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Vesa Mustonen
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Mappings of Monotone Type: Theory and Applications

VESA MUSTONEN, OULU

1. INTRODUCTION

Let X be a real reflexive Banach space and let X^* stand for its dual space with respect to the continuous pairing $\cdot\cdot$. A mapping T from its domain $D(T)$ in X to X^* is said to be monotone if

$$T(u) - T(v)u - v \geq 0 \qquad \text{for all } u, v \in D(T).$$

Mappings from a subset of X to X^* are rather natural framework for the approach to deal with the problems of calculus of variations. Indeed, if ϕ is a convex functional from a subset G of X to , then its derivative ϕ' is a mapping from G to X^* and in fact, ϕ is convex if and only if ϕ' is monotone. The basic result for monotone mappings obtained by Minty in early sixties says that every monotone hemicontinuous coercive mapping form X to X^* is surjective. This theorem has many extensions to more general classes of mappings of monotone type as was shown by Browder, Višik, Brezis, Leray and Lions, Hess, Gossez and others in the following two decades. The principal tool in proving such extension was a Galerkin method which uses approximations in finite dimensional spaces.

On the other hand, the degree of mapping has been one of the most important tools in nonlinear functional analysis applied to obtain existence and multiplicity results for solutions of functional equations. The classical degree for continuous mappings from a bounded open subset G of ^{N} to ^{N} was constructed by Brouwer in 1912. In the celebrated paper by Leray and Schauder in 1934 the degree was extended for mappings in infinite dimensional Banach spaces of the form $F = I + C$ where I is the identity map and C is compact. Since then a number of further generalizations have been introduced. Important recent contributions are due to Browder in the framework of his larger program of studying the mapping degree for general classes of mappings with domains in one Banach space X and ranges in another Banach space Y ([**Bro 3**,4,5]).

The present notes are concerned with the construction of approximative degree theories for some classes of mappings of monotone type and applications to nonlinear partial differential equations. We shall start in Section 2 by giving the definitions and basic properties of various classes of mappings of monotone type relevant for our applications. In Section 3 we survey the construction of the degree for mappings of class (S_+) and for quasimonotone mappings. This approach was introduced in ([**BM 1**], [**Be**]) and it is based on the Leray-Schauder theory rather than on Brouwer degree and Galerkin procedure used in the previous approach by Browder ([**Bro 4**,5]). Section 4 is devoted to some general theorems obtained by homotopy argument whenever a classical degree theory is available. The results are applied in Section 5 to the study of nonlinear elliptic partial differential operators in divergence form

$$A(u) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, u, \dots, D^m u).$$

In Section 6 we consider mappings defined in a Hilbert space H having a representation of the form

$$F = Qg + Pf,$$

where $H = M \oplus M^{\perp}$, Q and P are the orthogonal projections to M and $M^{\perp}$, respectively, $g = I + C$ is a Leray-Schauder map and f is a mapping of class (S_+). We show that the degree for such mappings can be constructed as an extension of the Leray-Schauder degree. The results of Section 6 are applied to the study of existence and multiplicity of periodic solutions for semilinear wave equations of the form

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - g(x,t,u) = h & \text{in } (0,\pi) \times (0,2\pi) \\ u(0,\cdot) = u(\pi,\cdot) = 0 \\ u(\cdot, t+2\pi) = u(\cdot,t) \end{cases}$$

as we shall show later in Section 7. These two sections survey the results of recent papers ([**BM 2,3,4**]). Finally, in Section 8 we deal with the extension of the degree for mappings of the form $F = T + S$, where T is a maximal monotone map from a subset $D(T)$ of X to X^* and S is of class (S_+). The case where $F = L + S$, L is a linear densely defined maximal monotone map and S is of class (S_+) or pseudomonotone with respect to $D(L)$, is discussed in Section 9. This makes possible to apply the results to the study of nonlinear parabolic operators of the form

$$\frac{\partial u}{\partial t} + \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^{\alpha} A_{\alpha}(x,t,u,\ldots,D^m u).$$

The material of these two sections is mainly based on Browder's paper ([**Bro 5**]) and on the ideas indicated therein.

2. Mappings of monotone type

Let X be a real reflexive Banach space. We can assume without loss of generality that X and its dual space X^* are locally uniformly convex (see [**De**]). We shall be dealing with mappings T acting from a subset $D(T)$ of X into X^*. The norm convergence in X and in X^* is denoted by $\to$ and the weak convergence by $\rightharpoonup$, respectively. We start with the following concepts of continuity. T is called *hemicontinuous* if $T(u + t_n v) \rightharpoonup T(u)$ as $t_n \to 0+$, and *demicontinuous* if $u_n \to u$ implies $T(u_n) \rightharpoonup T(u)$.

A mapping $T\colon D(T) \to X^*$ is called

- *monotone* (we denote $T \in (MON)$) if $\langle T(u) - T(v), u - v\rangle \geq 0$ for all $u, v \in D(T)$.
- *quasimonotone* ($T \in (QM)$) if for any sequence $\{u_n\}$ in $D(T)$ with $u_n \rightharpoonup u$, we have $\limsup \langle T(u_n), u_n - u\rangle \geq 0$.
- *pseudomonotone* ($T \in (PM)$) if for any sequence $\{u_n\}$ in $D(T)$ with $u_n \rightharpoonup u$ and $\limsup\langle T(u_n), u_n - u\rangle \leq 0$, we have $\lim\langle T(u_n), u_n - u\rangle = 0$, and if $u \in D(T)$, then $T(u_n) \rightharpoonup T(u)$.
- *of class* (S_+) ($T \in (S_+)$) if for any sequence $\{u_n\}$ in $D(T)$ with $u_n \rightharpoonup u$ and $\limsup\langle T(u_n), u_n - u\rangle \leq 0$, we have $u_n \to u$.
- *bounded* if it takes bounded subsets of $D(T)$ into bounded subsets of X^*.
- *locally bounded* if for each $u \in D(T)$ there exists a neighbourhood $\mathcal{U}$ of u such that $T(\mathcal{U})$ is bounded in X^*.
- *compact* ($T \in (COMP)$) if it is continuous and takes any bounded subset of $D(T)$ into a relatively compact subset of X^*.

Moreover, a monotone mapping T is *strictly monotone* if $\langle T(u) - T(v), u - v\rangle > 0$ for all $u \neq v$ in $D(T)$, and *strongly monotone* (denote $T \in (MON)_S$) if there exists a continuous strictly increasing function g from $[0,\infty)$ to $[0,\infty)$ with $g(0) = 0$ such that

$$\langle T(u) - T(v), u - v\rangle \geq g(\|u - v\|)\|u - v\| \qquad \text{for all } u, v \in D(T).$$

If we assume that all mappings are demicontinuous and defined in the whole space X, we have the following inclusions

$$\begin{array}{ccccccc} (MON)_S & \longrightarrow & (S_+) & \longrightarrow & (PM) & \longrightarrow & (QM) \\ & & & & \uparrow & & \uparrow \\ & & & & (MON) & & (COMP). \end{array}$$

For the sequel it will be important to observe that if $S \in (S_+)$ and $T \in (QM)$ are arbitrary demicontinuous mappings, then $T + S \in (S_+)$. Otherwise, all the classes defined above have at least a conical structure.

It is well known that the conditions

$$\|\mathcal{J}(u)\| = \|u\|, \quad \langle \mathcal{J}(u), u\rangle = \|u\|^2 \qquad \text{for all } u \in X,$$

determine a unique mapping $\mathcal{J}$ from X to X^*, which is called the *duality map*. In our case it is bijective bicontinuous strictly monotone and of class (S_+).

The concept of a monotone map has a natural generalization for multivalued maps. A subset $A \subset X \times X^*$ is called *monotone* if for each pair $[u_1, w_1]$, $[u_2, w_2] \in A$ we have $\langle w_1 - w_2, u_1 - u_2\rangle \geq 0$. Let T be a *multi* from $D(T) \subset X$ to 2^{X^*}, the subsets of X^*, where $D(T) = \{u \in X \mid T(u) \neq \emptyset\}$. Then T is called *monotone* if its graph

$$G(T) = \{[u, w] \subset X \times X^* \mid u \in D(T),\ w \in T(u)\}$$

is a monotone set in $X \times X^*$. A monotone multi T is said to be *maximal monotone* (denote $T \in (MM)$) if $G(T)$ is not a proper subset of any monotone set in $X \times X^*$, or equivalently, if $[u_0, w_0] \in X \times X^*$ and $\langle w_0 - w, u_0 - u\rangle \geq 0$ for all $[u, w] \in G(T)$, then $[u_0, w_0] \in G(T)$, i.e., $u_0 \in D(T)$ and $w_0 \in T(u_0)$. Any monotone multi T has the inverse T^{-1} from $\mathcal{R}(T) = \bigcup\{T(u) \mid u \in D(T)\}$ to 2^X defined by

$$T^{-1}(w) = \{u \in D(T) \mid w \in T(u)\},$$

which is also monotone. Moreover, T is maximal monotone if and only if T^{-1} is maximal monotone. Using the duality map $\mathcal{J}$ one can show that T is maximal monotone if and only if $\mathcal{R}(T + \lambda\mathcal{J}) = X^*$ for every $\lambda > 0$. The results above remainnaturally true if T is a single valued mapping. In particular, any single valued hemicontinuous monotone mapping from X to X^* is also maximal monotone.

Finally, if L is a linear monotone multi from $D(L)$ to 2^{X^*}, then its graph $G(L)$ is a subspace of $X \times X^*$. If $D(L)$ is dense in X, then L is necessarily single valued. We can also define the adjoint of L as a map $L^*\colon D(L^*) \to 2^{X^*}$ by

$$w \in L^*u \quad \text{if and only if} \quad \langle w, v\rangle = \langle Lv, u\rangle \quad \text{for all } v \in D(L).$$

A necessary and sufficient condition for the maximal monotony of L is that $G(L)$ is closed and L^* is monotone. For more details and proofs on multis we refer to [**De**].

3. The degree for mappings of class (S_+), (PM) and (QM)

The topological degree of mappings is one of the most effective tools for studying the existence and multiplicity of solutions of nonlinear equations. In our discussion the basic concept to start with will be the degree for mappings in infinite-dimensional Banach spaces of the form $F = I + C$ with C compact, which we shall call the LS-degree according to Leray and Schauder. For later reference we recall however the following more general definition of classical topological degree.

DEFINITION 3.1. *Let X and Y be topological spaces and $\mathcal{O}$ a family of open subsets of X. With each $G \in \mathcal{O}$ we associate a class $\mathcal{F}_G$ of mappings $F: \overline{G} \to Y$ and with each triplet (F, G, y) with $F \in \mathcal{F}_G$, $G \in \mathcal{O}$ and $y \in Y$, $y \notin F(\partial G)$ we associate an integer $d(F, G, y)$. The function d is said to be a classical topological degree if the following conditions are satisfied:*

(a) *If $d(F, G, y) \neq 0$, then there exists $u \in G$ such that $F(u) = y$.*
(b) *If $G_0, G \in \mathcal{O}$, $G_0 \subset G$ and $F \in \mathcal{F}_G$, then the restriction $F|_{\overline{G_0}} \in \mathcal{F}_{G_0}$. If $G_1, G_2 \in \mathcal{O}$ are disjoint subsets of $G \in \mathcal{O}$ such that $y \notin F(\overline{G} \setminus (G_1 \cup G_2))$, then*

$$d(F, G, y) = d(F, G_1, y) + d(F, G_2, y).$$

(c) *For an admissible class of homotopies $\mathcal{H}_G = \{F_t \in \mathcal{F}_G \mid 0 \leq t \leq 1\}$, $G \in \mathcal{O}$, and for a continuous curve $\{y(t) \mid 0 \leq t \leq 1\}$ in Y with $y(t) \notin F_t(\partial G)$ for all $t \in [0,1]$ we have $d(F_t, G, y(t))$ is constant for all $t \in [0,1]$.*
(d) *There exists a normalizing map $j: X \to Y$ such that $j|_{\overline{G}} \in \mathcal{F}_G$ for each $G \in \mathcal{O}$, and $d(j, G, y) = 1$ whenever $y \in j(G)$.*

The unique LS-degree function d_{LS} is obtained by choosing $X = Y$ a real Banach space, $\mathcal{O}$ all open bounded subsets of X, $\mathcal{F}_G$ all continuous mappings from $\overline{G}$ to X of the form $F = I + C$, C compact. The admissible class of homotopies $\mathcal{H}_G$ consists of *LS-homotopies* $F_t = I + C_t$, where $(t, u) \to C_t(u)$ is compact from $[0,1] \times \overline{G}$ to X. Obviously the normalizing map for LS-degree is I.

Our task now will be to introduce approximative procedures which extend the LS-degree to further classes of mappings of monotone type. From now on we assume that X is a real reflexive separable Banach space and that X and X^* are locally uniformly convex.

In virtue of the embedding theorem by Browder and Ton **[BT]** there exists a separable Hilbert space H and a linear compact injection $\Psi: H \to X$ such that $\Psi(H)$ is dense in X. We define a further map $\hat{\Psi}: X^* \to H$ by

$$(\hat{\Psi}(w), v) = \langle w, \Psi(v) \rangle, \quad v \in H, \ w \in X^*,$$

where $(\cdot, \cdot)$ stands for the inner product in H. It is obvious that $\hat{\Psi}$ is also linear compact injection. Let G be an open bounded subset in X. We denote

$$\mathcal{F}_G(S_+) = \{F: \overline{G} \to X^* \mid F \in (S_+), \text{ bounded and demicontinuous}\}$$

and

$$\mathcal{H}_G(S_+) = \{F_t: \overline{G} \to X^*, \ 0 \leq t \leq 1 \mid F_t \text{ bounded homotopy of class } (S_+)\},$$

where F_t is said to be a *bounded homotopy of class* (S_+) if it is uniformly bounded in $t \in [0,1]$ and if for any sequences $\{u_n\}$ in $\overline{G}$, $\{t_n\}$ in $[0,1]$ with $u_n \rightharpoonup u$ in X and $t_n \to t$ such that $\limsup \langle F_{t_n}(u_n), u_n - u \rangle \leq 0$, we have $u_n \to u$ in X and $F_{t_n}(u_n) \rightharpoonup F_t(u)$ in X^*. With each $F \in \mathcal{F}_G(S_+)$ we can now associate a family of mappings $\{F_\epsilon \mid \epsilon > 0\}$ defined by

$$F_\epsilon(u) = u + \frac{1}{\epsilon} \Psi \hat{\Psi} F(u), \qquad u \in \overline{G}. \tag{3.1}$$

For any fixed $\epsilon > 0$, F_ϵ maps $\overline{G}$ into X and has the form $I + C_\epsilon$ where $C_\epsilon = \frac{1}{\epsilon} \Psi \hat{\Psi} F$ is compact. Hence the LS-degree is defined for the triplets (F_ϵ, G, y) whenever $y \notin F_\epsilon(\partial G)$. We have the following basic

LEMMA 3.1. *Let $F \in \mathcal{F}_G(S_+)$, $A \subset \overline{G}$ a closed subset and $0 \notin F(A)$. Then there exists $\hat{\epsilon} > 0$ such that $0 \notin F_\epsilon(A)$ for all $0 < \epsilon < \hat{\epsilon}$. Moreover, if $0 \notin F(\partial G)$, there exists $\epsilon_0 > 0$ such that $d_{LS}(F_\epsilon, G, 0)$ is constant for all $0 < \epsilon < \epsilon_0$.*

PROOF: If the first assertion were false, there would exist sequences $\{\epsilon_n\}$ and $\{u_n\} \subset A$ such that $\epsilon_n \to 0+$ and $F_{\epsilon_n}(u_n) = 0$. Choosing subsequences, if necessary, (we will not change notations) we may assume that $u_n \rightharpoonup u$ in X and $F(u_n) \rightharpoonup w$ in X^*. Since $\Psi\hat{\Psi}$ is linear and compact, $\Psi\hat{\Psi}F(u_n) \to \Psi\hat{\Psi}w$ in X. On the other hand, $\Psi\hat{\Psi}F(u_n) = -\epsilon_n u_n \to 0$ implying that $w = 0$. Hence $F(u_n) \rightharpoonup 0$ and

$$\begin{aligned}\limsup\langle F(u_n), u_n - u\rangle &= \limsup\langle F(u_n), u_n\rangle \\ &= \limsup\langle F(u_n), -\frac{1}{\epsilon_n}\Psi\hat{\Psi}F(u_n)\rangle \\ &= \limsup\left\{-\frac{1}{\epsilon_n}\|\hat{\Psi}F(u_n)\|_H^2\right\} \leq 0.\end{aligned}$$

Since $F \in (S_+)$, we have $u_n \to u$ with $u \in A$. By demicontinuity $F(u_n) \rightharpoonup F(u)$ implying a contradiction $F(u) = 0$. Hence we may conclude the existence of $\hat{\epsilon} > 0$ such that $F_\epsilon(u) \neq 0$ for all $u \in A$ and $0 < \epsilon < \hat{\epsilon}$.

For the second assertion we assume that $\epsilon_0 = \hat{\epsilon}$ for $A = \partial G$, $0 < \epsilon_1 < \epsilon_2 < \epsilon_0$ and consider the following LS-homotopy

$$tF_{\epsilon_1} + (1-t)F_{\epsilon_2} = I + \left(\frac{t}{\epsilon_1} + \frac{1-t}{\epsilon_2}\right)\Psi\hat{\Psi}F := F_{\epsilon_t},$$

where $\epsilon_t = (\frac{t}{\epsilon_1} + \frac{1-t}{\epsilon_2})^{-1}$, $0 \leq t \leq 1$. Since $\epsilon_1 \leq \epsilon_t \leq \epsilon_2 < \epsilon_0$ for all $t \in [0,1]$ we have $F_{\epsilon_t}(u) \neq 0$ for all $u \in \partial G$ and $t \in [0,1]$. Since the LS-degree satisfies (c), we obtain

$$d_{LS}(F_{\epsilon_1}, G, 0) = d_{LS}(F_{\epsilon_2}, G, 0)$$

completing the proof. ▮

In view of Lemma 3.1 it is relevant to define

$$d_{S_+}(F, G, 0) = d_{LS}(F_\epsilon, G, 0) \qquad \text{where } 0 < \epsilon < \epsilon_0. \tag{3.2}$$

Moreover, for any $y \in X^*$ with $y \notin F(\partial G)$ we can define

$$d_{S_+}(F, G, y) = d_{S_+}(F - y, G, 0). \tag{3.3}$$

To convince ourselves that we have obtained a classical topological degree function d_{S_+} for mappings in $\mathcal{F}_G(S_+)$ the conditions (a) to (d) have to be verified. It is obviously sufficient to deal with the case $y = 0$ or $y(t) \equiv 0$.

(a) If $0 \notin F(\overline{G})$, it follows from Lemma 3.1 that $0 \notin F_\epsilon(\overline{G})$ for all $0 < \epsilon < \hat{\epsilon}$. Hence $d_{LS}(F_\epsilon, G, 0) = 0$ for all $0 < \epsilon < \hat{\epsilon}$ implying $d_{S_+}(F, G, 0) = 0$. Therefore $d_{S_+}(F, G, 0) \neq 0$ implies $0 \in F(G)$.

(b) If G_1 and G_2 are open disjoint subsets of G and $0 \notin F(\overline{G} \setminus (G_1 \cup G_2))$, we can apply again Lemma 3.1 with $A = \overline{G} \setminus (G_1 \cup G_2)$ and use the property (b) for the LS-degree to derive (b) for d_{S_+}.

(c) If $F_t \in \mathcal{H}_G(S_+)$ we can extend Lemma 3.1 for homotopies in the obvious way. The property (c) follows then from the corresponding property for LS-homotopies.

(d) To show that $\mathcal{J}$ plays the role of normalizing map we consider the affine LS-homotopy $(1-t)I + t\mathcal{J}_\epsilon$. Since $\mathcal{J}(u) = 0$ if and only if $u = 0$, and since

$$\langle \mathcal{J}(u), (1-t)u + t\mathcal{J}_\epsilon(u)\rangle = \|u\|^2 + \frac{t}{\epsilon}\|\hat{\Psi}\mathcal{J}(u)\|_H^2 > 0$$

for all $u \neq 0$ and $0 \leq t \leq 1$, we obtain

$$d_{S_+}(\mathcal{J}, G, 0) = \lim_{\epsilon \to 0+} d_{LS}(\mathcal{J}_\epsilon, G, 0) = d_{LS}(I, G, 0) = 1$$

whenever $0 \in \mathcal{J}(G)$.

Using the fact that LS-degree is unique it is not hard to show that also S_+-degree is unique (see [**Be**]). Thus we can conclude

THEOREM 3.1. *Let X be a separable reflexive Banach space, G an open bounded subset in X and $\mathcal{F}_G(S_+)$ the class of admissible mappings. Then there exists exactly one degree function d_{S_+}, the S_+-degree, satisfying the properties* (a) *to* (d) *with respect to $\mathcal{H}_G(S_+)$ and normalizing map $\mathcal{J}$.*

REMARK 3.1: Originally Browder [**Bro 4**] constructed the S_+-degree by using Galerkin approximations, for which in each finite dimensional subspace the Brouwer degree is defined. His approach works also when X is not separable.

REMARK 3.2: The S_+-degree can be constructed to all demicontinuous mappings of class (S_+), not only for bounded ones as above. It is essential that a demicontinuous (S_+)-mapping $F: X \to X^*$ is proper on bounded sets, i.e., for any bounded closed set $A \subset X$ and for any compact set $K \subset X^*$, $F^{-1}(K) \cap A$ is compact. For the detailed discussion in this direction we refer to [**Be**].

REMARK 3.3: The S_+-degree can be extended for quasimonotone mappings, i.e., to the class $\mathcal{F}_G(QM)$ by using the fact that $F + \epsilon\mathcal{J} \in (S_+)$ whenever $F \in (QM)$ and $\epsilon > 0$. However, we face here the difficulty that the image $F(A)$ of a closed subset $A \subset \overline{G}$ is no more closed. Therefore the QM-degree obtained through approximations

$$d_{QM}(F,G,y) = \lim_{\epsilon \to 0+} d_{S_+}(F + \epsilon\mathcal{J}, G, y) \tag{3.4}$$

is not a classical degree in the sense of Definition 3.1. For instance we have:

(a)′ If $d_{QM}(F,G,y) \neq 0$, then $y \in \overline{F(G)}$.

In fact, for the definition (3.4) we assume $y \notin \overline{F(\partial G)}$. For more details on weak degree theories we refer to [**Bro 4,5**], [**Be**].

REMARK 3.4: Since $(S_+) \subset (PM) \subset (QM)$, the QM-degree is defined for all mappings $F \in \mathcal{F}_G(PM)$. We shall see from applications that many results obtained for (S_+)-mappings hold true also for pseudomonotone mappings. This is based on the fact that for each $F \in \mathcal{F}_G(PM)$ the set $F(A)$ is closed whenever $A \subset \overline{G}$ is weakly closed. Indeed, if $\{w_n\} \subset F(A)$ with $w_n \to w$, then $w_n = F(u_n)$ for some $\{u_n\} \subset A$. Since G is bounded, $u_n \rightharpoonup u$ for some $u \in A$, at least for a subsequence. Thus

$$\limsup \langle F(u_n), u_n - u\rangle = 0$$

implying $F(u_n) \rightharpoonup F(u)$ and hence $w = F(u) \in F(A)$. In particular, if we assume that G is convex, then $\overline{G}$ is weakly closed implying that $F(\overline{G})$ is closed. Consequently, for $F \in \mathcal{F}_G(PM)$ and G convex, we can conclude instead of (a)′, that

$$d_{QM}(F,G,y) \neq 0 \quad \text{implies} \quad y \in F(\overline{G}).$$

4. SOME APPLICATIONS OF CONTINUATION METHOD

We shall describe some standard results obtained by a continuation method (homotopy argument) when a classical degree theory is available. We use here the S_+-degree but analogous results hold for other degree functions which will be introduced in our further discussion.

Let X be a real reflexive Banach space, G an open bounded subset in X and $F \in \mathcal{F}_G(S_+)$. If $y \in X^* \setminus F(\partial G)$, a sufficient condition for the solvability of the equation

$$F(u) = y$$

is that $d_{S_+}(F,G,y) \neq 0$. In many cases this can be shown by using the homotopy argument (c) for a suitable homotopy involving F and some *reference map* $T \in \mathcal{F}_G(S_+)$, i.e., an injection satisfying $d(T,G,w) \neq 0$ whenever $w \in T(G)$. Then the property (c) for the (S_+)-degree yields

THEOREM 4.1. *Let G be an open bounded subset in X, $T \in \mathcal{F}_G(S_+)$ a reference map and $F \in \mathcal{F}_G(S_+)$. If for a given $y \in X^*$ there exists $w \in T(G)$ such that*

$$tF(u) + (1-t)T(u) \neq ty + (1-t)w \qquad \text{for all } u \in \partial G,\ 0 \leq t \leq 1, \tag{4.1}$$

then $d_{S_+}(F, G, y) \neq 0$ and the equation $F(u) = y$ admits a solution u in G.

The obvious reference map for the (S_+)-degree is the duality map $\mathcal{J}$. Bearing in mind the properties of $\mathcal{J}$ we can rewrite condition (4.1) to obtain from Theorem 4.1 the following

THEOREM 4.2. *Let G be an open bounded subset in X and $F \in \mathcal{F}_G(S_+)$. If there exists $\bar{u} \in G$ such that*

$$\langle F(u) - y, u - \bar{u}\rangle > \|F(u) - y\| \|u - \bar{u}\| \qquad \text{for all } u \in \partial G, \tag{4.2}$$

then $d_{S_+}(F, G, y) = 1$ and the equation $F(u) = y$ admits a solution u in G.

For mappings F which satisfy some coercivity conditions we can derive surjectivity results

THEOREM 4.3. *Let $F \in \mathcal{F}_X(S_+)$ satisfy the conditions*

(i) *if $y \in X^*$ and $F(u_n) \to y$ in X^*, then $\{u_n\}$ is bounded in X*

(ii) *there exists $R > 0$ such that*

$$\frac{\langle F(u), u\rangle}{\|u\|} + \|F(u)\| > 0 \quad \text{for all } u \in X \text{ with } \|u\| \geq R.$$

Then $F(X) = X^*$.

PROOF: Let $y \in X^*$ be given. Then there exist $k > 0$ and $R' \geq R$ such that

$$\|F(u) - ty\| \geq k \qquad \text{for all } t \in [0,1] \text{ and } \|u\| \geq R'. \tag{4.3}$$

Indeed, if we assume the contrary, we find sequences $\{u_n\}$ in X and $\{t_n\}$ in $[0,1]$ such that $\|u_n\| \to \infty$, $\|F(u_n) - t_n y\| \to 0$ and $t_n \to t$. Hence $F(u_n) \to ty$ and by (i) we get a contradiction. From (4.3) we then obtain by the property (c)

$$d_{S_+}(F, B_{R'}, y) = d_{S_+}(F, B_{R'}, 0), \tag{4.4}$$

where $B_{R'} = \{u \in X \mid \|u\| < R'\}$. In view of (ii) we have $\langle F(u), u\rangle > -\|F(u)\|\|u\|$ for all $\|u\| = R'$. By Theorem 4.2 we get $d_{S_+}(F, B_{R'}, 0) = 1$ implying by (4.4) that $y \in F(B_{R'})$. ∎

REMARK 4.1: It is useful to observe that (i) is met if

$$\|F(u_n)\| \to \infty \qquad \text{whenever } \|u_n\| \to \infty \text{ in } X. \tag{i$_1$}$$

Moreover, if F satisfies the condition

$$\frac{\langle F(u), u\rangle}{\|u\|} + \|F(u)\| \to \infty \quad \text{as } \|u\| \to \infty, \tag{i$_2$}$$

then clearly both (i) and (ii) are met. Finally, the strong coercivity condition

$$\frac{\langle F(u), u\rangle}{\|u\|} \to \infty \quad \text{as } \|u\| \to \infty \tag{i$_3$}$$

implies (i)$_2$.

In view of the Remarks 3.3 and 3.4 we can expect that Theorems 4.1 to 4.3 have generalizations for mappings F in $\mathcal{F}_G(QM)$ and $\mathcal{F}_G(PM)$. Indeed, the following modifications are easily obtained (see [**BM 1**], [**Be**]).

THEOREM 4.4. *Let G be an open bounded subset in X and $F \in \mathcal{F}_G(QM)$. If there exists $\bar{u} \in G$ such that the condition* (4.2) *holds, then the equation $F(u) = y$ is almost solvable in the sense that $y \in \overline{F(\overline{G})}$. In particular, if G is convex and $F \in \mathcal{F}_G(PM)$, then $y \in F(\overline{G})$, i.e., the equation $F(u) = y$ admits a solution u in $\overline{G}$.*

THEOREM 4.5. *Let $F \in \mathcal{F}_X(QM)$ satisfy the conditions* (i) *and* (ii), *then $\overline{F(X)} = X^*$. In particular, if $F \in \mathcal{F}_X(PM)$, then $F(X) = X^*$.*

REMARK 4.2: For monotone demicontinuous mappings $F\colon X \to X^*$ one can show that $F(X) = X^*$ if and only if the condition (i) is satisfied.

For odd mappings we can obtain generalizations of Borsuk's theorem. If G is an open bounded symmetric set in X containing the origin and $F \in \mathcal{F}_G(S_+)$ with $F(-u) = -F(u)$ for all $u \in \partial G$, then also F_ϵ given by (3.1) is odd on ∂G for any $\epsilon > 0$. Hence Borsuk's theorem for LS-mappings ([**De**]) implies the existence of $u_\epsilon \in \overline{G}$ such that

$$F_\epsilon(u_\epsilon) = 0 \qquad \text{for any } \epsilon > 0.$$

Moreover, if $0 \notin F_\epsilon(\partial G)$, then $d_{LS}(F_\epsilon, G, 0)$ is odd. As in the proof of Lemma 3.1 we conclude $0 \in \overline{F(\overline{G})}$. For the case $0 \notin F(\partial G)$ we have also that $d_{S_+}(F, G, 0)$ is odd. If we assume that $F \in \mathcal{F}_G(QM)$ and F is odd on ∂G, we can consider its S_+-approximation $F + \epsilon\mathcal{J}$, which is also odd on ∂G for every $\epsilon > 0$. Thus Borsuk's theorem extends to the class $\mathcal{F}_G(QM)$ in an obvious way. In particular, we have the following standard surjectivity theorem, which can be proved as Theorem 4.3.

THEOREM 4.6. *Let $F \in \mathcal{F}_X(QM)$ satisfy the condition* (i) *and the condition*

(iii) *there exists $R > 0$ such that $F(-u) = -F(u)$ for all $\|u\| \geq R$.*

Then $\overline{F(X)} = X^*$. In particular, if $F \in \mathcal{F}_X(PM)$, then $F(X) = X^*$.

5. Applications to nonlinear elliptic problems

Since the contributions by Minty, Browder, Višik, Brezis, Leray and Lions in the sixties the theory of mappings of monotone type has become a standard frame work (see [**PS**], [**FK**]) for the study of boundary value problems for nonlinear elliptic partial differential operators in divergence form

$$Au(x) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, u(x), Du(x), \dots, D^m u(x)), \qquad x \in \Omega, \tag{5.1}$$

where Ω is an open subset in $\mathbf{R}^N$ ($N \geq 2$) and $m \geq 1$. The coefficients A_α are functions of the point $x \in \Omega$ and of $\xi = (\eta, \zeta) \in \mathbf{R}^{N_0}$ with $\eta = \{\eta_\beta \mid |\beta| \leq m-1\} \in \mathbf{R}^{N_1}$, $\zeta = \{\zeta_\beta \mid |\beta| = m\} \in \mathbf{R}^{N_2}$ and $N_1 + N_2 = N_0$. We assume that each $A_\alpha(x, \xi)$ is a *Carathéodory function*, i.e., measurable in x for fixed $\xi = (\eta, \zeta) \in \mathbf{R}^{N_0}$ and continuous in ξ for allmost all $x \in \Omega$. We shall assume here, for simplicity, that Ω is a bounded subset of $\mathbf{R}^N$. Then the familiar growth condition

($\mathbf{A}_1$) There exist $p > 1$, $c_1 > 0$ and $k_1 \in L^{p'}(\Omega)$, $p' = \frac{p}{p-1}$, such that

$$|A_\alpha(x, \xi)| \leq c_1(|\zeta|^{p-1} + |\eta|^{p-1} + k_1(x))$$

for all $x \in \Omega$, $\xi = (\eta, \zeta) \in \mathbf{R}^{N_0}$, $|\alpha| \leq m$, implies that the operator (5.1) gives rise to a bounded continuous mapping T from any Sobolev space X with $W_0^{m,p}(\Omega) \subset X \subset W^{m,p}(\Omega)$ into its dual space X^* by the rule

$$(5.2) \qquad \langle T(u), v\rangle = \int_\Omega \sum_{|\alpha|\le m} A_\alpha(x, \xi(u))D^\alpha v, \qquad u, v \in X.$$

In the sequel we shall consider the case $X = W_0^{m,p}(\Omega)$. Then for a given $h \in L^{p'}(\Omega) \subset X^*$ the solutions $u \in X$ of the equation

$$(5.3) \qquad T(u) = h$$

are called *weak solutions* of the Dirichlet boundary value problem

$$(5.4) \qquad \begin{cases} Au(x) = h(x) & \text{in } \Omega \\ D^\alpha u(x) = 0 & \text{on } \partial\Omega \text{ for all } |\alpha| \le m-1. \end{cases}$$

Hence we are in a position to apply the results of Section 4 to (5.4) as soon as T belongs to one of the classes of mappings of monotone type. A condition which obviously ensures that T is monotone is

$$(\mathrm{A}_2)_\mathrm{M} \qquad \sum_{|\alpha|\le m} \{A_\alpha(x,\xi) - A_\alpha(x,\xi^*)\}(\xi_\alpha - \xi_\alpha^*) \ge 0 \qquad \text{for all } x \in \Omega,\ \xi, \xi^* \in \mathbf{R}^{N_0}$$

A condition implying, that T is pseudomonotone, is the classical Leray-Lions condition **[LL]** (cf. **[LM]**)

$$(\mathrm{A}_2)_\mathrm{S} \qquad \sum_{|\alpha|=m} \{A_\alpha(x,\eta,\zeta) - A_\alpha(x,\eta,\zeta^*)\}(\zeta_\alpha - \zeta_\alpha^*) > 0 \qquad \text{for all } x \in \Omega,\ \eta \in \mathbf{R}^{N_1} \text{ and } \zeta \ne \zeta^* \text{ in } \mathbf{R}^{N_2}$$

A remarkable feature on the condition $(\mathrm{A}_2)_\mathrm{S}$ is that monotonicity is assigned only to the top order part

$$(5.5) \qquad A_1 u(x) = \sum_{|\alpha|=m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, u(x), Du(x), \ldots, D^m u(x))$$

and the lower order part

$$(5.6) \qquad A_2 u(x) = \sum_{|\alpha|\le m-1} (-1)^{|\alpha|} D^\alpha A_\alpha(x, u(x), Du(x), \ldots, D^m u(x))$$

is to obey the growth condition (A_1), only. It is interesting to observe that if we define the corresponding mappings T_1 and $T_2 \colon X \to X^*$ by

$$\langle T_1(u), v\rangle = \int_\Omega \sum_{|\alpha|=m} A_\alpha(x, \eta(u), \zeta(u))D^\alpha v, \qquad u, v \in X$$

and

$$\langle T_2(u), v\rangle = \int_\Omega \sum_{|\alpha|\le m-1} A_\alpha(x, \eta(u), \zeta(u))D^\alpha v, \qquad u, v \in X,$$

then of course (A_1) and $(\mathrm{A}_2)_\mathrm{S}$ imply $T_1 \in (PM)$ and (A_1) implies that $T_2 \in (QM)$. Actually, by the Sobolev embedding theorem $\lim\langle T_2(u_n), u_n - u\rangle = 0$ for any sequence $\{u_n\} \subset X$ with

$u_n \rightharpoonup u$. Moreover, T_1 remains in the class (PM) although we weaken the strict monotonicity condition $(\mathrm{A}_2)_\mathrm{S}$ to

$$(\mathrm{A}_2) \qquad \sum_{|\alpha|=m} \{A_\alpha(x,\eta,\zeta) - A_\alpha(x,\eta,\zeta^*)\}(\zeta_\alpha - \zeta_\alpha^*) \geq 0.$$

If we assume only the conditions (A_1) and (A_2), then $T = T_1 + T_2$ is in the class (QM) but no more necessarily in the class (PM), unless T_2 is weakly continuous which is related to linear behaviour of A_2 with respect to highest order derivatives. For a more detailed discussion with some further refinements we refer to **[GM]**.

In order to apply the existence and surjectivity results of Section 4 to the problem (5.4) a further condition is needed. If the operator A satisfies the condition

(A_3) There exists $c_2 > 0$ and $k_2 \in L^1(\Omega)$ such that

$$\sum_{|\alpha|\leq m} A_\alpha(x,\xi)\xi_\alpha \geq c_2|\zeta|^p - k_2(x)$$

for all $x \in \Omega$ and $\xi = (\eta,\zeta) \in \mathbf{R}^{N_0}$.

then T satisfies the strong coercivity condition $(\mathrm{i})_3$, since by the Poincare inequality in $X = W_0^{m,p}(\Omega)$,

$$\frac{\langle T(u),u\rangle}{\|u\|} \geq c_2 \frac{\|D^m u\|^p_{L^p(\Omega)}}{\|u\|} + \frac{\|k_2\|_{L^1(\Omega)}}{\|u\|} \to \infty$$

as $\|u\| \to \infty$. Consequently, we get from Theorem 4.3 and 4.5 the following

COROLLARY 5.1. *Let Ω be a bounded open subset in $\mathbf{R}^N$ and assume that the operator A satisfies the conditions (A_1) and (A_3).*

(a) *If A satisfies one of the conditions $(\mathrm{A}_2)_\mathrm{M}$ or $(\mathrm{A}_2)_\mathrm{S}$, then T is surjective and the problem (5.4) admits a weak solution u in $W_0^{m,p}(\Omega)$ for any given $h \in L^{p'}(\Omega)$.*

(b) *If A satisfies (A_2), then $T(W_0^{m,p}(\Omega))$ is dense in $W^{-m,p'}(\Omega)$ and the problem (5.4) is almost solvable for any given $h \in L^{p'}(\Omega)$.*

(c) *If A satisfies (A_2) and T_2 is weakly continuous, then T is surjective and the problem (5.4) admits a weak solution u in $W_0^{m,p}(\Omega)$ for any given $h \in L^{p'}(\Omega)$.*

REMARK 5.1: It is not difficult to see that in fact the conditions (A_1), $(\mathrm{A}_2)_\mathrm{S}$ and (A_3) imply that T belongs to the class (S_+). If we assume that the problem (5.4) has variational structure, i.e., there exists a real valued functional $f\colon X \to \mathbf{R}$ such that $f' = T$, and if T happens to be quasimonotone and $f(u) \to \infty$ as $\|u\| \to \infty$, then the result by Hess **[He]** tells us that f is weakly lower semicontinuous and has a minimum. This means that for such operators the assertion in the case (b) will be the same as in the cases (a) and (c).

6. THE DEGREE FOR A CLASS OF MAPPINGS IN A HILBERT SPACE

In this section we shall consider mappings of monotone type acting in a real separable Hilbert space H. The definitions of Section 2 are to be understood with respect to the inner product $(\cdot,\cdot)$ of H. Hence (LS) is a subclass of (S_+). We assume that $H = M \oplus M^\perp$ where both M and $M^\perp$ are infinite-dimensional. Let Q and P denote the orthogonal projections to M and $M^\perp$, respectively. If G is an open bounded subset of H, we denote

$$\mathcal{F}_G(LS;S_+) = \{F\colon \overline{G} \to H \mid F = Qg + Pf \text{ for some bounded demicontinuous } g \in (LS) \text{ and } f \in (S_+)\},$$

the class of admissible mappings and

$$\mathcal{H}_G(LS;S_+) = \{F_t \mid 0 \le t \le 1,\ F_t = Qg_t + Pf_t\},$$

the class of admissible homotopies, where g_t is a LS-homotopy and f_t a S_+-homotopy. For each $F \in \mathcal{F}_G(LS;S_+)$ we will find LS-approximations by a partial Galerkin procedure. Indeed, let $\{N_n\}$ be a sequence of finite-dimensional subspaces of $M^\perp$ such that $N_n \subset N_{n+1}$ for each n and $\cup N_n$ is dense in $M^\perp$. Denoting by P_n the orthogonal projection to N_n we associate with $F = Q(I+C) + Pf$ a sequence $\{F_n\}$ where

$$F_n = I + QC + P_n f - P_n, \qquad n \in \mathbb{N}. \tag{6.1}$$

Since $QC + P_n f - P_n$ is compact, $F_n \in (LS)$ for each $n \in \mathbb{N}$. Similarly, each $F_t \in \mathcal{H}_G(LS;S_+)$ admits an approximation

$$(F_t)_n = I + QC_t + P_n f_t - P_n, \qquad n \in \mathbb{N}. \tag{6.2}$$

For each $y \in H$ we also need the approximation

$$y_n = Qy + P_n y, \qquad n \in \mathbb{N}. \tag{6.3}$$

Then we have

LEMMA 6.1. *Let $F \in \mathcal{F}_G(LS;S_+)$ and $y \in H$ with $y \notin F(\partial G)$. Then there exists $n_0 \in \mathbb{N}$ such that $y_n \notin F_n(\partial G)$ for all $n \ge n_0$. Moreover, there exists $n_1 > n_0$ such that $d_{LS}(F_n, G, y_n)$ is constant for all $n \ge n_1$*

PROOF: Since $(F-y)_n = F_n - y_n$, we may assume that $y = 0$. If the first assertion were false, then there would exist sequences $\{u_k\} \subset \partial G$ and $\{n_k\} \subset \mathbb{N}$ such that $F_{n_k}(u_k) = 0$ and $n_k \to \infty$. Separating the components in M and $M^\perp$ we have

$$Qu_k + QC(u_k) = 0 \quad \text{and} \quad Pu_k + P_{n_k} f(u_k) - P_{n_k} u_k = 0. \tag{6.4}$$

Taking a subsequence we have $u_k \rightharpoonup u$, $C(u_k) \to z$ and $f(u_k) \rightharpoonup w$ in H. Hence $Qu_k \to Qu$ and $Pu_k = P_{n_k} u_k$ for all k implying that $P_{n_k} f(u_k) = 0$ for all k. By (6.4) we obtain $P_{n_k} f(u_k) \rightharpoonup Pw = 0$. Therefore $(f(u_k), Pu) \to 0$ and

$$\begin{aligned}
\limsup(f(u_k), u_k - u) &= \limsup(f(u_k), Pu_k - Pu + Qu_k - Qu) \\
&= \limsup(f(u_k), P_{n_k} u_k) \\
&= \limsup(P_{n_k} f(u_k), u_k) \\
&= 0.
\end{aligned}$$

Since $f \in (S_+)$, $u_k \to u$ with $u \in \partial G$ and $F_{n_k}(u_k) \rightharpoonup F(u) = 0$, a contradiction. Hence there exists $n_0 \in \mathbb{N}$ such that the LS-degree $d_{LS}(F_n, G, 0)$ is defined for all $n \ge n_0$.

Assume that the second assertion is false. Then there exists a sequence $\{n_k\} \subset \mathbb{N}$, $n_k \ge n_0$, $n_k \to \infty$ such that

$$d_{LS}(F_{n_k}, G, 0) \ne d_{LS}(F_{n_{k+1}}, G, 0)$$

for all $k = 1, 2, \ldots$. In view of the property (c) for the LS-degree we find sequences $\{u_k\} \subset \partial G$ and $\{t_k\} \subset (0,1)$ such that

$$(1 - t_k) F_{n_k}(u_k) + t_k F_{n_{k+1}}(u_k) = 0. \tag{6.5}$$

Bearing in mind that $F = Q(I+C) + Pf$ we get from **(6.5)**

$$u_k + QC(u_k) + (1-t_k)P_{n_k}(f(u_k) - u_k) + t_k P_{n_{k+1}}(f(u_k) - u_k) = 0.$$

Separating the components in M and $M^\perp$ we have

$$(6.6) \qquad \begin{cases} Qu_k + QC(u_k) = 0 \\ Pu_k + (1-t_k)P_{n_k}(f(u_k) - u_k) + t_k P_{n_{k+1}}(f(u_k) - u_k) = 0. \end{cases}$$

We can assume again that $u_k \rightharpoonup u$, $C(u_k) \to z$ and $f(u_k) \rightharpoonup w$ in H. As in the former proof we get $Qu_k \to Qu$, $Pu_k = P_{n_{k+1}} u_k$, $P_{n_k} f(u_k) = 0$ and $Pw = 0$. Moreover (6.6) also implies that

$$P_{n_{k+1}} f(u_k) = -\frac{(1-t_k)}{t_k}(P_{n_{k+1}} u_k - P_{n_k} u_k).$$

Hence

$$\begin{aligned} \limsup(f(u_k), u_k - u) &= \limsup(f(u_k), P_{n_{k+1}} u_k) \\ &= \limsup \left\{ -\frac{(1-t_k)}{t_k} \|P_{n_{k+1}} u_k - P_{n_k} u_k\|^2 \right\} \le 0. \end{aligned}$$

By the (S_+)-property of f we obtain $u_k \to u$ with $u \in \partial G$ and $F(u) = 0$, a contradiction. ∎

REMARK 6.1: The first part of Lemma 6.1 extends for admissible homotopies $F_t \in \mathcal{H}_G(LS; S_+)$ and continuous curves $\{y(t) \mid 0 \le t \le 1\}$ in H.

By virtue of Lemma 6.1 it is relevant to define

$$(6.7) \qquad d_H(F, G, y) = \lim d_{LS}(F_n, G, y_n)$$

for any given $F \in \mathcal{F}_G(LS; S_+)$ and $y \in H$ with $y \notin F(\partial G)$. It is not hard to verify that the function d_H defined by (6.7) satisfies the properties (a) to (d) of the classical topological degree with respect to $\mathcal{H}_G(LS; S_+)$ and the identity map I as the normalizing map. We indicate here how to verify (c), for example. Let $F_t \in \mathcal{H}_G(LS; S_+)$ and let $\{y(t) \mid 0 \le t \le 1\}$ be a continuous curve in H such that $y(t) \notin F_t(\partial G)$ for all $t \in [0,1]$. To show that $d_H(F_t, G, y(t))$ is constant for all $t \in [0,1]$ we first use the extension of Lemma 6.1 as mentioned in Remark 6.1. Hence there exists $n_0 \in \mathbb{N}$ such that $(y(t))_n \notin (F_t)_n(\partial G)$ for all $t \in [0,1]$ and $n \ge n_0$. For any pair $t_1, t_2 \in [0,1]$ we can then apply the definition 6.7 to find $k \ge n_0$ such that

$$d_H(F_{t_i}, G, y(t_i)) = d_{LS}((F_{t_i})_k, G, (y(t_i))_k)$$

for each $i = 1, 2$. Using the property (c) for the LS-degree the assertion follows for d_H.

For (d) it is sufficient to note that $I_n = I$ for all $n \in \mathbb{N}$ and $y_n \in G$ for all large n whenever $y \in G$. Therefore

$$d_H(I, G, y) = \lim d_{LS}(I, G, y) = 1$$

for all $y \in G$.

Finally the uniqueness of the degree d_H can also be shown (see [**BM 2**]) and hence we have

THEOREM 6.1. *Let H be a real separable Hilbert space, G a bounded open subset in H and $\mathcal{F}_G(LS; S_+)$ the class of admissible mappings. Then there exists exactly one degree function d_H satisfying the properties* (a) *to* (d) *with respect to $\mathcal{H}_G(LS; S_+)$ and normalizing map I.*

REMARK 6.2: A weak degree theory can be established here (cf. Remark 3.3) for mappings $\mathcal{F}_G(LS; QM)$ by using the approximations $f_\epsilon = f + \epsilon I$ for $f \in (QM)$.

It is now obvious from Section 4 how the existence and surjectivity results can be derived for admissible mappings F.

THEOREM 6.2. *Let G be an open bounded subset in H, $T \in \mathcal{F}_G(LS;S_+)$ a reference map and either $F \in \mathcal{F}_G(LS;S_+)$ or $F \in \mathcal{F}_G(LS;PM)$, G convex,and $F = Q(I + C_1 + C_2 f) + Pf$ with C_1, C_2 compact linear operators.*

If for a given $y \in H$ there exists $w \in T(G)$ such that

$$tF(u) + (1-t)T(u) \neq (1-t)w + ty \qquad \text{for all } u \in \partial G,\ t \in [0,1], \tag{6.8}$$

then the equation $F(u) = y$ admits a solution u in G.

PROOF: Since the case $F \in \mathcal{F}_G(LS;S_+)$ is obvious by Theorem 4.1, we consider the latter case. It is sufficient to show that $F(\overline{G})$ is closed. In order to show this, let $\{w_n\}$ be a sequence in $F(\overline{G})$ with $w_n \to w$. Then $w_n = F(u_n)$ for some $\{u_n\} \subset \overline{G}$ and since G is bounded and convex, $u_n \rightharpoonup u$ with $u \in \overline{G}$ for a subsequence. Using the representation of F we have

$$Qw_n = Qu_n + QC_1(u_n) + QC_2(f(u_n)) \quad \text{and} \quad Pw_n = Pf(u_n).$$

Since f is bounded and C_1, C_2 are compact linear operators, we can choose a further subsequence of $\{u_n\}$ such that $Qu_n \to Qu$ and $Pf(u_n) \to Pw$. Therefore

$$\begin{aligned}\limsup(f(u_n), u_n - u) &= \limsup\{(Pf(u_n), u_n - u) + (f(u_n), Qu_n - Qu)\}\\ &= 0.\end{aligned}$$

Using the fact that $f \in (PM)$ we have $f(u_n) \rightharpoonup f(u)$ implying $F(u_n) = Qu_n + QC_1(u_n) + QC_2(f(u_n)) + Pf(u_n) \rightharpoonup F(u)$. Hence $w = F(u)$ with $u \in \overline{G}$ completing the proof. ∎

We are now in a position to derive surjectivity results from Theorem 6.2 along the lines of Theorems 4.3 to 4.6. For our applications we shall need the following

THEOREM 6.3. *Let $F \in \mathcal{F}_H(LS;S_+)$ or $F \in \mathcal{F}_H(LS;PM)$ with a representation $F = Q(I + C_1 + C_2 f) + Pf$ where C_1, C_2 are compact linear operators. Assume that F satisfies the condition*

(i) *if $y \in H$ and $F(u_n) \to y$, then $\{u_n\}$ is bounded,*

and one of the conditions

(ii) *there exists $R > 0$ such that $\dfrac{(F(u),u)}{\|u\|} + \|F(u)\| > 0$ for all $\|u\| \geq R$,*

(iii) *there exists $R > 0$ such that $F(-u) = -F(u)$ for all $\|u\| \geq R$.*

Then $F(H) = H$.

7. APPLICATIONS TO SEMILINEAR WAVE EQUATIONS

We consider semilinear wave equation of the form

$$\begin{cases} u_{tt} - u_{xx} - g(x,t,u) = h & \text{in } (0,\pi) \times \mathbf{R} \\ u(0,\cdot) = u(\pi,\cdot) = 0 \\ u(\cdot, t + 2\pi) = u(\cdot, t), & t \in \mathbf{R} \end{cases} \tag{7.1}$$

where the function $(x,t,s) \to g(x,t,s)$ from $\Omega \times \mathbf{R}$ to $\mathbf{R}$ is measurable in $w = (x,t) \in \Omega = (0,\pi) \times (0,2\pi)$ for each $s \in \mathbf{R}$ and continuous in s for almost all $w \in \Omega$ and h is a given function in $L^2(\Omega)$. Moreover we assume that there are positive constants α and β that

$$\begin{cases} \alpha \leq \dfrac{g(\cdot,\cdot,s)}{s} \leq \beta & \text{for all } s \neq 0 \\ g(x,\cdot,s) \text{ is } 2\pi\text{-periodic and } g(\cdot,\cdot,0) = 0 \\ g(x,t,\cdot) \text{ is nondecreasing.} \end{cases} \tag{7.2}$$

Then it is clear that g gives rise to a Nemytskii operator N in $L^2(\Omega)$ by

$$N(u) = g(\cdot, \cdot, u), \quad u \in L^2(\Omega).$$

Moreover, N is continuous bounded and monotone. We denote in the sequel the Hilbert space $L^2(\Omega)$ by H. Let C^2 stand for twice continuously differentiable functions $v\colon [0,\pi] \times \mathbf{R} \to \mathbf{R}$ such that $v(0,\cdot) = v(\pi,\cdot) = 0$ and $v(x,t)$ is 2π-periodic in t. Then a function $u \in H$ is said to be ***a weak solution*** of (7.1) if

$$(u, v_{tt} - v_{xx}) - (N(u), v) = (h, v) \qquad \text{for all } v \in C^2.$$

Define

$$\phi_{lm} = \begin{cases} \frac{\sqrt{2}}{\pi} \sin(lx)\sin(mt), & l, m \in \mathbf{Z}_+ \\ \frac{1}{\pi}\sin(lx), & l \in \mathbf{Z}_+,\ m = 0 \\ \frac{\sqrt{2}}{\pi}\sin(lx)\cos(mt), & l, -m \in \mathbf{Z}_+. \end{cases}$$

Then the set $\{\phi_{lm} \mid l \in \mathbf{Z}_+,\ m \in \mathbf{Z}\}$ forms an orthonormal basis in H and each $u \in H$ has a representation

$$u = \sum_{l=1}^{\infty} \sum_{m=-\infty}^{+\infty} a_{lm}\phi_{lm}.$$

The wave operator has an abstract realization in H defined by

$$Lu = \sum_{l=1}^{\infty} \sum_{m=-\infty}^{+\infty} (l^2 - m^2) a_{lm}\phi_{lm}, \tag{7.3}$$

from $D(L) = \{u \in H \mid \sum_{l,m} |(l^2 - m^2)|^2 |a_{lm}|^2 < \infty\}$ to H. It can be shown that $u \in H$ is a weak solution of (7.1) if and only if

$$Lu - N(u) = h \qquad \text{with } u \in D(L). \tag{7.4}$$

The operator L is linear, densely defined, self adjoint and closed with closed range. In particular $H = \operatorname{Im} L \oplus \operatorname{Ker} L$ and L has a pure point spectrum of eigenvalues

$$\sigma(L) = \{\lambda_{lm} = l^2 - m^2 \mid l \in \mathbf{Z}_+,\ m \in \mathbf{Z}\}$$

with corresponding eigenvectors ϕ_{lm}. Here $\operatorname{Ker} L$ is infinite-dimensional, but all non-zero eigenvalues have finite multiplicity. If we denote by L_0 the restriction of L into $\operatorname{Im} L \cap D(L)$, then its inverse $L_0^{-1}\colon \operatorname{Im} L \to \operatorname{Im} L \cap D(L)$ is well-defined linear compact operator. To see the connection between the equation (7.4) and the class $\mathcal{F}_G(LS; S_+)$ we consider the mapping

$$F = Q(I - L_0^{-1}QN) + PN, \tag{7.5}$$

where Q and P are the orthogonal projections to $\operatorname{Im} L$ and $\operatorname{Ker} L$, respectively. If we assume that N is strongly monotone, i.e.,

$$\frac{g(\cdot,\cdot,s) - g(\cdot,\cdot,t)}{s - t} \geq \mu, \qquad s \neq t$$

for some constant $\mu > 0$, then obviously $F \in \mathcal{F}_G(LS; S_+)$. Otherwise we only have $F \in \mathcal{F}_G(LS; PM)$, where F admits a representation $F = Q(I + C_2 N) + P$ with $C_2 = -L_0^{-1}Q$ a compact linear operator as required in Theorems 6.2 and 6.3. Hence the results of Section 6 are available for the equation (7.4) as soon as we observe that $(L-P)(L_0^{-1}Q-P) = Q+P = I$ implying the equivalence of the equations

$$(7.6) \qquad Lu - N(u) = h \qquad \text{with } u \in D(L) \cap \overline{G}$$

$$(7.7) \qquad F(u) = y \qquad \text{with } u \in \overline{G},\ y = (L_0^{-1}Q - P)h,$$

where F is given by (7.5).

The obvious reference map in the class $\mathcal{F}_G(LS; S_+)$ is the identity map I corresponding to $L-P$ in the setting of (7.6). If we assume that G is an open bounded (convex in the case $N \in (PM)$) subset in H with $0 \in G$, the condition (6.8) of Theorem 6.2 with $w = 0$ and $T = I$ becomes

$$(7.8) \qquad Lu - tN(u) - (1-t)Pu \neq th \qquad \text{for all } u \in \partial G,\ 0 \le t \le 1.$$

We get the following well known result ([**BN**], [**Ma**])

COROLLARY 7.1. *If g satisfies* (7.2) *with $0 < \alpha < \beta < 1$, then the equation* (7.1) *admits a weak solution u in $D(L)$ for any given $h \in H$.*

PROOF: Let $h \in H$ be given. It suffices to show that there exists R such that (7.8) holds for $\|u\| = R$, $0 \le t \le 1$. We argue by contradiction. Assume there exist sequences $\{u_n\} \subset D(L)$, $\{t_n\} \subset [0,1]$ with $\|u_n\| \to \infty$ and $t_n \to t$ such that

$$(7.9) \qquad Lu_n - t_nN(u_n) - (1-t_n)Pu_n = t_nh, \qquad n \in \mathbb{N}.$$

Hence $\|Lu_n\| = t_n\|Qh + QN(u_n)\| \le t_n(\|h\| + \|N(u_n)\|)$. By (7.3) we have $(Lu_n, u_n) \le \|Lu_n\|^2$ and by (7.2) we get $\frac{1}{\beta}\|N(u_n)\|^2 \le (N(u_n), u_n)$ and $\|u_n\| \le \frac{1}{\alpha}\|N(u_n)\|$. Using also the fact that $0 \le t_n^2 \le t_n \le 1$ we obtain from (7.9)

$$\begin{aligned} t_n^2\|N(u_n)\|^2 + 2t_n\|h\|\|N(u_n)\| + \|h\|^2 &\ge (L(u_n), u_n) \\ &= t_n(N(u_n), u_n) + (1-t_n)(Pu_n, u_n) + t_n(h, u_n) \\ &\ge \frac{1}{\beta}t_n^2\|N(u_n)\|^2 + (1-t_n)\|Pu_n\|^2 - t_n\|h\|\frac{1}{\alpha}\|N(u_n)\|. \end{aligned}$$

Since $0 < \alpha < \beta < 1$, we can conclude that $\{t_n\|N(u_n)\|\}$ and $\{(1-t_n)\|Pu_n\|^2\}$ remain bounded. In the case that $t > 0$ we obtain a contradiction from $\|u_n\| \le \frac{1}{\alpha}\|N(u_n)\|$. If $t = 0$, then $\{\|Pu_n\|\}$ is bounded and by (7.9) also $\{\|Lu_n\|\}$ is bounded. Since $\|Qu_n\| \le \|L_0^{-1}\|\|Lu_n\|$, a contradiction follows from $\|u_n\| \le \|Qu_n\| + \|Pu_n\|$. ∎

The result of Corollary 7.1 can be easily extended to cover the non-resonance cases where we assume that $[\alpha, \beta] \cap \sigma(L) = \emptyset$. This is shown by replacing in (7.8) the reference map $L-P$ by $A_1 = L - \tau I$, where $\tau = \frac{1}{2}(\alpha + \beta)$, for example. The corresponding reference map in the setting of the equation is $F_1 = Q(I - \tau L_0^{-1}Q) + \tau P$.

COROLLARY 7.2. *If g satisfies* (7.2) *with $0 < \alpha < \beta$ and $[\alpha, \beta] \cap \sigma(L) = \emptyset$, then the equation* (7.1) *admits a weak solution u in $D(L)$ for any given $h \in H$.*

The resonance problem around an eigenvalue $\lambda \in \sigma(L)$, $\lambda > 0$, can be tackled by choosing the reference map $A_2 = -\lambda I + P_\lambda$, where P_λ stands for the orthogonal projection into $\mathrm{Ker}(L - \lambda I)$. For the results in this direction we refer to [**BM 2**], where the condition (7.2) is also given in a more general form. Similar results by different methods can be found in [**BN**], [**Ma**], [**MW**], for example.

Next we indicate briefly how the problem of multiple periodic solutions for semilinear wave equations can be dealt with using the framework of Section 6. We are interested in the case where the nonlinearity g interacts on the spectrum of L. For convenience we consider an autonomous wave equation of the form

$$\text{(7.10)}\qquad \begin{cases} u_{tt} - u_{xx} = g(u) & \text{in } (0,\pi)\times \mathbf{R} \\ u(0,\cdot) = u(\pi,\cdot) = 0 & \\ u(\cdot, t+2\pi) = u(\cdot, t), & t \in \mathbf{R}, \end{cases}$$

where g is continuous non-decreasing with $g(0) = 0$ and satisfies the condition

$$\text{(7.11)}\qquad 0 < \tilde{\alpha} = \inf_{s\neq 0} \frac{g(s)}{s} < \sup_{s\neq 0} \frac{g(s)}{s} = \tilde{\beta} < \infty$$

where $(\tilde{\alpha}, \tilde{\beta}) \cap \sigma(L) \neq \emptyset$. We also assume that

$$\text{(7.12)}\qquad \lambda_{l-1} < \tilde{\alpha} \le h^*(\infty) < \lambda_l \le \cdots \le \lambda_m < h_*(0) \le \tilde{\beta} < \lambda_{m+1},$$

where $h^*(\infty) = \limsup_{|s|\to\infty} \frac{g(s)}{s}$ and $h_*(0) = \liminf_{|s|\to 0} \frac{g(s)}{s}$. The condition (7.12) means that $h(s) = \frac{g(s)}{s}$ crosses finitely many positive eigenvalues when s goes from 0 to ∞. The idea is to use the homotopy argument similar to (7.8) with a suitable reference map $F_0 = L - A$ with $A \in (S_+)$ and find open bounded convex sets G in H such that for some $w \in F_0(G \cap D(L))$,

$$\text{(7.13)}\qquad Lu - tN(u) - (1-t)A(u) \neq (1-t)w \qquad \text{for all } u \in \partial G \cap D(L),\ 0 \le t \le 1.$$

Then we can conclude that the equation (7.10) admits at least one solution u in $G \cap D(L)$. Since (7.10) obviously has the trivial solution, we must require that $0 \notin \overline{G}$.

To deal with the simplest possible case we assume that only one simple eigenvalue $\lambda_m = \bar{\lambda}$ is crossed. (In fact, for the wave operator such eigenvalues are $\lambda_1 = 1$ and $\lambda_3 = 4$, only.) Let $\bar{\phi}$ be the eigenvector associated with $\bar{\lambda}$. We denote $H_2 = \mathrm{sp}\{\bar{\phi}\}$, $H_1 = \overline{\mathrm{sp}}\{\phi_{lm} \mid l^2 - m^2 < \bar{\lambda}\}$ and $H_3 = \overline{\mathrm{sp}}\{\phi_{lm} \mid l^2 - m^2 > \bar{\lambda}\}$. Then $H = H_1 \oplus H_2 \oplus H_3$ and each $u \in H$ has a representation $u = P_1u + P_2u + P_3u$ where P_i is the orthogonal projection to H_i, $i = 1,2,3$. A suitable homotopy will be $F_0 = L - \bar{\lambda}I + P_2$, i.e., $A = \bar{\lambda}I - P_2$ in (7.13). Thus we are led to find a priori estimates for the solutions of the equation

$$\text{(7.14)}\qquad Lu - tN(u) - (1-t)(\bar{\lambda}u - P_2u) = (1-t)w, \qquad u \in D(L),\ 0 \le t \le 1,$$

where $w = \bar{\phi}$ or $w = -\bar{\phi}$. It is shown in [**BM 4**] that using (7.11) and (7.12) such estimates can be obtained and we get the following

COROLLARY 7.3. *Let $\bar{\lambda} \in \{1,4\}$ and let g satisfy the conditions* (7.11) *and* (7.12) *with $\lambda_l = \lambda_m = \bar{\lambda}$. Then there exist constants ρ and R with $0 < \rho < 1 < R$ such that* (7.10) *admits at least one nontrivial solution in each of the sets $G_+ = \{u \in D(L) \mid \|u\| < R,\ P_2u = \theta\bar{\phi},\ \rho < \theta < R\}$ and $G_- = \{u \in D(L) \mid \|u\| < R,\ P_2u = -\theta\bar{\phi},\ \rho < \theta < R\}$.*

It turns out that $\dim H_2$ plays a crucial role when we look for sets G where to apply homotopy argument. The case where $\dim H_2 = 2$ and $\bar{\lambda} = l^2 - m^2$ for some $l, m \in \mathbf{Z}_+$ can be dealt with as shown in [**BM 3**] (cf [**Hi**]) but all other cases where $\dim H_2 \ge 3$ seem to be open.

On the other hand, we can generalize Corollary 7.3 by constructing subspaces V of H which are invariant under L and N and apply Corollary 7.3 to the reduced problem

$$L_V u = N_V(u), \qquad u \in D(L) \cap V,$$

where L_V and N_V are the restrictions of L and N into V, respectively. This idea for wave equations was used by Vejvoda [**Ve**] and Coron [**Co**]. For more general results we refer to [**BM 3,4**].

8. The degree for perturbations of maximal monotone mappings

We shall assume again that X is a real reflexive Banach space and that X and X^* are locally uniformly convex. We consider mappings $F = T + S$ where T is a maximal monotone multi (or a single valued map) from $D(T) \subset X$ to 2^{X^*} with $0 \in T(0)$, and S is a bounded demicontinuous mapping from X to X^* of class (S_+). With T we associate the family of generalized Yosida approximations

$$T_\epsilon(u) = (T^{-1} + \epsilon\mathcal{J}^{-1})^{-1}(u), \qquad u \in X,\ \epsilon > 0.$$

It is well-known (see **[De]**, **[Bro 5]**) that for each $\epsilon > 0$, T_ϵ is a bounded continuous maximal monotone single valued mapping from all of X to X^* with $T_\epsilon(0) = 0$. Moreover, $T_\epsilon(u) \rightharpoonup T^0(u)$ on $D(T)$ as $\epsilon \to 0+$, where $T^0(u)$ denotes the unique element of the set $T(u)$ having minimal norm, i.e., $\|T^0(u)\| = \operatorname{dist}(0, T(u))$. On the other hand, $\|T_\epsilon(u)\| \to \infty$ as $\epsilon \to 0+$, if $u \notin \overline{D(T)}$.

We shall sketch the extension of the degree function for the mappings $F = T + S$ by using approximations $F_\epsilon = T_\epsilon + S$ in the class (S_+). Let G be an open bounded set in X. We denote

$$\begin{aligned}\mathcal{F}_G(MM; S_+) = \{F = T + S \mid T\colon X \supset D(T) \to 2^{X^*} &\text{ maximal monotone}\\ \text{with } 0 \in T(0) \text{ and } S\colon \overline{G} \to X^* &\text{ bounded demicontinuous}\\ \text{of class } (S_+)\}.\end{aligned}$$

The obvious idea now is to show that we can define the degree function d_{MM} for all admissible triplets (F, G, y), $F \in \mathcal{F}_G(MM; S_+)$, $y \notin F(\partial G)$, as the limit

$$d_{MM}(F, G, y) = \lim_{\epsilon \to 0+} d_{S_+}(F_\epsilon, G, y). \tag{8.1}$$

To this end we start with the following (cf. **[Bro 5]**)

Lemma 8.1. *Let T be a maximal monotone multi: $X \supset D(T) \to 2^{X^*}$ with $0 \in T(0)$ and let $\{u_n\}$, $\{\epsilon_n\}$, $\{\delta_n\}$, $\{s_n\}$ be sequences such that $u_n \rightharpoonup u$ in X, $\epsilon_n \to 0+$, $\delta_n \to 0+$, $0 \le s_n \le 1$. Let $v_n = T_{\epsilon_n}(u_n)$, $z_n = T_{\delta_n}(u_n)$ and $w_n = (1 - s_n)v_n + s_n z_n$ and assume that $w_n \rightharpoonup w$ in X^* and $\limsup\langle w_n, u_n\rangle \le \langle w, u\rangle$. Then $u \in D(T)$, $w \in T(u)$ and $\lim\langle w_n, u_n\rangle = \langle w, u\rangle$.*

Proof: Since $v_n = T_{\epsilon_n}(u_n)$ and $z_n = T_{\delta_n}(u_n)$, $v_n \in T(u_n - \epsilon_n\mathcal{J}^{-1}(v_n))$ and $z_n \in T(u_n - \delta_n\mathcal{J}^{-1}(z_n))$. Since T is monotone and $0 \in T(0)$, we have

$$\langle v_n, u_n - \epsilon_n\mathcal{J}^{-1}(v_n)\rangle \ge 0 \quad \text{and} \quad \langle z_n, u_n - \delta_n\mathcal{J}^{-1}(z_n)\rangle \ge 0.$$

Therefore

$$\epsilon_n\langle v_n, \mathcal{J}^{-1}(v_n)\rangle = \epsilon_n\|v_n\|^2 \le \langle v_n, u_n\rangle$$

and similarly

$$\delta_n\|z_n\|^2 \le \langle z_n, u_n\rangle.$$

Multiplying these inequalities by $(1 - s_n)$ and s_n, respectively, and adding we get

$$(1 - s_n)\epsilon_n\|v_n\|^2 + s_n\delta_n\|z_n\|^2 \le \langle w_n, u_n\rangle \qquad \text{for all } n \in \mathbb{N}.$$

Since $\{\langle w_n, u_n\rangle\}$ remains bounded, we conclude that $(1 - s_n)\epsilon_n\|v_n\| \to 0$ and $s_n\delta_n\|z_n\| \to 0$. Let $[x, y] \in G(T)$ be arbitrary. By the monotony of T we have

$$\langle v_n - y, u_n - \epsilon_n\mathcal{J}^{-1}(v_n) - x\rangle \ge 0 \quad \text{and} \quad \langle z_n - y, u_n - \delta_n\mathcal{J}^{-1}(z_n) - x\rangle \ge 0.$$

Hence

$$\langle v_n - y, u_n - x\rangle \ge -\epsilon_n\|y\|\|v_n\| \quad \text{and} \quad \langle z_n - y, u_n - x\rangle \ge -\delta_n\|y\|\|z_n\|.$$

By multiplying and adding as above again we get

$$\langle w_n - y, u_n - x\rangle \geq -\|y\| \{(1-s_n)\epsilon_n\|v_n\| + s_n\delta_n\|z_n\|\}$$

implying

$$\liminf\langle w_n - y, u_n - x\rangle \geq 0.$$

Bearing in mind the assumption we obtain

$$\langle w, u\rangle \geq \limsup\langle w_n, u_n\rangle \geq \liminf\langle w_n, u_n\rangle \geq \langle w, x\rangle + \langle y, u\rangle - \langle y, x\rangle.$$

Hence $\langle w-y, u-x\rangle \geq 0$ and since $T \in (MM)$ we conclude $u \in D(T)$ and $w \in T(u)$. Substituting $[u, w]$ for $[x, y]$ above we also get $\lim\langle w_n, u_n\rangle = \langle w, u\rangle$. ∎

Now we are in the position to continue in the familiar way.

LEMMA 8.2. *Let $F \in \mathcal{F}_G(MM, S_+)$ and $y \in X^*$ with $y \notin F(\partial G)$. Then there exists $\hat{\epsilon} > 0$ such that $y \notin F_\epsilon(\partial G)$ for all ϵ with $0 < \epsilon < \hat{\epsilon}$. Moreover, there exists $\epsilon_0 < \hat{\epsilon}$ such that $d_{LS}(F_\epsilon, G, y)$ is constant for all ϵ with $0 < \epsilon < \epsilon_0$.*

PROOF: Assume that the first assertion is false. Then there exist sequences $\{u_n\} \subset \partial G$ and $\{\epsilon_n\}$ such that $u_n \rightharpoonup u$, $\epsilon_n \to 0+$ and

$$F_{\epsilon_n}(u_n) = T_{\epsilon_n}(u_n) + S(u_n) = y \qquad \text{for all } n \in \mathbb{N}.$$

Since S is bounded, we can assume that $v_n = T_{\epsilon_n}(u_n) = y - S(u_n) \rightharpoonup v$. Since $S \in (S_+) \subset (QM)$ we also have

$$\limsup\langle v_n, u_n - u\rangle = \limsup\langle y - S(u_n), u_n - u\rangle \leq 0$$

and we can employ Lemma 8.1 with the specialization $s_n = 0$ for all $n \in \mathbb{N}$. Consequently $u \in D(T)$, $v \in T(u)$ and $\langle v_n, u_n\rangle \to \langle v, u\rangle$ implying that

$$\limsup\langle S(u_n), u_n - u\rangle = 0.$$

Using the fact that $S \in (S_+)$ we get $u_n \to u$ with $u \in \partial G$ and $S(u_n) \rightharpoonup S(u)$. Thus $v = y - S(u) \in T(u)$ and $y \in (T+S)(u)$, a contradiction.

If the second assertion were false, we could find, using the property (c) of the S_+-degree, sequences $\{u_n\} \subset \partial G$, $\{\epsilon_n\}$, $\{\delta_n\}$ and $\{s_n\}$ such that $u_n \rightharpoonup u$, $\epsilon_n \to 0+$, $\delta_n \to 0+$, $0 \leq s_n \leq 1$ and

$$(1-s_n)F_{\epsilon_n}(u_n) + s_nF_{\delta_n}(u_n) = y \qquad \text{for all } n \in \mathbb{N}.$$

Thus

$$(1-s_n)T_{\epsilon_n}(u_n) + s_nT_{\delta_n}(u_n) + S(u_n) = y \qquad \text{for all } n \in \mathbb{N}.$$

Denoting $v_n = T_{\epsilon_n}(u_n)$, $z_n = T_{\delta_n}(u_n)$ and $w_n = (1-s_n)v_n + s_nz_n$ we can assume that $w_n = y - S(u_n) \rightharpoonup w$ in X^*. Hence we can invoke Lemma 8.1 if we know that $\limsup\langle w_n, u_n\rangle \leq \langle w, u\rangle$. But this is again a consequence of the fact that $S \in (S_+)$. Thus we may conclude that $u \in D(T)$, $w \in T(u)$ and $\lim\langle w_n, u_n\rangle = \langle w, u\rangle$, which implies as above that $\limsup\langle S(u_n), u_n - u\rangle = 0$ and $u_n \to u$ with $u \in \partial G$. A contradiction is achieved by the fact that $y \in (T+S)(u)$. ∎

By Lemma 8.2 the definition (8.1) is relevant. To verify that d_{MM} is a classical degree function we need the class of admissible homotopies. We shall call a homotopy F_t, $0 \leq t \leq 1$, permissible if $F_t = T_t + S_t$, where S_t is a bounded homotopy of class (S_+) and T_t is a *permissible homotopy of maximal monotone mappings* in the sense that $T_t \in (MM)$ for each $t \in [0,1]$ and the mapping $t \to (T_t + \mathcal{J})^{-1}(v)$ is continuous from $[0,1]$ to the strong topology of X. In fact, it is shown by Browder ([**Bro 5**]) that an equivalent condition for T_t is the following generalized pseudomonotonicity condition: if $t_n \to t$ in $[0,1]$, $[u_n, w_n] \in G(T_{t_n})$ with $u_n \rightharpoonup u$, $w_n \rightharpoonup w$ and $\limsup\langle w_n, u_n\rangle \leq \langle w, u\rangle$, then $u \in D(T_t)$, $w \in T_t(u)$ and $\limsup\langle w_n, u_n\rangle = \langle w, u\rangle$. However, for applications and even for uniqueness of the degree function (8.1) it seems adequate to deal with the class of affine homotopies of the form

$$\mathcal{H}_G(MM; S_+) = \{F_t = tF_0 + (1-t)S_1 \mid F_0 \in \mathcal{F}_G(MM; S_+),\ S_1 \in \mathcal{F}_G(S_+)\}$$

Then we have

THEOREM 8.1. *Let X be a real reflexive Banach space, G an open bounded subset in X and $\mathcal{F}_G(MM; S_+)$ the class of admissible mappings. Then there exists exactly one degree function d_{MM} satisfying the properties* (a) *to* (d) *with respect to $\mathcal{H}_G(MM; S_+)$ and normalizing map $\mathcal{J}$.*

For the proof we need generalized versions of Lemmas 8.1 and 8.2 for homotopies as shown by Browder [**Bro 5**]. We check only the property (a). Indeed, assume $d_{MM}(T + S, G, y) \neq 0$. Then $d_{S_+}(T_\epsilon + S, G, y)$ is constant and non-zero for ϵ small enough, and there exist sequences $\{u_n\} \subset G$ and $\{\epsilon_n\}$ such that $T_{\epsilon_n}(u_n) + S(u_n) = y$ for all $n \in \mathbb{N}$, $u_n \rightharpoonup u$ and $\epsilon_n \to 0+$. We can proceed exactly the same way as in proving Lemma 8.2 to conclude that $u_n \to u \in \overline{G}$ and $y \in T(u) + S(u)$.

Existence and surjectivity results for mappings $F = T + S \in \mathcal{F}_G(MM; S_+)$ can be derived along the lines of Section 4. For example we get from Theorem 4.3 the following

COROLLARY 8.1. *Let $F = T + S \in \mathcal{F}_X(MM; S_+)$ satisfy the conditions*

(i) *F^{-1} is locally bounded*
(ii) *there exists $R > 0$ such that*

$$\frac{\langle w + S(u), u\rangle}{\|u\|} + \|w + S(u)\| > 0 \qquad \text{for all } \|u\| \geq R,\ w \in T(u).$$

Then $\mathcal{R}(T + S) = X^*$.

REMARK 8.1: It can be shown by different methods that for $T \in (MM)$, $\mathcal{R}(T) = X^*$ if and only if T^{-1} is locally bounded (cf. Remark 4.2). We also know that $\mathcal{R}(T + \lambda\mathcal{J}) = X^*$ for all $\lambda > 0$ if and only if $T \in (MM)$ (for more details consult [**De**]).

REMARK 8.2: The degree function obtained above can be extended to the class $\mathcal{F}_G(MM; QM)$ as in the previous cases by approximations $T + S + \epsilon\mathcal{J}$, $T \in (MM)$, $S \in (QM)$, $\epsilon > 0$. On the other hand we may also use approximations $T_\epsilon + S$ for which the weak degree is already defined. To indicate the results in this direction we have

COROLLARY 8.2. *Let $F = T + S$, where T is maximal monotone with $0 \in T(0)$ and $S : X \to X^*$ is bounded demicontinuous pseudomonotone and strongly coercive. Then $\mathcal{R}(T + S) = X^*$.*

PROOF: We know that T_ϵ is bounded demicontinuous monotone mapping from X to X^* for each $\epsilon > 0$. Hence $T_\epsilon \in (PM)$ and $F_\epsilon = T_\epsilon + S \in (PM)$. Since $T_\epsilon(0) = 0$, we have

$$\frac{\langle F_\epsilon(u), u\rangle}{\|u\|} = \frac{\langle T_\epsilon(u), u\rangle}{\|u\|} + \frac{\langle S(u), u\rangle}{\|u\|} \to \infty \qquad \text{as } \|u\| \to \infty.$$

By Theorem 4.5, $F_\epsilon(X) = X^*$ for each $\epsilon > 0$. Let $y \in X^*$ be arbitrary and $\{\epsilon_n\}$ a sequence with $\epsilon_n \to 0+$. For each $n \in \mathbb{N}$ there exists $u_n \in X$ such that

$$T_{\epsilon_n}(u_n) + S(u_n) = y.$$

Since S is strongly coercive, $\{u_n\}$ is bounded and we can assume that $u_n \rightharpoonup u$ and $S(u_n) \rightharpoonup h$. Denoting $v_n = T_{\epsilon_n}(u_n)$ we also have $v_n \rightharpoonup y - h = v$. Since $S \in (PM)$ we obtain $\limsup\langle v_n, u_n - u\rangle \leq 0$ and we can invoke Lemma 8.1 to derive $u \in D(T)$, $v \in T(u)$ and $\lim\langle v_n, u_n\rangle = \langle v, u\rangle$. Hence $\limsup\langle S(u_n), u_n - u\rangle = 0$ implying $S(u_n) \rightharpoonup S(u)$ and $v = y - S(u)$, i.e., $y \in (T + S)(u)$. ∎

9. PERTURBATIONS OF LINEAR MAXIMAL MONOTONE MAPPINGS AND APPLICATIONS TO PARABOLIC PROBLEMS

The study of perturbations of maximal monotone multis is motivated for instance by the connections to generalized Hammerstein equations, variational inequalities and subdifferentials of lower semicontinuous functionals. We are interested here in the parabolic initial-boundary value problems for differential operators of the form

$$\text{(9.1)} \quad \frac{\partial u(x,t)}{\partial t} + \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha A_\alpha(x,t,u(x,t),D_x u(x,t),\ldots,D_x^m u(x,t)), \qquad (x,t) \in Q$$

where $Q = \Omega \times (0,T)$, Ω an open bounded subset in $\mathbf{R}^N$, $m \ge 1$ and the coefficients A_α are functions of $(x,t) \in Q$ and of $\xi = (\eta,\zeta) \in \mathbf{R}^{N_0}$ with $\eta \in \mathbf{R}^{N_1}$ and $\zeta \in \mathbf{R}^{N_2}$, $N_1 + N_2 = N_0$, as in Section 5. We assume also that each $A_\alpha(x,t,\xi)$ is a Carathéodory function and satisfies the polynomial growth condition (A_1) for some $c_1 > 0$, $1 < p < \infty$ and $k_1 \in L^{p'}(Q)$. The latter part of the operator (9.1) gives rise to a bounded continuous mapping $S\colon \mathcal{V} \to \mathcal{V}^*$ by the rule

$$\text{(9.2)} \qquad \langle S(u), v\rangle = \int_Q \sum_{|\alpha|\le m} A_\alpha(x,t,\xi(u)) D^\alpha v, \qquad u, v \in \mathcal{V},$$

where $\mathcal{V} = L^p(0,T;X)$ with $X = W_0^{m,p}(\Omega)$, for example. If we assume, for simplicity, that $p \ge 2$ and if we give also an initail value $u(0) = 0$, then the former part of (9.1) has a realization L as a maximal monotone closed linear mapping from

$$D(L) = \left\{ v \in \mathcal{V} \mid \frac{\partial v}{\partial t} \in \mathcal{V}^*,\ v \in C(0,T;L^2(\Omega)),\ v(0) = 0 \right\}$$

to $\mathcal{V}^*$, (see [Li]). Hence the initial-boundary value problem for the operator (9.1) admits a weak formulation

$$\text{(9.3)} \qquad Lu + S(u) = h, \qquad u \in D(L),$$

where h is a given element in $\mathcal{V}^* = L^{p'}(0,T;X^*)$. If we knew that S is coercive and pseudomonotone we can apply the results of Section 8. Indeed, if the coefficients A_α satisfy the monotonicity condition

$$(\mathrm{A}_2)_\mathrm{M} \qquad \sum_{|\alpha|\le m} \{A_\alpha(x,t,\xi) - A_\alpha(x,t,\xi^*)\}(\xi_\alpha - \xi_\alpha^*) \ge 0 \quad \text{for all } (x,t) \in Q \text{ and } \xi,\xi^* \in \mathbf{R}^{N_0}$$

and the strong coercivity condition

$$(\mathrm{A}_3)_\mathrm{S} \qquad \sum_{|\alpha|\le m} A_\alpha(x,t,\xi)\xi_\alpha \ge c_2|\xi|^p - k_2(x,t) \qquad \text{for all } (x,t) \in Q,\ \xi \in \mathbf{R}^{N_0} \text{ with } c_2 > 0 \text{ and } k_2 \in L^1(Q),$$

then S is monotone and hence also pseudomonotone. By Corollary 8.2 the equation (9.3) has a solution for any $h \in \mathcal{V}^*$. In order to deal with more refined monotonicity conditions like

$$(\mathrm{A}_2)_\mathrm{S} \qquad \sum_{|\alpha|=m} \{A_\alpha(x,t,\eta,\zeta) - A_\alpha(x,t,\eta,\zeta^*)\}(\zeta_\alpha - \zeta_\alpha^*) > 0$$
$$\text{for all } (x,t) \in Q,\ \eta \in R^{N_1} \text{ and } \zeta \ne \zeta^* \in \mathbf{R}^{N_2}$$

we need a further extension of the degree for mappings of the form $F = L + S$ where L is a linear maximal monotone closed densely defined operator from $D(L)$ in $\mathcal{V}$ into $\mathcal{V}^*$ and S is *of class* (S_+) or *pseudomonotone with respect to* $D(L)$, i.e., for any sequence $\{u_n\}$ in $D(L)$ with $u_n \rightharpoonup u$ in $\mathcal{V}$, $Lu_n \rightharpoonup Lu$ in $\mathcal{V}^*$ and $\limsup\langle S(u_n), u_n - u\rangle \le 0$, we have $u_n \to u$ in $\mathcal{V}$, or $S(u_n) \rightharpoonup S(u)$ in $\mathcal{V}^*$ and $\langle S(u_n), u_n\rangle \to \langle S(u), u\rangle$, respectively.

LEMMA 9.1. *If the operator* (9.1) *satisfies conditions* (A_1) *and* $(A_2)_S$, *then the mapping* S *defined by* (9.2) *is pseudomonotone with respect to* $D(L)$. *If also the condition* $(A_3)_S$ *is satisfied, then* S *is of class* (S_+) *with respect to* $D(L)$.

PROOF: For the proof of the first part we refer to [**Mu**] which deals with the case where Ω may be unbounded. Assume that the conditions (A_1), $(A_2)_S$ and $(A_3)_S$ are satisfied. Hence for any sequence $\{u_n\}$ in $D(L)$ with $u_n \rightharpoonup u$ in $\mathcal{V}$, $Lu_n \rightharpoonup Lu$ in $\mathcal{V}^*$ and $\limsup\langle S(u_n), u_n - u\rangle \leq 0$ we have $S(u_n) \rightharpoonup S(u)$ in $\mathcal{V}^*$ and $\langle S(u_n), u_n\rangle \to \langle S(u), u\rangle$. Moreover we know by the proof of first part that $D^\alpha u_n \to D^\alpha u$ in $L^p(Q)$ for all $|\alpha| \leq m-1$ and $D^\alpha u_n(x,t) \to D^\alpha u(x,t)$ a.e. in Q for all $|\alpha| = m$, at least for a subsequence. Denoting $h_n(x,t) = \sum_{|\alpha|\leq m} A_\alpha(x,t,\xi(u_n(x,t)))D^\alpha u_n(x,t) + k_2(x,t)$, $h(x,t) = \sum_{|\alpha|\leq m} A_\alpha(x,t,\xi(u(x,t)))D^\alpha u(x,t) + k_2(x,t)$, we have $h_n(x,t) \geq 0$ a.e. in Q, $\|h_n\|_{L^1(Q)} \to \|h\|_{L^1(Q)}$ and $h_n(x,t) \to h(x,t)$ a.e. in Q. Hence $h_n \to h$ in $L^1(Q)$ and there exists a function $k \in L^1(Q)$ such that

$$\sum_{|\alpha|\leq m} A_\alpha(x,t,\xi(u_n(x,t)))D^\alpha u_n(x,t) \leq k(x,t) \qquad \text{a.e. in } Q$$

for some further subsequence. Using now $(A_3)_S$ we can conclude by the dominated convergence theorem that $u_n \to u$ in $\mathcal{V}$. ∎

In order to get suitable approximations for the map $F = L+S$ we denote $Y = D(L)$ equipped with the norm $\|u\|_Y = \|u\|_{\mathcal{V}} + \|Lu\|_{\mathcal{V}^*}$. Y is also a reflexive Banach space with continuous embedding $j\colon Y \to \mathcal{V}$. For each $\epsilon > 0$ we then define (cf. [**Li**, Chapitre 3])

$$a_\epsilon(u,v) = \epsilon\langle \mathcal{J}^{-1}(Lu), Lv\rangle + \langle Lu, v\rangle + \langle S(u), v\rangle, \qquad u,v \in D(L), \tag{9.4}$$

where $\mathcal{J}$ stands for the duality map: $\mathcal{V} \to \mathcal{V}^*$. It is easy to see that $v \to a_\epsilon(u,v)$ is a bounded linear functional in Y and therefore (9.4) defines a mapping $F_\epsilon\colon Y \to Y^*$ by

$$\langle F_\epsilon(u), v\rangle = a_\epsilon(u,v), \qquad u,v \in Y. \tag{9.5}$$

If we assume that S is of class (S_+) with respect to $D(L)$, then F_ϵ is of class (S_+) for each $\epsilon > 0$. Indeed, let $\{u_n\}$ be a sequence in Y with $u_n \rightharpoonup u$ in Y and $\limsup\langle F_\epsilon(u_n), u_n - u\rangle \leq 0$. Hence $u_n \rightharpoonup u$ in $\mathcal{V}$, $Lu_n \rightharpoonup Lu$ in $\mathcal{V}^*$ and

$$\limsup\{\epsilon\langle \mathcal{J}^{-1}(Lu_n) - \mathcal{J}^{-1}(Lu), Lu_n - Lu\rangle + \langle Lu_n - Lu, u_n - u\rangle + \langle S(u_n), u_n - u\rangle\} \leq 0. \tag{9.6}$$

Since $\mathcal{J}^{-1}$ is strictly monotone and L is monotone, we have

$$\limsup\langle S(u_n), u_n - u\rangle \leq 0$$

implying $u_n \to u$ in $\mathcal{V}$. Hence also

$$\lim\langle \mathcal{J}^{-1}(Lu_n) - \mathcal{J}^{-1}(Lu), Lu_n - Lu\rangle = 0.$$

Since $\mathcal{J}^{-1}$ has the properties of the duality map: $\mathcal{V}^* \to \mathcal{V}^{**}$, $\|Lu_n\| \to \|Lu\|$ and thus $Lu_n \to Lu$ in $\mathcal{V}^*$. Hence F_ϵ is in the class (S_+) for any $\epsilon > 0$.

Using the approximations $\{F_\epsilon \mid \epsilon < 0\}$ we can now proceed in the familiar way to obtain a degree function for mappings $F = L + S$. However, some further work is to be done due to the fact that the bounded sets in Y and $\mathcal{V}$ are not the same. A more detailed discussion on this extension will appear elsewhere. We close these notes by a remark that existence results for the equation (9.4) under the conditions (A_1), $(A_2)_S$ and $(A_3)_S$ can be obtained also directly by using approximating equations $F_\epsilon(u) = j^*h$ in Y, where $j^*\colon \mathcal{V}^* \to Y^*$ denotes the adjoint of j.

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