomenon. In the second part of this exposition we will analyze more deeply how this bifurcation occurs. For this, we will use singularity theory in Banach space. The idea is to study the singular set of the mapping $-\Delta - f$ and try to use this information to describe the image of $-\Delta - f$. In fact, this approach was already taken by A. Ambrosetti - G. Prodi to prove the quoted theorem. Recently, singularity theory in Banach space has been developped further by Berger - Church - Timourian [4], Lazzeri - Micheletti [20], Cafagna - Donati [5] and others. The results we present here are not complete and require further research. However, they indicate that the bifurcations of solutions mentioned above occur as global cusps forming in the image space of $-\Delta - f$.

II. Asymptotic crossing of eigenvalues and bifurcation

1. A nonlinear eigenvalue problem

It has been noticed by S. Fučik [12] and E.N. Dancer [7] that the following positive homogeneous equation (2.1) is crucial for the study of equation (1.1) under assumptions (1.5), (1.6) or (1.7):

$$(2.1) \qquad \begin{cases} -\Delta v - \mu v^+ + \eta v^- = 0 , & \text{in } \Omega \\ v = 0 , & \text{on } \partial\Omega , \end{cases}$$

where $v^+ = \max\{v,0\}$ and $v^- = v^+ - v$. By setting $\gamma = \mu - \eta$, equation (2.1) can be written as

$$(2.2) \qquad -\Delta v - \gamma v^- = \mu v$$

or equivalently

$$(2.3) \qquad -\Delta v - \gamma v^+ = \eta v .$$

Fixing $\gamma > 0$, equations (2.2) and (2.3) can be viewed as nonlinear eigenvalue problems. We note that $(\mu,v) \in \mathbb{R} \times H_o^1(\Omega)$ is a solution of (2.2) if and only if (η,v) with $\eta = \mu-\gamma$ is a solution of (2.3). In general, equation (2.1) (or equivalently (2.2) or (2.3)) is difficult to solve; in fact, it is not even known if in general the nonlinear eigenvalues μ and η (for fixed γ) are isolated. The following results are known.

Proposition 2.1. (Gallouët-Kavian [14], Ruf [22]). Assume that λ_k is a simple eigenvalue of (1.2) with corresponding eigenfunction e_k and that

$$(2.4) \qquad 0 < \gamma < \min\{\lambda_{k+1} - \lambda_k, \lambda_k - \lambda_{k-1}\} .$$

Then there exist exactly two eigenvalues μ^1, μ^2 in the interval $]\lambda_k,\lambda_{k+1}[$ such that (2.2) has nontrivial solutions v^1, v^2 ; these solutions satisfy

$$(2.5) \qquad (v^1,e_k) > 0 \quad , \quad (v^2,e_k) < 0 \; .$$

For the proof of this result we refer to [14,22]. Viewing the term $\gamma\cdot v^-$ as a perturbation of the linear equation (1.2), we can say that under a perturbation $\gamma > 0$ the eigenvalue λ_k "splits" into the two nonlinear eigenvalues $\mu^1(\gamma),\mu^2(\gamma)$, and similarly, the eigenspace $\{se_k, s \in \mathbb{R}\}$ is "bent" into the two eigenrays $\{sv^1, s \in \mathbb{R}^+\}$ and $\{sv^2, s \in \mathbb{R}^+\}$. In fact, one has for $\gamma \to 0$ that $\mu^{1,2}(\gamma) \to \lambda_k$ and $v^1 \to e_k$, $v^2 \to -e_k$. Finally, we remark that in general $\mu^1(\gamma) \neq \mu^2(\gamma)$ for $\gamma > 0$, but that $\mu^1(\gamma) = \mu^2(\gamma)$ can occur, e.g. due to a symmetry (see the result of the ODE below).

For the corresponding Sturm-Liouville problems

$$(2.6) \qquad \begin{cases} -v''-\gamma v^- = \mu v \; , \; \text{in} \quad (0,\pi) \\ \quad v(0) = v(\pi) = 0 \end{cases}$$

$$(2.7) \qquad \begin{cases} -v''-\gamma v^+ = \eta v \; , \; \text{in} \quad (0,\pi) \\ \quad v(0) = v(\pi) = 0 \end{cases}$$

one has a complete result:

<u>Proposition 2.2.</u> [23]. For any fixed $\gamma > 0$ there exists a sequence of nonlinear eigenvalues of (2.6), ordered us follows

$$0 < \mu_1^1 \equiv \lambda_1 < \lambda_1 + \gamma \equiv \mu_1^2 < \mu_2^1 = \mu_2^2 < \mu_3^1 < \mu_3^2 < \mu_4^1 = \mu_4^2 < \mu_5^1 < \ldots \to +\infty \; .$$

To μ_k^1 corresponds an eigenfunction v_k^1 of (2.6) with $k-1$ nodes and $(v_k^1)'(0) > 0$ and to μ_k^2 an eigenfunction v_k^2 with $k-1$ nodes and $(v_k^2)'(0) < 0$.

The *proof* of this result relies on the observation that on the intervals where $v > 0$ resp. $v < 0$, v satisfies $-v'' = \mu v$, resp. $-v'' = (\mu-\gamma)v$, and hence $v(x) = \alpha \sin(\sqrt{\mu}\, x + a)$, $\alpha > 0$, resp. $v(x) = \beta \sin(\sqrt{\mu-\gamma}\, x + b)$, $\beta < 0$. For any given number of sign changes one can construct solutions with $v'(0) > 0$, resp. $v'(0) < 0$, by the appropriate choice of the constants α, a, β, b . Finally, one notes that for even eigenvalues (i.e. for an odd number of sign changes) one has $v_{2k}^1(x) = v_{2k}^2(\pi-x)$, and therefore $\mu_{2k}^1 = \mu_{2k}^2$, $\forall\, k \in \mathbb{N}$.

2. <u>The crossing of</u> λ_1 <u>and</u> λ_2

We now return to the inhomogeneous equation (1.1) under assumption (1.5). Let $|\cdot|$ and $(\cdot,\cdot)$ denote the L^2-norm and L^2-inner product.

Theorem 2.3. Assume that f satisfies assumption (1.5). Then, for every given $h_1 \in C^{0,\alpha}(\Omega)$ with $(h_1,e_1)=0$ there exist constants $T_0(h_1) \geq T_1(h_1)$ such that for $h = te_1 + h_1$ with

$t > T_0(h_1)$, (1.1) has no solution

$t < T_1(h_1)$, (1.1) has at least four solutions.

Proof. This result was obtained by H. Hofer [17] and E.N. Dancer [8] by topological methods. We give here the idea of an alternative proof which is based on bifurcation arguments. We restrict the attention to the *model equation*

$$(2.8) \qquad -\Delta u - f^+u^+ + f^-u^- = te_1$$

where $f^+ = f'(+\infty)$, $f^- = f'(-\infty)$. The result for the general equation (1.1) is obtained from the results for (2.8) by a perturbation argument, see [24].

For equation (2.8) we have $T_0(0) = T_1(0) = 0$. In fact, the nonexistence result follows in this case easily: Assume first that u is a solution of (2.8) with $u^- \not\equiv 0$, and let $\Omega^- \subset \Omega$ be a subdomain where $u < 0$. Then

$$(-\Delta(-u^-) + f^-u^-, -u^-) = t(e_1,-u^-)$$

implies

$$\lambda_1(\Omega^-)\,\|u^-\|^2 - f^-\|u^-\|^2 \leqq -t(e_1,u^-) ,$$

where $\lambda_1(\Omega^-)$ denotes the first eigenvalue of $-\Delta\big|_{\Omega^-}$ (with Dirichlet boundary values). Since $\lambda_1(\Omega^-) \geq \lambda_1$, we conclude that $t < 0$.
Assume now that $u > 0$ in Ω . Multiplying (2.8) by e_1 we get

$$\lambda_1(u,e_1) - f^+(u,e_1) = t ,$$

i.e. again $t < 0$. Hence, (2.8) cannot have a solution for $t > 0$.

We now transform (2.8) as above into

$$(2.9) \qquad -\Delta u - \gamma u^- = f^+u + te_1 , \quad \gamma = f^+ - f^- .$$

However, instead of studying equation (2.9), we consider the following equation

$$(2.10) \qquad -\Delta y - \gamma(y+\alpha e_1)^- = \lambda y .$$

Note that if y is a solution of (2.10) for $\lambda = f^+$, then $u = \alpha e_1 + y$ solves (2.9) for $t = (\lambda_1 - f^+)\alpha$. It therefore suffices to find solutions of (2.10). We consider equation (2.10) in the space $E = \{u \in C^{1,\tau}(\Omega)\ ;\ u/\partial\Omega = 0\}$. Note that for $\alpha > 0$ equation (2.10) admits $y \equiv 0$ as (the trivial) solution, and that

$$\frac{\gamma(y+\alpha e_1)^-}{\|y\|_E} \xrightarrow[\|y\|_E \to 0]{} 0 \quad \text{in } E ,$$

since $e_1 > 0$ in Ω with $\frac{\partial e_1}{\partial n}\Big|_{\partial\Omega} > 0$. Hence, (2.10) can be considered as a bifurcation problem in $(\lambda,y) \in \mathbb{R}\times E$. By the global bifurcation results of P. Rabinowitz [21] we therefore have two global bifurcation branches C_k^1 , C_k^2 bifurcating from each eigenvalue λ_k , $k \in \mathbb{N}$. We note that the branches bifurcating from λ_1 can be explicitely calculated. In fact, it is easily verified that

$$C_1^1 = \{(\lambda_1, se_1);\ s \geq 0\}$$

$$C_1^2 = \{(\lambda_1 + \gamma(\frac{s+\alpha}{s})^+,\ se_1);\ s \leq 0\}\ .$$

Note that the second branch connects $(\lambda_1, 0)$ with $(\lambda_1+\gamma, \infty)$ in $\mathbb{R}\times E$. Considering now the branches C_2^1 and C_2^2 , one shows that they cannot meet the branches C_1^1 , C_1^2 , since on C_2^1, C_2^2 , the solutions change sign. Hence, they must go to infinity or to a higher eigenvalue λ_k , $k \geq 3$. Assume that they go to infinity, i.e. that there is a sequence of solutions $(\lambda_n, y_n) \in C_2^1$ (or C_2^2) with $\|(\lambda_n, y_n)\|_{\mathbb{R}\times E} \to \infty$. Then $\liminf_{n\to\infty} \lambda_n > \lambda_1 + \gamma$, since otherwise one obtains from (2.10) by dividing with $\|y_n\|_E$ and going to the limit $-\Delta z - \gamma z^- = \lambda z$; but this equation has no solution which changes sign for $\lambda \leq \lambda_1 + \gamma$. Finally, we note that the condition $f^- < \lambda_1 < \lambda_2 < f^+$ implies

$$\lambda_2 < f^+ < f^+ + \lambda_1 - f^- = \lambda_1 + \gamma\ .$$

Collecting these statements we can draw the following bifurcation diagram.

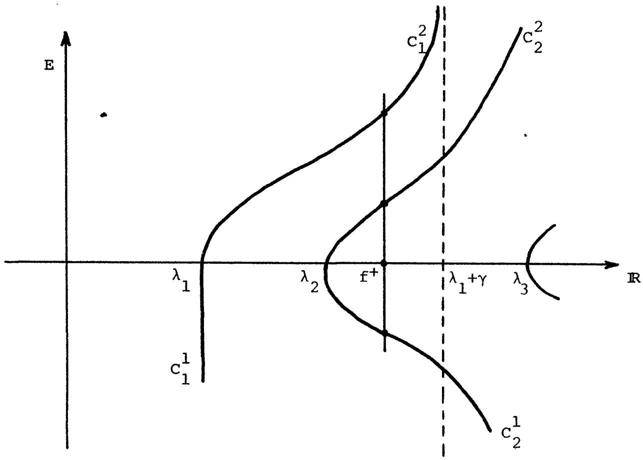

Figure 1

The four solutions of theorem 2.3 are now found by the intersection of $\{f^+\}\times E$ with C_1^2, C_2^1, C_2^2 and $\{(\lambda,0);\lambda\in\mathbb{R}\}$, the line of trivial solutions. ∎

We will see that for the corresponding ODE this approach works for any number of crossed eigenvalues. For the PDE, the difficulty to extend this result lies in the problem that the bifurcation branches might join each other. In fact, in a recent paper E.N. Dancer [9] has given an example which shows that this can actually occur. We note that the above result can be extended to the case where also the third eigenvalue is crossed, but just by some small $\varepsilon > 0$:

$$f^- < \lambda_1 < \lambda_2 < \lambda_3 < f^+ < \lambda_3 + \varepsilon \quad .$$

S. Solimini [27] has shown that in this case one can find six solutions. This result can also be obtained from the bifurcation diagram by observing that the branches C_2^1, C_2^2 and C_3^1, C_3^2, cannot join in a neighbourhood of $(\lambda_3,0)$.

3. The crossing of λ_k , $k \geq 2$

The next theorem reveals an interesting structure in the case that a higher eigenvalue λ_k is crossed. Assume that γ satisfies (2.4) and let $\mu^1(\gamma)$, $\mu^2(\gamma) \in]\lambda_k,\lambda_{k+1}[$ denote the eigenvalues of equation (2.2) which are asserted by proposition 2.1. We denote by h_1, $h_2 \in C^{0,\alpha}(\Omega)$ functions satisfying $\int h_1e_1dx = 0$, $\int h_2e_2dx = 0$, respectively.

Theorem 2.4. Assume that f satisfies assumptions (1.6) and (2.4) for $\gamma = f^+ - f^-$. We distinguish the following three cases.

a) $f^+ \in]\lambda_k,\mu^1(\gamma)[$. Then (1.1) has at least

1 solution for all $h \in C^{0,\alpha}(\Omega)$

3 solutions for $h = h_1 + te_1$ with t sufficiently negative.

b) $f^+ \in]\mu^1(\gamma),\mu^2(\gamma)[$. Then (1.1) has

2 solutions for $h = h_1 + te_1$ with $|t|$ sufficiently large

2 solutions for $h = h_2 + te_2$ with t sufficiently negative

0 solutions for $h = h_2 + te_2$ with t sufficiently positive.

c) $f^+ \in]\mu^2(\gamma),\lambda_{k+1}[$. Then (1.1) has at least

1 solution for all $h \in C^{0,\alpha}(\Omega)$

3 solutions for $h = h_1 + te_1$ with t sufficiently positive.

Proof: We outline the idea of the proof which consists of two parts.

1. Because of conditions (1.6) and (2.4) one can perform a Lyapunov-Schmidt reduction to reduce equation (1.1) to an equation in one dimension: Let

$P : C^{o,\alpha} \to [e_2]$, $Pu = e_2\int_\Omega ue_2dx$ denote the orthogonal projection onto $[e_2]$, and $Q = I-P$. Then, for $h_2 \in [e_2]^\perp$ given, there exists a unique solution $w(\alpha,h_2)$ of

$$-\Delta w - Q\gamma(\alpha e_2+w)^- - f^+ w = h_2 .$$

The existence of $w(\alpha,h_2)$ is obtained by the Leray-Schauder principle, while the uniqueness follows by direct estimates, using (1.6) and (2.4). Noting that $w(\alpha,h_2)$ is continuous in α , one then considers the continuous function

$$\Gamma(\alpha) = (\lambda_2 - f^+)\alpha - \int_\Omega \gamma(\alpha e_2+w(\alpha))^- \cdot e_2dx .$$

Using proposition 2.1 one shows that

$$\lim_{\alpha\to\pm\infty} \Gamma(\alpha) = \begin{cases} \pm\infty & \text{in case a)} \\ -\infty & \text{in case b)} \\ \mp\infty & \text{in case c)} \end{cases}$$

This yields for any $h_2 + te_2$ at least one solution in case a) and c), and the alternative of at least two or zero solutions in case b). We refer for this result to T. Gallouët-O. Kavian [14] and B. Ruf [22].

2. One can also in this situation consider the bifurcation equation (2.10). One obtains as above bifurcation branches C^1 and C^2 emanating from $(\lambda_k,0)$. By asymptotic estimates (using that $\int u\cdot e_k dx \neq 0$ for all solutions $(\lambda,u) \in C^1 \cup C^2$) one shows that these branches meet $(\mu^1,\infty) \in \mathbb{R}\times E$ and $(\mu^2,\infty) \in \mathbb{R}\times E$, respectively. This yields the bifurcation branches on the left in the diagram below.

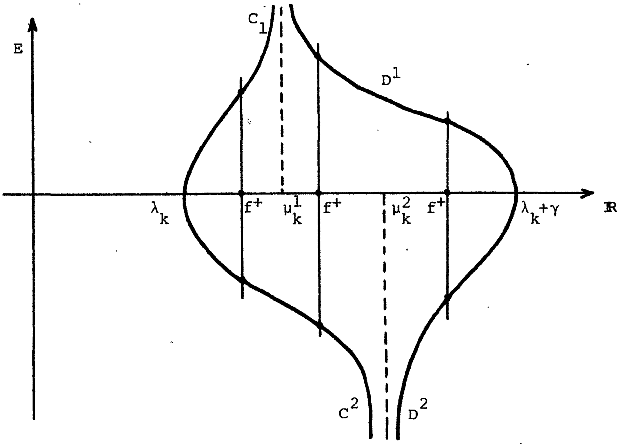

Figure 2

Now we note that instead of equation (2.10) one can also consider the equation

$$(2.11) \qquad -\Delta w - \gamma(w-\alpha e_1)^+ = \eta w \ , \ \alpha > 0 \ .$$

In fact, we have again that if w is a solution for $\eta = f^-$, then $u = -\alpha e_1 + w$ is a solution of (2.8) with $t = -\alpha(\lambda_1 - f^-)$. Therefore, equation (2.10) yields solutions of (2.8) for $t = \alpha(\lambda_1 - f^+) < 0$, while (2.11) yields solutions of (2.8) for $t = -\alpha(\lambda_1 - f^-) > 0$ (since $f^- > \lambda_k - \gamma > \lambda_1$).

Equation (2.11) can again be viewed as a bifurcation equation in (η, w), and we find again bifurcation branches D^1, D^2, emanating from $(\lambda_k, 0)$ and ending asymptotically in $(\mu^1-\gamma,\infty)$, $(\mu^2-\gamma,\infty)$, respectively. Setting $\lambda(w) = \eta(w) + \gamma$ we can draw the branches D^1, D^2 into the same diagram as C^1 and C^2, see fig. 2.

We now see that the set $\{f^+\} \times E$ with $f^+ \notin \{\mu^1, \mu^2\}$ intersects $C^1 \cup C^2 \cup D^1 \cup D^2$ at least twice. Depending on the position of f^+ with respect to μ^1 and μ^2 this yields either 2 (nontrivial, i.e. $u \neq \alpha e_1$) solutions for t negative (if $f^+ \in]\lambda_k, \mu^1[$), or 1 solution for t negative and 1 solution for t positive (if $f^+ \in]\mu^1, \mu^2[$), or 2 solutions for t positive (if $f^+ \in]\mu^2, \lambda_k + \gamma[$. For this result (in different notation) we refer also to S. Solimini [26]). ∎

Counting also the trivial solutions (i.e. the solutions of the form $u = \alpha e_1$) we can collect the statements of part b) in the following

<u>Corollary 2.5.</u> Assume that f satisfies (1.6), (2.4), and $f^+ \notin \{\mu^1, \mu^2\}$. Then the *sum* of the number of solutions of (2.8) for $t > 0$ and $t < 0$ is at least four.

<u>Proof:</u> Adding the trivial solutions $\pm \alpha e_1$ (corresponding to a negative, resp. positive t) to the solutions obtained above yields the statement. ∎

<u>Remark 2.6.</u> If $f^+ = \mu^1$ or $f^+ = \mu^2$, then the equation is *in resonance,* that is, in this case the homogenous eigenvalue problem (2.1) has nontrivial solutions. As is seen from the diagram, one has for $\mu^1 \neq \mu^2$ still a nontrivial solution for either $t < 0$ or $t > 0$ (depending whether $f^+ = \mu^1$ or $f^+ = \mu^2$) ; one could say that in this case the equation is in *half- resonance.*

4. The Sturm-Liouville problem

A general result for the Sturm-Liouville problem with f crossing an arbitrary finite number of eigenvalues is most easily formulated in the form of Corollary 2.5.

<u>Theorem 2.6.</u> (Hart-Lazer-McKenna [16]). Assume that f satisfies

$$\lambda_j < f^- < \lambda_{j+1} < \ldots < \lambda_{j+k} < f^+ < \lambda_{j+k+1}$$

i.e. f crosses k eigenvalues, and that

$$f^+ \notin \{\lambda_k, \mu_k^1(\gamma), \mu_k^2(\gamma),\ k \in \mathbb{N}\} \quad ,$$

where $\gamma = f^+ - f^-$ and $\mu_k^1(\gamma)$, $\mu_k^2(\gamma)$ denote the nonlinear eigenvalues of equation (2.6) given by proposition 2.2.

Then we have for equation

$$(2.12) \qquad \begin{cases} -u'' - f(u) = h_1 + te_1 \ , & \text{in } \ ]0,\pi[\\ u(0) = u(\pi) = 0 \ : \end{cases}$$

For given $h_1 \in C^{0,\alpha}(0,\pi)$ with $\int h_1 e_1 dx = 0$ the *sum* of the number of solutions for $h = h_1 + te_1$ with t large positive and t large negative is at least $2k+2$.

<u>Proof</u>. The proof follows the same lines as the proof of theorem 2.4, that is, one obtains for the bifurcation equations

$$(2.13) \qquad -y'' - \gamma(y+\alpha e_1)^- = \lambda y$$

$$(2.14) \qquad -w'' - \gamma(w-\alpha e_1)^+ = \eta w$$

bifurcation branches C_k^1 and C_k^2 , respectively D_k^1 and D_k^2 , which bifurcate from $(\lambda_k, 0) \in \mathbb{R} \times E$, for all $k \in N$. Using the nodal properties of Sturm-Liouville equations one shows that C_k^1 , C_k^2 end asymptotically in $\mu_k^1(\gamma)$, $\mu_k^2(\gamma)$, and D_k^1, D_k^2 end asymptotically in $\mu_k^1(\gamma) - \gamma$, $\mu_k^2(\gamma) - \gamma$. Setting again $\lambda(w) = \eta(w) + \gamma$ one obtains the following diagram:

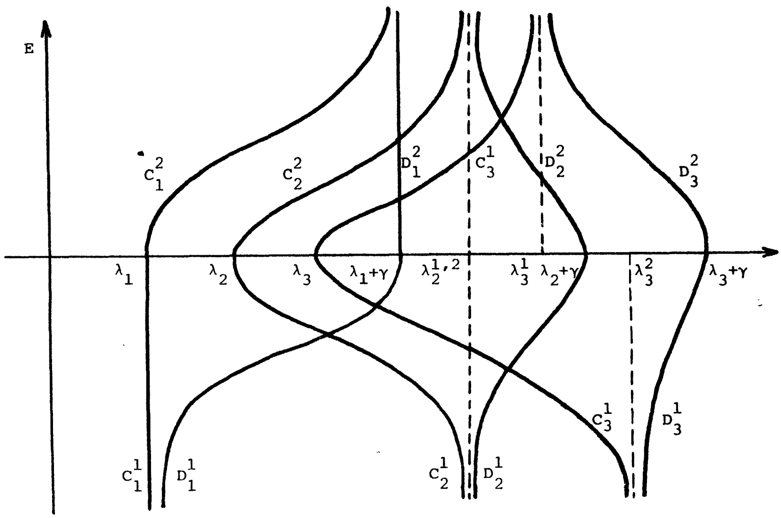

Figure 3

The proof of the theorem follows now from the observation that if f crosses asymptotically an eigenvalue, i.e. if $f^- < \lambda_m < f^+$, then

$$\{f^+\} \times E \cap [C_m^1 \cup C_m^2 \cup D_m^1 \cup D_m^2] \neq \emptyset . \tag{2.15}$$

In fact, $f^- < \lambda_m < f^+$ implies

$$\lambda_m < f^+ < f^+ + \lambda_m - f^- = \lambda_m + \gamma ,$$

and since $f^+ \neq \mu_m^i$ $(i=1,2)$ by assumption, the claim follows. But (2.15) implies that equation (2.13) and (2.14) have *together* 2 solutions with $m-1$ nodes. From this results clearly that if f crossses k eigenvalues, then equations (2.13) and (2.14) have together $2k$ solutions. Adding the two trivial solutions (of the form $\pm \alpha e_1$) we obtain the claimed result. ■

Remark 2.7. From the diagram one can also read off the number of solutions for the individual equations (2.13) resp. (2.14), and hence obtain the number of solutions for equation (2.12) for t large negative, respectively for t large positive. Also, if $f^+ = \mu_k^i$, for some $k \in \mathbb{N}$, $i \in \{1,2\}$, one still will have many solutions, as is again seen from the diagram. A half resonance, i.e. $f^+ = \mu_k^i$ with $\mu_k^1 \neq \mu_k^2$, causes the loss of one solution, while a full resonance, i.e. $f^+ = \lambda_k$, or $f^+ = \lambda_k + \gamma$, or $f^+ = \mu_k^1 = \mu_k^2$, causes the loss of two solutions.

III. Singularity theory and the geometry of nonlinear differential operators

In the last section it was seen that the multiplicity results for equation (1.1) (under assumptions (1.5), (1.6), or (1.7)) can be understood as a bifurcation phenomenon. This was achieved by introducing an additional bifurcation parameter (e.g. λ in (2.10)). From the bifurcation diagrams (figures 1, 2, 3) one then can read off the (minimal) number of solutions for data of the form $h = h_1 + te_1$ with t large. In praxis one is of course more interested in the solution behaviour for bounded data. We will give here some results in this direction by analyzing more deeply the geometry of the image of the given nonlinear operator. This will be done by the means of singularity theory. We point out that this approach can yield very precise information, but that on the other hand it requires restrictive assumptions on the nonlinearity f. For general expositions of singularity theory we refer e.g. to H. Whitney [28] and Golubitsky-Guillemin [15].

1. The singular set of nonlinear differential operators

We consider the operator $\Phi \equiv -\Delta - f : E \to F$, where $E = \{u \in C^{2,\alpha}(\Omega),\ 0<\alpha<1,\ u|_{\partial\Omega} = 0\}$, $F = C^{0,\alpha}(\Omega)$, and $f \in C^k(\Omega)$, $k \geq 1$.

Definition 3.1. We say that $u \in E$ is a *singular point* of Φ , if there exists a nontrivial $v \in E$ such that the Fréchet-derivative of Φ satisfies:

$$\Phi'(u)[v] = -\Delta v - f'(u)v = 0 \ . \tag{3.1}$$

We note that $f'(u) \in C^0(\Omega)$. We denote by $\{\mu_i(u)\}_{i\in\mathbb{N}}$ the spectrum of $-\Delta - f'(u) : E \to F$:

$$\Phi'(u)[v_i] = -\Delta v_i - f'(u)v_i = \mu_i(u)v_i \ , \ i \in \mathbb{N} \ . \tag{3.2}$$

Then, (3.1) amounts to saying that 0 is an eigenvalue of $-\Delta - f'(u)$.

The importance of the singular set for understanding the solution structure of a nonlinear equation becomes clear from the following theorem.

Theorem 3.2. Let E, F and Φ be as above, and let $S = \{u \in E;\ \Phi'(u)[v] = 0$ for some $0 \neq v \in E\}$ denote the singular set of Φ .

Then, denoting by

$$N(y) = \# \{u \in E | \Phi(u) = y \ , \ y \in F\}$$

the number of preimages of y , the function $N(y)$ is constant on every connected component of $F \setminus \Phi(S)$.

We remark that Theorem 3.2 holds under very general hypothesis (see Ambrosetti-Prodi [2]).

The theorem shows that we know completely the solution structure of the equation $\Phi(x) = y$, if we have a precise description of $\Phi(S)$. For this we need to know the precise structure of S , that is, how S looks as a subset of E , and what kind of singular points (in the sense of R. Thom) the set S contains.

First, we are interested in a condition which garantees that the *singular set* of Φ , $S = \{u \in E,\ \Phi'(u)[v] = 0$, for some $0 \neq v \in E\}$, is a "nice" set.

Lemma 3.3. Let $f \in C^k(\mathbb{R})$, $k \geq 2$, and $u \in E$ such that $f''(u) \neq 0$ and that 0 is a simple eigenvalue of $\Phi'(u)$. Then the singular set is locally a C^{k-1}-manifold of codimension 1 , i.e. there exists an open neighbourhood U of u such that $U \cap S$ is a C^{k-1}-manifold of codimension 1.

Proof: Note that $S = \{y \in E | \mu_i(y) = 0$ for some $i \in \mathbb{N}\}$. By assumption we have in the point u

$$\mu_{i-1}(u) < 0 = \mu_i(u) < \mu_{i+1}(u) \ , \text{ for some } i \in \mathbb{N} \ .$$

This implies that locally μ_i is C^{k-1}. We drop in the sequel the subscript i, and denote by $v(u)$ the eigenfunction corresponding to $\mu(u) = \mu_i(u)$. Let $\|\cdot\|$ and $(\cdot,\cdot)$ denote the L^2-norm and the L^2-innerproduct. We have to show that $D\mu(u)[z] \neq 0$, for some $z \in E$. But

$$\begin{aligned} D\mu(u)[z] &= D[\|\nabla v\|^2 - (f'(u)v,v)][z] = 2(\nabla v, \nabla Dv[z]) - 2(f'(u)v, Dv[z]) \\ &\quad - (f''(u)zv,v) = -(f''(u)v^2,z) \ . \end{aligned}$$

We choose $z = f''(u)v^2$ (if $z \notin E$, we can approximate it by $\tilde{z} \in E$). Then

$$D\mu(u)[f''(u)v^2] = -\|f''(u)v^2\|^2 < 0 \ .$$

The result now follows from the implicit function theorem. ∎

We now come to the classification of the singular points. We restrict ourselves to the so-called fold points and the cusp points.

<u>Proposition 3.4</u>. Assume that 0 is a simple eigenvalue of $\Phi'(u)$, and assume that

$$\int f''(u)v^3 \neq 0 \ , \tag{3.3}$$

where $v(u)$ is the eigenfunction corresponding to $\mu(u) \equiv \mu_i(u) = 0$.

Then there exists a neighbourhood U of u such that $\Phi(U \cap S) \subset F$ is a manifold of codimension 1. Furthermore, $v(u)$ is the normal vector to $\Phi(U \cap S)$ in $\Phi(u)$, and we can find a $\varepsilon > 0$ such that for any vector $z \in E$ which is transversal to $\Phi(U \cap S)$ in $\Phi(u) = \bar{y}$

a) $\forall y \in]\bar{y}, \bar{y} + \varepsilon z[$ the equation $\Phi(u) = y$ has exactly two solutions in U.

b) $\forall y \in]\bar{y}, \bar{y} - \varepsilon z[$ the equation $\Phi(u) = y$ has no solution in U.

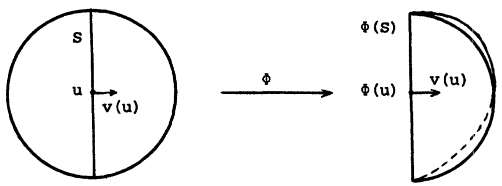

Figure 4

it is tangential to S :

$$\Phi'(u)Dv[z] - f''(u)vz = \mu(u)Dv(u)[z] + D\mu(u)[z]v(u) = 0$$

and hence $Dv(u)[z] = (\Phi'(u))^{-1}(f''(u)vz)$. ∎

If we strengthen condition (3.5) to

$$3\int f''(u)v^2(\Phi'(u))^{-1}(f''(u)v^2) + \int f'''(u)v^4 \neq 0 \,, \tag{3.7}$$

then we can characterize also the image $\Phi(U)$ of a neighbourhood U of u .

Proposition 3.6. Assume that $f \in C^k$, $k \geq 3$, and that $u \in E$ is such that 0 is a simple eigenvalue of $\Phi'(u)$ and that (3.4) and (3.7) hold.

Then, for a suitable neighbourhood U of u , we have with the notation $S = S_1 \cup C \cup S_2$ (see Lemma 3.5):

$\Phi(U \cap S_i)$, $i = 1,2$, are C^{k-1}-manifolds of codimension 1

$\Phi(U \cap C)$ is a C^{k-2}-manifold of codimension 2

Furthermore, for every vector z which is transversal to $\Phi(U \cap C)$ in $\Phi(u) = \bar{y}$ and satisfies $\int zv(u) = 0$ one has for all $\varepsilon > 0$ sufficiently small:

(a) $\forall y \in]\bar{y}, \bar{y}+\varepsilon z[$ the equation $\Phi(u) = y$ has exactly 3 solutions in U .

(b) $\forall y \in]\bar{y}, \bar{y}-\varepsilon z[$ the equation $\Phi(u) = y$ has exactly 1 solution in U .

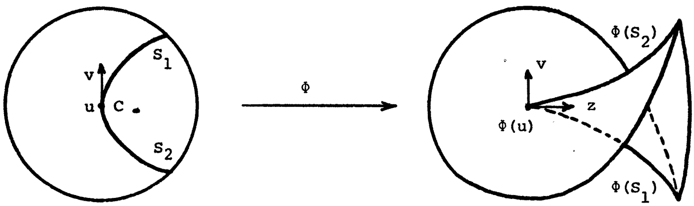

Figure 5

A point u satisfying the assumptions of proposition 3.6 is called a *cusp point* for Φ ; the name derives from the cusp which is formed by the image of the singular set.

As for the fold one can also in this case express the result of proposition

A point u satisfying the assumptions of proposition 3.3 is called a *fold point* for Φ as is justified by figure 4.

Another way to express this result is that under the assumptions of proposition 3.3 the mapping Φ is in u locally diffeomorphic to the mapping

$$\psi : B\times]-1,1[\to B\times]-1,1[$$
$$(x,t) \longmapsto (x,t^2)$$

where B is the unit ball in some Banach space. The mapping ψ is the *normal form* of the fold-mapping.

For the proof of proposition 3.3 we refer to Ambrosetti-Prodi [2], Berger-Church [3]. We just note that for the vector $z = f''(u)v^2$, which is transversal to $\Phi(U\cap S)$ (by (3.3) and since $v(u)$ is normal to $\Phi(U\cap S)$), the statements a) and b) follow from

$$\begin{aligned}\Phi(u+\alpha v(u)) &= \Phi(u) + \alpha\Phi'(u)[v(u)] + \\ &\quad + \frac{\alpha^2}{2}\Phi''(u)[v(u),v(u)] + O(\alpha^3) \\ &= \Phi(u) - \frac{\alpha^2}{2} f''(u)v^2(u) + O(\alpha^3) \qquad \blacksquare\end{aligned}$$

A more complicated situation occurs if $\int f''(u)v^3 = 0$. We again give first a condition garanteeing that the subset of such points in S is nice.

Lemma 3.5. Assume that $f\in C^k$, $k\geq 3$, that $u\in E$ is such that $0=\mu_i(u)\equiv\mu(u)$ is a simple eigenvalue of $\Phi'(u)$ and that

$$\int f''(u)v^3(u)dx = 0 \, , \qquad (v(u) \text{ as above}) \, , \tag{3.4}$$

$$3\int f''(u)v^2(\Phi'(u))^{-1}(f''(u)vz) + \int f'''(u)v^3z \neq 0 \, , \text{ for some } z\in E \, . \tag{3.5}$$

Then the set $C=\{u\in S\,|\,(3.4)\text{ holds}\}$ is locally a C^{k-2}-manifold of codimension 2 (with respect to E) such that $S\setminus C$ consists of two C^{k-1}-manifolds S_1, S_2 .

Proof: The set C is given by

$$C = \{u\,|\,\mu(u) = 0 \, , \int f''(u)v^3(u)dx = D\mu(u)[v(u)] = 0\}$$

(see Lemma 3.3). Hence, C is locally a codimension 2 manifold if

$$D(\int f''(u)v^3(u)dx)[z] \neq 0 \, , \text{ for some } z\in T_uS \, , \tag{3.6}$$

where T_uS denotes the tangent space to S in the point u . From (3.6) we obtain

$$3\int f''(u)v^2(u)Dv(u)[z]dx + \int f'''(u)v^3(u)z\,dx \neq 0 \, .$$

From $\Phi'(u)[v] = \mu v = 0$ we get by taking the derivative in the direction z, since

3.6 as a normal form theorem:

Under the assumptions of proposition 3.6 the mapping Φ is in u locally diffeomorphic to the mapping

$$\Gamma : B \times]-1,1[x]-1,1[\to B \times]-1,1[x]-1,1[$$

$$(x,s,t) \longmapsto (x,s,t^3-st)$$

where B is the unit ball in a Banach space.

For the proof of proposition 3.6 we refer to Berger-Church-Timourian [4], Lazzeri-Micheletti [20]. Here, we give a qualitative explanation of the described situation.

First we note that since in proposition 3.6 the third derivative is involved, it is not sufficient to consider Φ along the lines $u+\alpha v(u)$ (as in proposition 3.4), but one has also to take into account the variation of $v(u)$ along the path. Hence, we are led to consider the integral curves along the vector field $v(z)$, for $z \in U(u)$, i.e. the solution curves of

$$(3.8) \qquad \begin{cases} \frac{d}{dt} z(t,u) = v(z(t,u)) \\ z(0,u) = u \ , \end{cases}$$

where $v(z)$ is the eigenvector to $\mu(z) \equiv \mu_i(z)$: $\Phi'(z)v(z) = \mu(z)v(z)$; see also section III.3. Relevant for the description of $\Phi(U)$ is the behaviour of the projection onto $[v(u)]$ of $\Phi(z(t))$, the image of the integral curves.

First, we consider the integral curve through u, $z(t,u)$. Denoting $z_t(t) = \frac{d}{dt} z(t)$, we can write

$$(3.9) \qquad \begin{aligned} \Phi(z(t)) &= \Phi(u) + t\Phi'(u)z_t(0) \\ &\quad + \frac{t^2}{2}[\Phi''(u)z_t^2(0) + \Phi'(u)z_{tt}(0)] \\ &\quad + \frac{t^3}{3!}[\Phi'''(u)z_t^3(0) + 3\Phi''(u)z_{tt}z_t(0) + \Phi'(u)z_{ttt}(0)] + \text{h.o.} \\ &= \Phi(u) + \frac{t^3}{3!}[\Phi'''(u)v^3(u) + 3\Phi''(u)v(u)v_t(u) \\ &\qquad + \Phi'(u)v_{tt}(u)] + \text{h.o.} \end{aligned}$$

because $z_t(0) = v(u)$ by (3.8), and hence $\Phi'(u)z_t(0) = 0$, and

$$\Phi'(u)v_t + \Phi''(u)v^2 = \mu v_t + \mu_t v = 0 ,$$

by assumption. Finally, the projection onto $[v(u)]$ of the last term in (3.9) gives

$$\frac{t^3}{3!}\,[(f'''(u)v^3,v) + 3(f''(u)v^2,v_t)]\ .$$

Since the term in the bracket does not vanish by assumption, we therefore have a cubic curve through $\Phi(u)$ with a degenerate point at $t = 0$ (the cusp point).

If we now move the starting point u a little in a direction tangential to $\{y|\mu_v(y) = 0\}$ (and hence transversal to $\{y|\mu(y) = 0\}$) , i.e. if we consider the integral curves

$$z(t,u+sx)\ ,\ \text{with}\ \ x \in T_u\{\mu_v = 0\}\ ,\quad |s| < \varepsilon$$

we obtain, writing $z(t,s) = z(t,u+sx)$, $v(s) = v(z(0,s))$,

$$\begin{aligned}
\Phi(z(t,s)) &= \Phi(z(0,s)) + t\Phi'(z(0,s))v(s) + \\
&\quad + \frac{t^2}{2}\,[(\Phi''(z(0,s))v^2(s) + \Phi'(z(0,s))v_t(s)] \\
&\quad + \frac{t^3}{3!}\,[\Phi'''(z(0,s))v^3(s) + 3\Phi''(z(0,s))v(s)v_t(s) \\
&\qquad\qquad + \Phi'(z(0,s))v_{tt}(s)] + \text{h.o.} \\
&= \Phi(z(0,s)) + t\mu(s)v(s) + \frac{t^2}{2}\,\mu(s)v_t(s) \\
&\quad + \frac{t^3}{3!}\,[\Phi'''(z(0,s))v^3(s) + 3\Phi''(z(0,s))v(s)v_t(s) \\
&\qquad\qquad + \Phi'(z(0,s))v_{tt}(s)] + \text{h.o.}
\end{aligned}$$

The projection of this onto $[v(s)]$ gives

$$\begin{aligned}
\gamma(t,s) &\equiv (\Phi(z(0,s)),v(s)) + t\mu(s) + \\
&\quad + \frac{t^3}{3!}\,[(f'''(z(0,s)),v^4(s)) + 3(f''(z(0,s))v^2(s),v_t(s)) + \\
&\qquad\qquad + \mu(s)(v_{tt}(s),v(s))] + \text{h.o.}
\end{aligned}$$

We have $\mu_s\big|_{s=0} \neq 0$ (since x is transversal to $\{\mu = 0\}$). Furthermore, for s small, the coefficient for the cubic term does not vanish. Hence, the curve $\gamma(t,s)$ is a cubic curve in t with a positive or negativ linear term (depending on the sign of s). In the first case the curve contains no singular point (where $\gamma_t(t,s) = 0$), while in the second case it contains exactly two singular points.

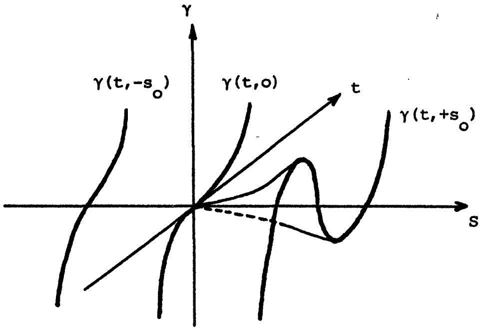

Figure 6

The projection of this figure into the (γ,s)-plane gives the described behaviour.

2. Asymptotic crossing of eigenvalues and the form of the singular set

In the introduction we have mentioned that if f does not interact with the spectrum of the Laplacian, then the mapping $-\Delta-f$ is globally invertible. In fact, one sees easily that $f'(t) \neq \lambda_i$, $\forall i \in \mathbb{N}$, $\forall t \in \mathbb{R}$, implies that the singular set of $-\Delta-f$ is empty, and hence all points in E are regular.

Here we will show that asymptotic crossing of eigenvalues together with a convexity assumption on f yields nice singular sets in the form of smooth hypersurfaces.

As in section II we consider the following three situations.

a) $\Phi = -\Delta-f : E \to F$, where $f \in C^k$, $k \geq 2$, satisfies

(3.10) $$f'(-\infty) < \lambda_1 < \lambda_2 < f'(+\infty) < \lambda_3 .$$

b) $\Phi = -\Delta-f$ and $f \in C^k$, $k \geq 2$, satisfies

(3.11) $$\lambda_{k-1} < f'(-\infty) < \lambda_k < f'(+\infty) < \lambda_{k+1}$$

c) $\Phi = -\dfrac{d^2}{dx^2} - f$, and $f \in C^k$, $k \geq 2$, satisfies

(3.12) $$\lambda_j < f'(-\infty) < \lambda_{j+1} < \ldots < \lambda_{j+k} < f'(+\infty) < \lambda_{j+k+1} .$$

Theorem 3.7. Assume situation a), b) or c), and assume that f satisfies in addition

(3.13) $$f''(t) > 0\ ,\quad \forall t \in \mathbb{R}\ .$$

Then the singular set of Φ consists in case a) of two disjoint C^{k-1}-hypersurfaces, in case b) of one C^{k-1}-hypersurface, and in case c) of k disjoint C^{k-1}-hypersurfaces.

Proof: Let $E_1 = \{u \in E; \int ue_1 dx = 0\}$, and consider for any given $u \in E_1$ the line $u + \alpha e_1$, $\alpha \in \mathbb{R}$. Now consider the continuous functions $\mu_i(\alpha) \equiv \mu_i(u+\alpha e_1)$, where $\mu_i(w)$ is the i-th eigenvalue of $\Phi'(w)$. Since

$$f'(u+\alpha e_1) \to f'(\underline{+}\infty) \quad \text{for} \quad \alpha \to \underline{+}\infty\ ,$$

we find that

$$\mu_i(\alpha) \xrightarrow[\alpha \to \underline{+}\infty]{} \lambda_i - f'(\underline{+}\infty)\ .$$

Therefore, the functions $\mu_i(\alpha)$ (with $i = 1,2$ in case a), $i = k$ in case b), and $i = j+1,\dots,j+k$ in case c)) have zeroes. From

$$\frac{d}{d\alpha}\mu_i(u+\alpha e_1) = 2(\Phi'(u+\alpha e_1)\frac{dv_i}{d\alpha}, v_i) +$$

$$+ (\Phi''(u+\alpha e_1)e_1 v_i, v_i)$$

$$= -(f''(u+\alpha e_1)v_i^2, e_1) < 0$$

follows that the functions μ_i are monotonically decreasing in α, and hence the zeroes obtained above are unique. In other words, on each line $u+\alpha e_1$, $\alpha \in \mathbb{R}$, exist in case a) exactly two values $\alpha_1(u)$, $\alpha_2(u)$ with $\mu_1(\alpha_1(u)) = \mu_2(\alpha_2(u)) = 0$, in case b) exactly one value $\alpha_k(u)$ with $\mu_k(\alpha_k(u)) = 0$, and in case c) exactly k values $\alpha_{j+1}(u),\dots,\alpha_{j+k}(u)$ with $\mu_{j+1}(\alpha_{j+1}(u)) = \dots = \mu_{j+k}(\alpha_{j+k}(u)) = 0$. Finally, we have in case a) that $\alpha_1(u) < \alpha_2(u)$, $\forall u \in E_1$, since $\mu_1(u+\alpha e_1)$ is always a simple eigenvalue, and in case c) that $\alpha_{j+1}(u) < \dots < \alpha_{j+k}(u)$, since for Sturm-Liouville problems all eigenvalues are simple. From this one now obtains easily that S consists of disjoint hypersurfaces. The differentiability is obtained by direct verification or by the implicit function theorem. ■

3. The structure of the singular set

The next task is to classify the singular points. From section III.2 it follows that if f satisfies (3.13) and

(3.14) $$f'(-\infty) < \lambda_1 < f'(+\infty)$$

then the singular set S contains a hypersurface S_1 given by $S_1 = \{u \in E;\ \mu_1(u) = 0\}$. The following proposition shows that the structure of S_1 is particularly simple, namely that it consists entirely of fold points. This was first observed by Ambrosetti-Prodi [2].

Proposition 3.8. If $f \in C^k$, $k \geq 2$, satisfies (3.13) and (3.14), then the subset $S_1 = \{u \in E; \mu_1(u) = 0\}$ of S (the singular set of $\Phi = -\Delta - f$) consists entirely of fold points.

Proof: Let $u \in S_1$. Then $0 = \mu_1(u)$ is a simple eigenvalue of $\Phi'(u)$, and

$$\int f''(u) v_1^3(u) > 0 ,$$

by (3.13) and since $v_1(u) > 0$, $\forall u \in E$. Hence u is a fold point (see proposition 3.4). ∎

The situation is more complicated for the subsets $S_i \subset S$, $i \geq 2$, given by $S_i = \{u \in E ; \mu_i(u) = 0\}$. In section III.2 we have seen that these sets are nice hypersurfaces under the assumptions (3.10), (3.11), or (3.12). We show next that under these assumptions the sets S_i , $i \geq 2$, always contain "higher" singularities, i.e. points u with $\int f''(u) v_i^3(u) = 0$. To show this, we use as in section III.1 the integral curves along the vectorfields on E given by the eigenvectors $v_i(u)$. One has the following

Lemma 3.9. Assume that f satisfies (3.10), (3.11), or (3.12). Then the vectorfields given by $v_i(u)$ (with $i = 1,2$ in case a), $i = k$ in case b), $i = j+1,\dots,k+j$ in case c)) are C^{k-1} . Furthermore, the integral curves $z_i(t,u)$, $t \in \mathbb{R}$, i.e. the solutions to the equation

$$(3.15) \qquad \begin{cases} \dfrac{d}{dt} z_i(t,u) = v_i(z_i(t,u)) \\ z_i(0,u) = u \end{cases}$$

exist globally for every $u \in E$.

We will use these integral curves to prove the following propositions.

Proposition 3.10. Assume that f satisfies (3.10) or (3.12). Then the nonempty sets $S_i = \{u \in E; \mu_i(u) = 0\}$, $i \geq 2$, contain higher singularities.

Proof: Assume that $S_i \neq \emptyset$.
We consider the i-th eigenvalue of $\Phi'(z_i(t,u))$, $\mu_i(z_i(t,u))$. In the proof of theorem 3.7 we have seen that

$$(3.16) \qquad \begin{aligned} &\mu_i(u+\alpha e_1) > 0 \quad \text{for } \alpha \text{ large negative} \\ &\mu_i(u+\alpha e_1) < 0 \quad \text{for } \alpha \text{ large positive .} \end{aligned}$$

On the other hand, one has for all given $y \in E$

(3.17) $$\mu_i(z_i(t,y)) \xrightarrow[|t|\to\infty]{} \bar{\mu}_i > 0 .$$

This follows from the equation

$$-\Delta v_i(t) - f'(z_i(t,y))v_i(t) = \mu_i(t)v_i(t)$$

by taking the limits $t \to \pm\infty$; in fact, one shows that $\lim\limits_{t\to+\infty} \frac{z_i(t,y)}{t} = \lim\limits_{t\to+\infty} v_i(t) = v$ (and analoguously for $t\to-\infty$), and hence the limit equation is

(3.18) $$-\Delta v - f^+v^+ + f^-v^- = \bar{\mu}_i v , \quad i \geq 2 .$$

$$\Leftrightarrow$$

(3.19) $$-\Delta v - (f^+-f^-)v^- = (f^+ + \bar{\mu}_i)v =: \mu v ,$$

i.e. μ is a nonlinear eigenvalue for (3.19) with an eigenfunction v which changes sign. This implies that $\mu > \lambda_1+\gamma = \mu_1^2$ (see section II). From

$$f^+ < \lambda_1 + \gamma = \mu_1^2 < \mu , \; i \geq 2 ,$$

we find that $\bar{\mu}_i = \mu - f^+ > 0$. Let now

$$m(y) = \inf_{t\in\mathbb{R}} \mu_i(z_i(t,y)) , \; y \in E .$$

One checks that m is continuously dependent on y , and if $m(y) < \mu_i^1 - f^+$, then the infimum is assumed by a $\bar{t} \in \mathbb{R}$. Finally, by (3.16), we find that $m(u+\alpha e_1) \lessgtr 0$ $\alpha \to \pm\infty$. Hence, there exists a $\bar{\alpha}$ such that $0 = m(u+\bar{\alpha}e_1) = \mu_i(z_i(\bar{t},u+\bar{\alpha}e_1))$. In this point we have

$$\begin{aligned} 0 = \frac{d}{dt}\mu_i(z_i(\bar{t},u+\bar{\alpha}e_1)) &= D\mu_i(z_i(\bar{t},u+\bar{\alpha}e_1))\frac{d}{dt}z_i(\bar{t}) \\ &= D\mu_i(z_i(\bar{t},u+\bar{\alpha}e_1))v_i(\bar{t}) \\ &= \int f''(z_i(\bar{t}))v_i^3(\bar{t}) \quad . \blacksquare \end{aligned}$$

<u>Proposition 3.11</u>. Assume that f satisfies (3.11) and that $f^+ \in]\lambda_k,\mu_k^1[$ or $f^+ \in]\mu_k^2,\lambda_k+\gamma[$ (see proposition 2.1). Then $S = S_k = \{u \in E;\ \mu_k(u) = 0\}$ contains points y such that $\int f''(y)v_k^3(y) = 0$.

<u>Proof</u>: The proof proceeds as above: one shows that

$$\mu_k(z_k(t,y)) \xrightarrow[|t|\to\infty]{} \begin{cases} \bar{\mu}_k > 0 , & \text{if } f^+ \in]\lambda_k,\mu_k^1[\\ \bar{\mu}_k < 0 , & \text{if } f^+ \in]\mu_k^2,\lambda_k+\gamma[\end{cases}$$

by observing that in the limit equation (3.19) $\bar{\mu}_k = \mu - f^+ > 0$ (resp. < 0) if

$f^+ \in]\lambda_k,\mu_k^1[$ (resp. $f^+ \in]\mu_k^2,\lambda_k+\gamma[$) . From this one concludes the existence of points with $0 = \mu_k(u) = D\mu_k(u)[v_k(u)]$, as above. ∎

For the case (3.11) with $f^+ \in]\mu_k^1,\mu_k^2[$ it is not possible to proceed as above, since $\lim_{t\to+\infty} \mu_k(z_k(t,y)) > 0$ and $\lim_{t\to-\infty} \mu_k(z_k(t,y)) < 0$. If one could establish that $\mu_k(z_k(t,y))$ is monotonically decreasing in t for all $y \in E$, then S_k would contain only fold points. This would give rise to the zero or two solutions as obtained asymptotically in theorem 2. However, we are not able to give conditions which imply the desired monotonicity for all $y \in E$.

Next, we give a condition which implies that the subsets $C_i \subset S_i$ consisting of higher singularities, i.e. $C_i = \{u \in E;\ 0 = \mu_i(u) = D\mu_i(u)[v_i] = 0\}$, are codimension 2 submanifolds of E :

$$(3.20) \qquad f'''(t) < 0 , \ \forall t \in \mathbb{R} .$$

This condition is very restrictive, since it implies under the assumption $f^+ = f'(+\infty) < +\infty$ that $f^- = f'(-\infty) = -\infty$. Hence, it does not apply to the situation (3.11).

<u>Proposition 3.12</u>. Assume that f satisfies (3.10) or (3.12), and (3.20). Then the subsets $C_i \subset S_i$, where $C_i = \{u \in E;\ 0 = \mu_i(u) = D\mu_i(u)[v_i] = 0\}$ are unbounded C^{k-2}-manifolds of codimension 2 such that $S_i \setminus C_i$ consists of two unbounded C^{k-1}-manifolds of codimension 1.

<u>Proof</u>: By lemma 3.5 C_i is locally a C^{k-2}-manifold in u if there exists a $z \in E$ such that $\int f''(u)v_i^2 z = 0$ and

$$(3.21) \qquad 3\int f''(u)v_i^2(\Phi'(u))^{-1}(f''(u)v_i z) + \int f'''(u)v_i^3 z \neq 0 .$$

We set $z = \dfrac{v_1(u)}{f''(u)v_i(u)}$. Then $\int f''(u)v_i^2 z = \int v_i(u)v_1(u) = 0$ and (3.21) becomes

$$3\int f''(u)v_i^2 \frac{1}{\mu_1} v_1 + \int f'''(u)v_i^2 \frac{v_1}{f''(u)} < 0 ,$$

since $f''(u) > 0$, $f'''(u) < 0$, and $\mu_1(u) < 0$, $v_1(u) > 0$. Note that $z \notin E$; however, z can be approximated by elements $\tilde{z}$ of E , and for a good enough approximation (3.21) will also hold for $\tilde{z}$.

Finally, considering the integral curves along $z(u)$ (respectively the approximating $\tilde{z}(u) \in E$), i.e. the solutions (which remain on S_i) of

$$\begin{cases} \dfrac{d}{dt}\gamma(t,u) = z(\gamma(t,u)) \\ \gamma(0,u) = u \in S_i , \end{cases}$$

one finds that on each of these curves there exists exactly one $\bar{t}$ with $0 = \mu_i(\gamma(\bar{t},u))$, and $0 = \frac{d}{dt}\mu_i(\gamma(\bar{t},u)) = D_{\mu_i}(\gamma(\bar{t},u))v_i(u)$; from this one concludes that C_i is a codimension 2 manifold such that $S_i \setminus C_i$ has two unbounded components. ■

We still need an additional step, namely the characterization of the set C_i. We restrict ourselves to C_2 and we wish to prove that it consists entirely of cusp points; this requires a sharpening of condition (3.20). The condition we impose is

$$|f''(t)|^2 < \frac{f^+-\lambda_3}{3} f'''(t) \ , \ t > \bar{t} > -\infty \ . \tag{3.21}$$

We note that this assumption is very restrictive. In fact, it implies that if $f'(+\infty) < \infty$ and $f''(+\infty) = 0$, then $f'(t) \to -\infty$ for $t \downarrow \bar{t} > -\infty$, i.e. the function is forced to blow up in a finite point $\bar{t}$. On the other hand, it cannot blow up too fast, more precisely, $f'(t)$ cannot go to $-\infty$ faster than $f'(t) \cong \frac{\lambda_3-f^+}{3} \ell g(t-\bar{t})$ for $t \downarrow \bar{t}$, as is easily checked. We will further discuss these restrictions below. First we show

Proposition 3.13. Assume that f satisfies (3.10) and (3.21). Then $C_2 \cap \{u|u(x) > \bar{t}\}$, where $C_2 \subset S_2$ is given by proposition 3.12, consists entirely of cusp points.

Proof: We have to show that for all $u \in C_2$ we have

$$3\int f''(u)v_2^2(\Phi'(u))^{-1}(f''(u)v_2^2) + \int f'''(u)v_2^4 \neq 0 \ . \tag{3.22}$$

Note that $(\Phi'(u))^{-1}f''(u)v_2^2 = \sum_{k\neq 2} \frac{1}{\mu_k}(f''(u)v_2^2,v_k)v_k$, which gives

$$3 \sum_{k\neq 2} (f''(u)v_2^2,v_k)^2 \frac{1}{\mu_k} + \int f'''(u)v_2^4 \leq$$

$$\leq 3 \frac{1}{\mu_2}\|f''(u)v_2^2\|^2 + \int f'''(u)v_2^4 < 0$$

since $\mu_1 < 0$ and $\mu_2 < \lambda_3 - f^+$. ■

4. The image of $\Phi = -\Delta-f$

We are now in position to give a complete characterization of the image of $-\Delta-f$ restricted to the subdomain where f satisfies condition (3.21).

We first consider the case where $f'(t)$ does blow up in a finite point $\bar{t}$, and we choose $\bar{t} = 0$ (see the remarks above). In this case it seems natural to consider the problem in the domain $\{u \in C^{2,\alpha}(\Omega) \ ; \ \frac{\partial u}{\partial n}\Big|_{\partial\Omega} = 0 \ , \ u > 0 \ \text{in} \ \Omega\}$, i.e. we look for positive solutions of the equation

$$(3.23)\qquad \begin{cases} -\Delta u - f(u) = h \ , & \Omega\subset \mathbb{R}^n \\ \dfrac{\partial u}{\partial n} = 0 \ , & \partial\Omega \end{cases}$$

An example of a function which is admissible in the following theorem is:

$$f(x) = \begin{cases} \alpha x-\beta \ell g x \ , & x \geq 1 \\ \beta(x\ell g x - x) \ , & 0 < x < 1 \end{cases}$$

with $\lambda_2 < \alpha < \lambda_3$, and $0 < \beta < \frac{\lambda_3-\alpha}{3}$.

(note that f is only C^2 in 1 , with a finite jump in the third derivative. One can check that this is admissible in the above theorems, or smooth out f appropriatly in the point $x = 1$)

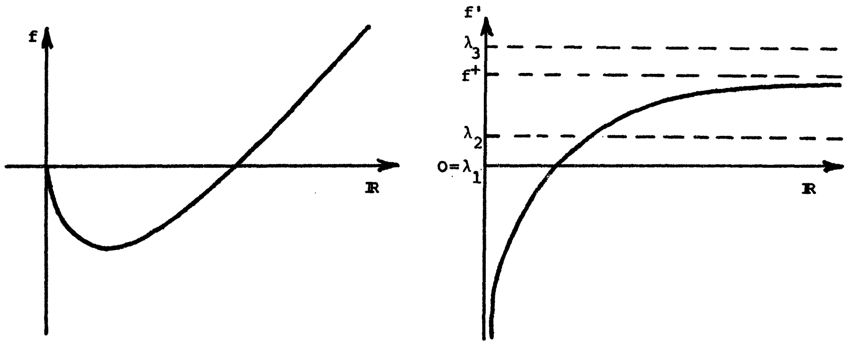

Figure 7

<u>Theorem 3.14</u>. Assume that $f \in C^3(\mathbb{R}^+,\mathbb{R})$ satisfies

$$(3.24)\qquad f'(0) = -\infty < \lambda_1 < \lambda_2 < f'(+\infty) < \lambda_3$$

and (3.21) in $\mathbb{R}^+$ (with $f^+ = f'(+\infty)$) . Let $P = \{u \in C^{2,\alpha}(\Omega,\mathbb{R}^+);\ \frac{\partial u}{\partial n}\Big|_{\partial\Omega} = 0\}$.

Then the singular set S of $\Phi = -\Delta - f$ consists of $S_1 = \{u \in P;\ \mu_1(u) = 0\}$ and $S_2 = \{u \in P;\ \mu_2(u) = 0\}$ and the following holds

(a) S_i , $i = 1,2$, are disjoint C^{k-1}-manifolds of codimension 1 which intersect the boundary ∂P .

(b) S_1 consists entirely of fold points, while S_2 contains (in general) a C^{k-2}-manifold C_2 of codimension 2 (with respect to E) of cusp points; $S_2\setminus C_2$ has two components $S_{1,1}$ and $S_{1,2}$ which consist of fold points.

(c) $\Phi(S_1)$ and $\Phi(S_{2,i})$, $i = 1,2$, are C^{k-1}-manifolds of codimension 1, and $\Phi(C_2)$

is a C^{k-2}-manifold of codimension 2.

(d) $\Phi(S_1) \cap \Phi(S_2) = \emptyset$ and $\Phi(S_{2,1}) \cap \Phi(S_{2,2}) = \emptyset$.

(e) $\Phi(P)$ is asymptotically cone shaped, and it has at least 5 bounded components and exactly one unbounded component. The number of solutions in each component is easily established. Figure 8 gives an idea of P and $\Phi(P)$ with the corresponding singular and critical sets. The numbers in the different components of $\Phi(P) \setminus \Phi(S)$ denote the exact numbers of positive solutions for h lying in the respective components.

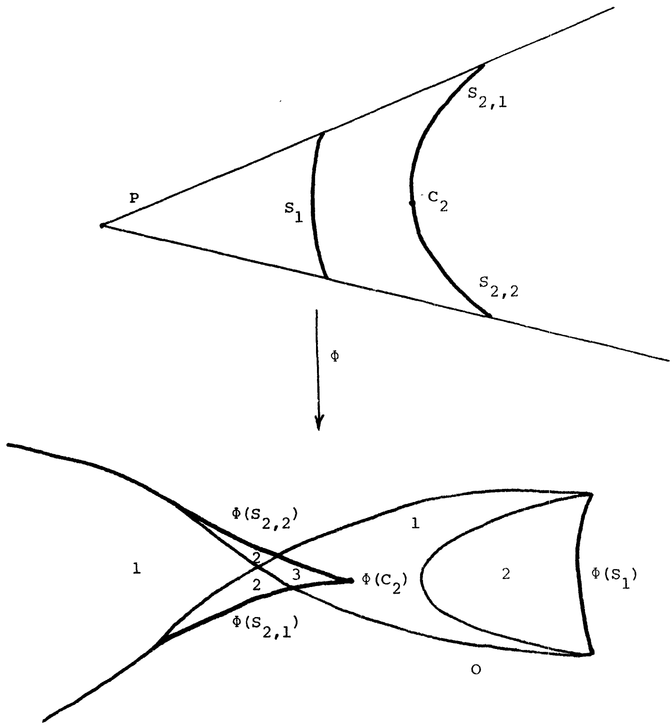

Figure 8

Proof: (a) follows from theorem 3.7, and (b) from propositions 3.8, 3.12, 3.13. Note that if S_2 is bounded, we cannot use proposition 3.10 to show that $C_2 \neq \emptyset$. However, C_2 is not empty if $\int e_2^3 = 0$ (this is e.g. the case if Ω is an interval, a ball or a cube in $\mathbb{R}^n$, or in general if Ω has a suitable symmetry): in fact, since $e_1 = \text{const.}$, we have $v_2(\alpha e_1) = e_2$, $\forall \alpha \in \mathbb{R}^+$, and hence for $\alpha_2 e_1 \in S_2 : 0 = \mu_2(\alpha_2 e_1)$ and $D\mu_2(\alpha_2 e_1)[e_2] = -\int f''(\alpha_2 e_1) e_2^3 = c\int e_2^3 = 0$. It is clear that $C_2 \neq \emptyset$ in many other cases, but we cannot prove it in general.

To prove (c) we use again a Lyapunov-Schmidt reduction to reduce (3.23) to an equation in two dimensions. Let $F_2 = \text{span}\{e_1, e_2\}$, $H_2 = \{u \in F; \int u e_i = 0,\ i = 1,2\}$, and $p_2 : F \to F_2$, $q_2 : F \to H_2$ the orthogonal projections. Then we can write equation (3.23) as

$$(3.25) \qquad -\Delta u - q_2 f(u+w) = q_2 h \equiv h_2$$

$$(3.26) \qquad -\Delta w - p_2 f(u+w) = p_2 h ,$$

where $u \in q_2 E$ and $w \in p_2 E$. Because $f'(t) < \lambda_3$ for all $t \in \mathbb{R}^+$, one can solve uniquely (3.25) for fixed h_2 and w (if $w \in p_2 E$ is such that $u + w \in P$; this is the case if $\int w e_1 dx$ is large positive). Then the problem is reduced to find a solution w of (3.26) with $u = u(w, h_2)$.

We now linearize equation (3.26) as above to obtain eigenvectors $x_i(w) \in E_2$, $i = 1,2$, of

$$-\Delta x_i(w) - p_2 f'(w+u)(x_i(w) + u_{x_i}(w)) = \eta_i(w) x_i(w) \quad , \ i = 1,2 ,$$

where $u_{x_i}(w) = D_w u(w, h_2)[x_i]$. Let $P_2 = \{w \in F_2 | w + u(w) \in P\}$. The singular set of the mapping

$$(3.27) \qquad \tilde{\Phi} \equiv -\Delta p_2 - p_2 f(\cdot + u(\cdot, h_2)) : P_2 \to F_2$$

consists of the set $\tilde{S} = \{w \in P_2 | \eta_i(w) = 0 , \quad i = 1 \text{ or } 2\}$. One notes that $\tilde{S}$ corresponds to the singular set S of $\Phi = -\Delta - f$, in fact one has

$$w \in \tilde{S} \Leftrightarrow w + u(w, h_2) \in S ,$$

since $-\Delta u_{x_i} - q_2 f'(w+u)(x_i + u_{x_i}) = 0$. From this one concludes that $\tilde{S} = \tilde{S}_1 \cup \tilde{S}_2$ consists of two disjoint C^{k-1} curves in E, where $\tilde{S}_i = \{\eta_i(w) = 0\}$, $i = 1,2$. Since also the classifications of the singular points in $\tilde{S}$ and S correspond, one has by b) that $\tilde{S}_1$ consists of fold points, and that $\tilde{S}_2$ consists of fold points except in at most one point, which is a cusp point.

We know by propositio.. 3.4 that $\tilde{\Phi}(\tilde{S}_1)$ is locally a C^{k-1}-curve and that $x_1(w)$ is the normal vector to $\tilde{\Phi}(\tilde{S}_1)$ in $\tilde{\Phi}(w)$. Note that $(x_1(w), e_1) = (x_1 + u_{x_1}, e_1) > 0$, $\forall w \in \tilde{S}_1$. This implies that $\tilde{\Phi}(\tilde{S}_1)$ cannot have any selfintersections. From this we get that $\Phi(S_1)$ is a C^{k-1}-manifold of codimension 1.

Similarly, one shows that $\tilde{\Phi}(\tilde{S}_{2,i})$, $i = 1,2$, are C^{k-1}-curves and hence $\tilde{\Phi}(\tilde{S}_{2,i})$, $i = 1,2$, are C^{k-1}-manifolds of codimension 1, while $\Phi(C_2)$ is a codimension 2 manifold, since $\tilde{C}_2(h_2)$ (for $h_2 \in H_2$ fixed) consists of at most one point, which is C^{k-2}-dependent on $h_2 \in H_2$.

To prove (d), one considers the integral curves $z_i(t,u)$, $i = 1,2$, along the vector fields $v_i(u)$, $i = 1,2$ (see lemma 3.9). The sets S_1 and S_2 can be parametrized by the curves $z_1(t,u)$, $u \in E_1 = \{u \in E; \int ue_1 = 0\}$, that is, for every $u \in E_1$ there exist exactly two values $t_1(u) < t_2(u)$ with $z_i(t_i,u) \in S_i$, $i = 1,2$. Since $\mu_1(z_1(t,u)) < 0$ for $t \in]t_1(u),t_2(u)]$ one finds that Φ maps the path $z_1(t,u)$, $t_1(u) \leq t \leq t_2(u)$, into a path $\Phi(z_1(t,u))$ with tangent vector $\mu_1 v_1(z_1(t,u)) < 0$. From this, one deduces that $\Phi(S_1) \cap \Phi(S_2) = \emptyset$.

To show $\Phi(S_{2,1}) \cap \Phi(S_{2,2}) = \emptyset$ one uses $z_2(t,u)$. Again, $S_{2,1}$ and $S_{2,2}$ can be parametrized by these curves, since each $z_2(t,u)$ intersects $S_{2,1}$ and $S_{2,2}$ at most once, say in $t_1(u) < t_2(u)$. Since $\mu_2(z_2(t,u)) < 0$ for $t_1(u) < t < t_2(u)$, one deduces also in this case the desired result.

From these statements one infers (e) and the illustrative figure 8. We remark that the bounded region next to $\Phi(S_1)$ can for certain nonlinearities f be larger than drawn, so that it overlaps the set $\Phi(C_2)$. In this case there will be a region with exactly four solutions. ∎

In theorem 3.14 we have obtained a complete characterization of the positive solutions of equation (3.23). In particular, one sees that the crossing of λ_1 by the nonlinearity f gives rise to a fold, while the crossing of λ_2 gives a global cusp.

The situations a), b) or c) mentioned in the beginning of section III.2 cannot be completely described with these results. In fact, we believe that higher singularities than cusps are involved in a complete characterization of these cases. But our results can serve to give information also in these situations. For instance, if we assume that in case a) we have the assumptions

(3.28) $$f'(-\infty) < \lambda_1 < \lambda_2 < f'(+\infty) < \lambda_3 \ , \ f'(0) < \lambda_2$$

(3.29) $$f''(t) > 0 \ , \quad \forall t \in \mathbb{R}$$

(3.30) $$|f''(t)|^2 < \frac{f^+-\lambda_3}{3} f'''(t) \ , \text{ for } t \geq 0 \ ,$$

then we can say the following for equation (3.23). We set $E = \{u \in C^{2,\alpha}(\Omega) \ ; \ \frac{\partial u}{\partial n}\Big|_{\partial\Omega} = 0\}$ and $P = \{u \in E; \ u(x) > 0 \text{ in } \Omega\}$.

<u>Theorem 3.15</u>. Assume that $f \in C^k$, $k \geq 3$, satisfies (3.28-30). Then the singular set S of $\Phi = -\Delta - f : E \to F$ consists of S_1 and S_2 , which are C^{k-1}-manifolds of co-

dimension 1. Furthermore, we have

a) $\Phi(S_1)$ is a C^{k-1}-manifold of codimension 1, and $\Phi(S_2 \cap P)$ is as in theorem 3.14.

b) $F \setminus \Phi(S_1) =: F_o \cup F_2$ has two unbounded components such that for

$h \in F_o$, (3.23) has no solution

$h \in F_2$, (3.23) has at least 2 solutions (and exactly two solutions if h is "close" to $\Phi(S_1)$ in F_2) .

c) Let $F_4 = \Phi(\{u \in P | \mu_2(u) < 0\})$, i.e. the image of the points in P which lie on the right of S_2 (see figure 8). Then, for

$h \in F_4$, (3.23) has at least four solutions (with the exact number of *positive* solutions given by figure 8).

The proof of this result follows the same lines as that of theorem 3.14. For b) one uses that $\Phi(S_1)$ is a hypersurface in F , and that $\mu_1(u)$ along the integral curves $z_1(t,u)$ to the vectorfield $v_1(u)$ satisfies

$$\lim_{t \to \pm\infty} \mu_1(z_1(t,u)) \lesseqgtr 0 .$$

This implies that $(\Phi(z_1(t,u)),v_1) \xrightarrow[|t|\to\infty]{} -\infty$

that is, Φ covers the region on "the left" of $\Phi(S_1)$ at least twice: once by $\Phi(\{\mu_1(u) < 0\})$ and once by $\Phi(\{\mu_1(u) < 0\})$. Similarly, one notes for c) that $\mu_2(u)$ along the integral curves $z_2(t,u)$ to the vectorfield $v_2(u)$ satisfies

$$\lim_{t \to \pm\infty} \mu_2(z_2(t,u)) > 0 ,$$

while $\mu_2(u) < 0$ in the set $\{u \in P | \mu_2(u) < 0\}$. This implies that $\Phi(\{\mu_1(u) > 0\})$ covers the region $\Phi(P)$ at least three times. Since also $\Phi(\{\mu_1(u) < 0\})$ covers $\Phi(P)$, we obtain that $\Phi(P)$ is covered at least four times.

∎

As pointed out in the beginning there is more research needed to obtain complete pictures. As was seen, the application of singularity theory requires strong assumptions on the nonlinearities. These assumptions guarantee that no higher singularities occur, which is equivalent to say that no secondary (or higher) bifurcations occur. For other problems with such properties we refer to [20, 25].

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