ON THE SHAPE OF SOLUTIONS TO SOME VARIATIONAL PROBLEMS

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The first spring school that I ever attended was organized by the late Svatopluk Fučík in 1978 in Horní Bradlo. During the conference he jokingly remarked that he might not be able to attend the next spring school, because it would be on free boundary problems and this is a topic which might be misunderstood by the authorities. Unfortunately, he did not even have time to find out that it became a regular institution. I dedicate my four lectures during the spring school 1994 in Praha to his memory.

The lectures will deal with the following three problems:

1. A free boundary problem with fuzzy free boundary.
2. A conjecture of Saint Venant's on points of maximal stress.
3. Newton's principle of minimal resistance.

First Problem

Let $\Omega \subset \mathbb{R}^2$ be the cross-section of a long cylindrical bar, whose torsional rigidity we want to minimize subject to some side constraints. The outer shape of Ω is prescribed as well as the proportions and physical properties (shear moduli) of the materials which fill up Ω. Suppose that μ_i^{-1} denotes the shear modulus of material $i = 1, 2$. The shear modulus measures how much one has to pull sideways at the top of an elastic block, which is fixed at the bottom, in order to shear it by a standard amount. Therefore "strong" material will have a high shear modulus, that is a small value for μ. I shall assume that $0 < \mu_1 < \mu_2 < \infty$.

Maximizing torsional rigidity amounts to solving

$$m = \min_{v \in H_0^{1,2}(\Omega)} \int_\Omega \left\{ \frac{\mu(x)}{2} |\nabla v(x)|^2 - v(x) \right\} dx. \tag{1.1}$$

If u solves (1.1), then

$$\int_\Omega \{\mu(x)\nabla u(x)\nabla\varphi(x) - u(x)\varphi(x)\}\,dx = 0 \quad \text{for all } \varphi \in C_0^\infty(\Omega), \tag{1.2}$$

and consequently

$$m = -\frac{1}{2}\int_\Omega u(x)\,dx \tag{1.3}$$

Now the torsional rigidity is defined as $-2m$, see [PS, p. 88].

If Ω is a circular disc, then every engineer will immediately guess the solution to our shape optimization problem. One has to put the stronger material in an annulus with outer boundary $\partial\Omega$ and the softer material into the interior disc of the annulus. In this case the free boundary between those two materials consists of a circle, a line of constant (and prescribed) curvature.

What if the disc is slightly perturbed to an ellipse, say? Then I claim that there is no optimal solution to this shape optimization problem. There is, however, an almost optimal solution with a very fuzzy free boundary. This boundary consists among other things of many (nonclosed) circular arcs of identical (and prescribed) curvature. There the two materials are intertwined like two curved combs that have been pushed against each other. In other words, they form a sandwich-like homogenized material.

To prove this claim will take up the rest of this lecture. Let $\Omega_1 = \{x \in \Omega \mid \mu(x) = \mu_1\}$ and $\Omega_2 = \{x \in \Omega \mid \mu(x) = \mu_2\}$, let u be a solution of (1.1), set $u_i(x) = u|_{\Omega_i}$ and suppose first that $\varphi \in C_0^\infty(\Omega_i)$. Then (1.2) leads to the Euler equations

$$-\Delta u_i = \frac{1}{\mu_i} \quad \text{in } \Omega_i, \quad i = 1, 2. \tag{1.4}$$

Next suppose that Ω_1 and Ω_2 have a common boundary Γ of class C^1 and that n_i is the outward normal to Ω_i along Γ. Then (1.2), integration by parts and (1.4) lead to

$$\int_\Gamma \left\{\mu_1\frac{\partial u_1}{\partial n_1} + \mu_2\frac{\partial u_2}{\partial n_2}\right\}\varphi\,ds = 0. \tag{1.5}$$

But (1.5) represents the continuity of flux across the free boundary. For Ω a ball and radial u, condition (1.5) translates into

$$\mu_1\frac{\partial u_1}{\partial r} = \mu_2\frac{\partial u_2}{\partial r} \tag{1.6}$$

The following lemma will be useful.

Lemma 1.1. *Suppose that the shape optimization problem has a Lipschitz continuous solution which is of class $C^1(\Omega_i)$ and which has a free boundary Γ of class C^1. Then there exists a $\lambda \in \mathbb{R}$ such that*

$$|\nabla u_2(x)| \leq \sqrt{2\lambda\mu_1/\mu_2} := \tau_1 \quad \text{in } \Omega_2, \tag{1.7}$$

$$|\nabla u_1(x)| \geq \sqrt{2\lambda\mu_2/\mu_1} := \tau_2 \quad \text{in } \Omega_1. \tag{1.8}$$

I postpone the proof of this lemma and derive some immediate consequences. The estimates (1.7) and (1.8) imply

$$\mu_2|\nabla u_2| \leq \sqrt{2\lambda\mu_1\mu_2} \leq \mu_1|\nabla u_1|. \tag{1.9}$$

If we split ∇u into a tangential and normal derivative, we obtain, using (1.9) (1.5) and the continuity of u along Γ,

$$\begin{aligned}\mu_2^2\left|\frac{\partial u_2}{\partial n}\right|^2 + \mu_2^2\left|\frac{\partial u_2}{\partial t}\right|^2 &\leq \mu_1^2\left|\frac{\partial u_1}{\partial n}\right|^2 + \mu_1^2\left|\frac{\partial u_1}{\partial t}\right|^2 \\ = \mu_2^2\left|\frac{\partial u_2}{\partial n}\right|^2 + \mu_1^2\left|\frac{\partial u_2}{\partial t}\right|^2 &\leq \mu_2^2\left|\frac{\partial u_2}{\partial n}\right|^2 + \mu_2^2\left|\frac{\partial u_2}{\partial t}\right|^2.\end{aligned}$$

Since there is equality and since $\mu_1 < \mu_2$ we conclude

$$\frac{\partial u_1}{\partial t} = \frac{\partial u_2}{\partial t} = 0 \quad \text{on } \Gamma, \tag{1.10}$$

i.e.

$$u = \text{const} \quad \text{along } \Gamma. \tag{1.11}$$

But now (1.10), (1.9) and (1.5) imply

$$\mu_i|\nabla u_i| = \mu_i\left|\frac{\partial u_i}{\partial n_i}\right| = \sqrt{2\lambda\mu_1\mu_2} \text{ on } \Gamma. \tag{1.12}$$

Now let D be a connected component of Ω_2 which is compactly contained in Ω. Such is the case if Ω is a ball, and we might expect continuous dependence of Γ under small changes of Ω. Then

$$\begin{aligned} -\Delta u_2 &= 1/\mu_2 && \text{in } D, \\ u_2 &= c_2 && \text{on } \partial D, \\ \frac{\partial u_2}{\partial n} &= C_2 && \text{on } \partial D. \end{aligned} \tag{1.13}$$

But now, according to a famous result of Serrin [Se], D must be a circular disc. By analyticity of u_1, Ω must be a circular disc, too. Thus we have shown

Proposition 1.2. *Unless Ω is a circular disc, the shape optimization problem cannot have a smooth solution.*

It remains to prove Lemma 1.1. Notice that so far I have not brought the condition into play, that the size of Ω_1 is prescribed and that its shape is optimal. The condition on the size of Ω_1 can be restated as

$$\int_\Omega \mu(x)\,dx = C_0, \qquad \mu(x) \in \{\mu_1, \mu_2\} \tag{1.14}$$

and it can be entered into the variational problem (1.1) by means of a Lagrange parameter. We look for

$$\begin{aligned}&\inf_{\mu\in\{\mu_1,\mu_2\},\int_\Omega \mu = C_0} \quad \inf_{v\in H_0^{1,2}(\Omega)} \int_\Omega \left\{\frac{\mu}{2}|\nabla v|^2 - v\right\} dx \\ &= \inf_{v,\mu} \sup_{\lambda\in\mathbb{R}} \int_\Omega \left\{\frac{\mu}{2}|\nabla v|^2 - v - \lambda\mu\right\} dx + C_0\lambda.\end{aligned}$$

In fact, if $\int_\Omega \mu\,dx \neq C_0$, the sup over λ will be infinite.

Let us relax the condition $\mu(x) \in \{\mu_1, \mu_2\}$ for a moment to $\mu(x) \in [\mu_1, \mu_2]$. Then the inf is taken over a convex set of admissible functions μ and v, and the above functional is convex in v and μ and concave in λ. Therefore we may exchange the sup and inf and study

$$\sup_\lambda \inf_{v,\mu} \int_\Omega \mu\left\{\frac{1}{2}|\nabla v|^2 - \lambda\right\} dx + C_0\lambda - \int_\Omega v\,dx. \tag{1.15}$$

Let us consider the curly bracket in (1.15). If it is negative we want μ to be as large as possible, i.e. $\mu = \mu_2$, and if it is positive we want $\mu = \mu_1$. So the inf over $\mu(x) \in [\mu_1, \mu_2]$ will be attained in $\{\mu_1, \mu_2\}$. We set

(1.16)

$$g_\lambda(|\nabla v|) := \inf_{\mu\in[\mu_1,\mu_2]} \mu\left(\frac{1}{2}|\nabla v|^2 - \lambda\right) = \begin{cases} \mu_2(\frac{1}{2}|\nabla v|^2 - \lambda) & \text{if } |\nabla v|^2 \le 2\lambda \\ \mu_1(\frac{1}{2}|\nabla v|^2 - \lambda) & \text{if } |\nabla v|^2 \ge 2\lambda. \end{cases}$$

Therefore (1.15) can be rewritten as

$$\sup_\lambda \inf_v \int_\Omega \{g_\lambda(|\nabla v|) - v\}\,dx + C_0\lambda, \tag{1.17}$$

and in particular for fixed λ we are faced with

$$\inf_{v \in H_0^{1,2}(\Omega)} \int_\Omega \{g_\lambda(|\nabla v|) - v\}\, dx, \tag{1.18}$$

where g_λ is defined in (1.16) and depicted in Figure 1.1.

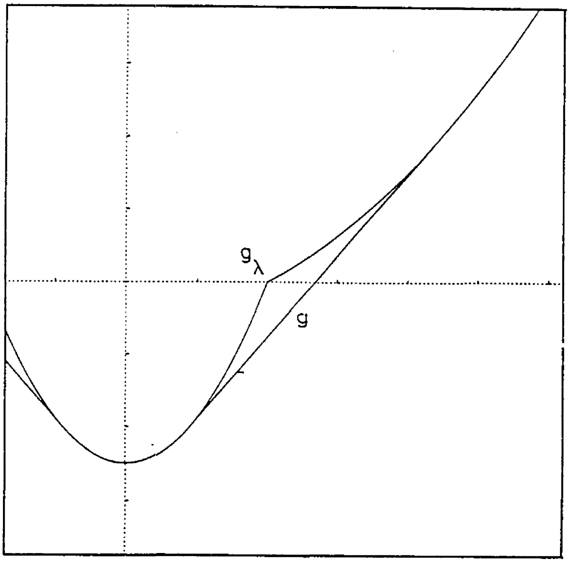

Figure 1.1 g_λ and g

Suppose we can solve (1.18) for every λ. Then (1.17) amounts to maximizing a function of a single variable λ only. But for fixed v the functional in (1.17) is concave in λ, i.e.

$$tg_\lambda(s) + (1-t)g_\nu(s) \le g_{\lambda t+(1-t)\nu}(s) \tag{1.19}$$

for $s \ge 0$, $t \in [0,1]$ and λ, $\nu \in \mathbb{R}$; and the functional tends to $-\infty$ as $|\lambda|$ tends to ∞. The derivation of (1.19) is a simple calculus exercise that has to distinguish four different cases. Therefore the maximization with respect to λ will pose no problem.

Unfortunately problem (1.18) is nonconvex, and therefore we cannot use the direct method in the calculus of variations to derive existence of a solution. This is in accordance with our observation, that in general there will be no solution of the shape optimization problem. However, if we convexify (1.18), i.e. if we replace g_λ by its convex envelope g, which is depicted in Figure 1.1, then we could at least solve the relaxed problem

$$\inf_{v\in H_0^{1,2}(\Omega)} \int_\Omega \{g(|\nabla v|) - v\}\, dx. \tag{1.20}$$

What happens in the case that Ω is a circular disc? We may replace any minimizer u, μ of (1.15) by their circular means

$$\underline{u}(r) = \frac{1}{2\pi}\int_0^{2\pi} u(r,\vartheta)\, d\vartheta, \quad \underline{\mu}(r) = \frac{1}{2\pi}\int_0^{2\pi} \mu(r,\vartheta)\, d\vartheta.$$

This will lower the functional in (1.15), since it is convex in u and μ. Without loss of generosity we may therefore assume that minimizers are radial; and by the reasoning that leads to (1.16) we can be sure that $\underline{\mu}(r) \in \{\mu_1, \mu_2\}$ and $\underline{\mu}(r) \notin (\mu_1, \mu_2)$. Let us now see why $|\nabla u| \notin (\tau_1, \tau_2)$ in the radial case. If $|\partial u/\partial r| \in (\tau_1, \tau_2)$ on a set of positive measure, we can modify u to a zig-zag-function u_ε on this set, so that u_ε approximates u in $L^\infty(\Omega)$ and so that $|\nabla u_\varepsilon| \notin (\tau_1, \tau_2)$. This is illustrated in Figure 1.2.

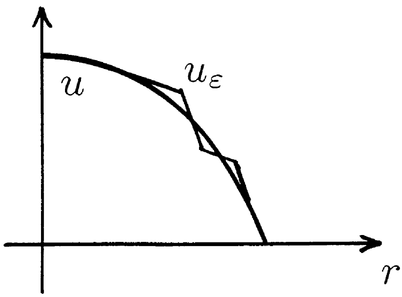

Figure 1.2 u and u_ε

Therefore the inequality

$$\int_\Omega g_\lambda(|\nabla u_\varepsilon|)\, dx = \int_\Omega g(|\nabla u_\varepsilon|)\, dx = \int_\Omega g(|\nabla u|)\, dx < \int_\Omega g_\lambda(|\nabla u|)\, dx \tag{1.21}$$

would lead to a contradiction.

The same argument can be applied in the nonradial case. Now the level lines of u serve as lines of discontinuity for $|\nabla u_\varepsilon|$, but along each line of steepest descent of u, the approximating function u_ε looks like the one in Figure 1.2. Again (1.21) leads to a contradiction and proves Lemma 1.1. $\square$

Inequality (1.21) has another consequence. Any minimizer of (1.18) will also be a minimizer of the relaxed problem (1.20).

Let us therefore from now on consider the relaxed variational problem (1.20). This problem has a solution $u \in H_0^{1,2}(\Omega)$, which satisfies the Euler equation

$$0 = \int_\Omega \{g'(|\nabla u|)\frac{\nabla u}{|\nabla u|}\ \nabla\phi - \phi\}\, dx, \qquad \text{for any } \phi \in C_0^\infty(\Omega),$$

or formally

$$-\operatorname{div}\left\{\frac{g'(|\nabla u|)}{|\nabla u|}\nabla u\right\} = 1.$$

We split Ω into $\Omega_1 := \{x \in \Omega \mid |\nabla u| > \tau_2\}$, $\Omega_2 := \{x \in \Omega \mid |\nabla u| < \tau_1\}$ and $H = \Omega \backslash (\Omega_1 \cup \Omega_2)$. In Ω_i we have (1.4), and in H we have $g' = \sqrt{2\lambda\mu_1\mu_2}$, i.e.

$$\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = 1/\sqrt{2\lambda\mu_1\mu_2}. \tag{1.22}$$

If we rewrite (1.22) in curvilinear coordinates, tangent and normal to level lines of u, it is rewritten as

$$\kappa = 1/\sqrt{2\lambda\mu_1\mu_2}, \tag{1.23}$$

where κ denotes the curvature of a level line of u for plane domains (or the mean curvature of a level surface of u for higher-dimensional domains).

In H the type of differential equation switches from elliptic, i.e. Δ_2-operator, to degenerate elliptic, i.e. Δ_1-operator. Here $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. Relation (1.23) tells us, that level lines of u are, as long as they run through H, circular arcs of constant curvature $c_1 = 1/\sqrt{2\lambda\mu_1\mu_2}$. The derivation of (1.23), however, has to be justified via regularity considerations.

Theorem 1.3 (Regularity). *Let u be a solution of* (1.20). *Then*

i) $u \in L^\infty(\Omega)$.

ii) $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ *and, provided* $\partial\Omega$ *is smooth,* $u \in C^\alpha(\overline{\Omega})$.

iii) $u \in C^\infty(\mathrm{int}\Omega_i)$, $i = 1, 2$.

iv) *If the level lines* $\{x \in \mathrm{int}\, H \mid u(x) = t\}$ *are locally Lipschitz continuous,then they are* C^∞ *for a.e.* $t \in \mathbb{R}$ *and have constant (mean) curvature* $c_1 = 1/\sqrt{2\lambda\mu_1\mu_2}$.

Proof. To prove i) one compares the solution of (1.20) on Ω with a solution of the same problem on Ω^*, a ball of the same volume as Ω. On Ω^* the solution is explicitly known and is L^∞, see [Ta]. Statement ii) follows from a result in [CE], iii) is a consequence of (1.4). To prove *iv)*, let us fix u on $\Omega_1 \cup \Omega_2$ and vary it only in H. Then we have to minimize

$$J_H(v) = \int_H \{c_1|\nabla v| - v\}\, dx,$$

and the coarea formula yields

$$J_H(v) = \int_0^{v_{\max}} \{c_1 \text{ Perim}\{v > t\} \text{ in } H - \text{Area}\{v > t\} \text{ in } H\}\, dt.$$

So for a.e. $t \in \mathbb{R}$ we minimize $(N-1)$-dimensional perimeter minus N-dimensional volume for $N = 2$, a regular elliptic problem in $N-1$ dimensions (since level lines were Lipschitz by assumption). Therefore iv) follows. Notice that this proof of *iv)* does not use the implicit function theorem and regularity of u. In fact, I do not even know that $u \in C^2(H)$. $\square$

What can be said about uniqueness of solutions to (1.20)? Since we are dealing with a convex problem, there is hope to have uniqueness. But g is not strictly convex, so there may as well be nonuniqueness. In fact, consider the one-dimensional problem

$$\min_{v(0)=0,v(1)=(\tau_1+\tau_2)/2} \int_0^1 g(|v'|)\, dx. \tag{1.24}$$

This problem has $u(x) = (\tau_1 + \tau_2)x/2$ as one solution, but there are other ones depicted in Figure 1.3 with slope alternating between τ_1 and τ_2.

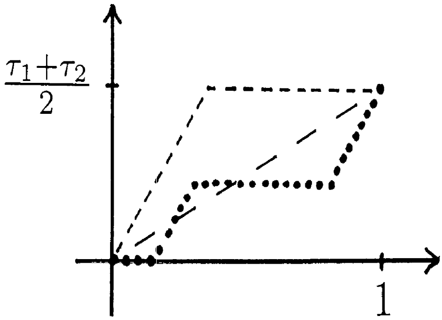

Figure 1.3 Solutions of (1.24) for $\tau_1 = 0$

Theorem 1.4 (Partial Uniqueness). *Let u, v be solutions of* (1.20). *Then*

$$\nabla u|\nabla v| = \nabla v|\nabla u| \quad \text{a.e. in } \Omega, \tag{1.25}$$

i.e. ∇u and ∇v are parallel. Moreover the sets $\Omega_1 = \{x \mid |\nabla u| > \tau_2\}$ and $\Omega_2 = \{x \mid |\nabla u| < \tau_1\}$ are uniquely determined (modulo nullsets) and

$$\nabla u = \nabla v \quad \text{a.e. in } \Omega_i, \quad i = 1, 2. \tag{1.26}$$

A proof of this theorem can be found in [KSW].

Remark 1.5. If u, v are different solutions of (1.20), we may assume without loss of generality that $u \leq v$. In fact $w_1 = \min\{u, v\}$ and $w_2 = \max\{u, v\}$ are solutions of (1.20).

Lemma 1.6. *Suppose that* (1.24) *holds and that the level lines of u and v are Lipschitz-continuous. Then $u - v = \text{const}$ on every component $L_a(u)$ of the set $L_a(u) = \{x \in \Omega \mid u(x) = a\}$ for a.e. a. In particular, we have that $v(x) - u(x) = c$ for some $x \in \ell_a(u)$ implies $v(x) = a + c$ for every $x \in \ell_a(u)$.*

Proof. ∇v and ∇v exist a.e. on $L_a(u)$ and are parallel. Therefore the tangent vector on $\ell_a(u)$ is defined a.e., and it coincides with the tangent vector on $\ell_a(v)$. $\square$

Lemma 1.7. *Under the assumptions of Lemma 1.6, every nonempty component $\ell_a(u)$ of a level set $L_a(u)$ has nonempty intersection with $\overline{\Omega_1 \cup \Omega_2}$.*

Proof. Else there exists a level C and a component $\ell_c(u)$ with positive distance to $\overline{\Omega_1 \cup \Omega_2}$. Therefore an entire neighborhood of $\ell_c(u)$ is contained in int H. According to Theorem 1.3.iv), $\ell_c(u)$ has to have constant curvature c_1. Since $|\nabla u| \in (\tau_1, \tau_2)$ in H, there must be an adjacent level d such that $\ell_d(u) \subset \operatorname{int} H$ for another level line of constant curvature c_1. Without loss of generality we may assume $d > c$. But now $\ell_d(u)$ and $\ell_c(u)$ are both closed circles of radius $\sqrt{2\lambda\mu_1\mu_2}$, and $\ell_c(u)$ does not intersect $\ell_d(u)$ for reasons of continuity of u, a contradiction. $\square$

Theorem 1.8 (Uniqueness). *Suppose that the relaxed variational problem* (1.20) *has a solution u with starshaped level sets $\{u > t\}$ and Lipschitzian level lines $\{u = t\}$, and $\partial\Omega_1, \partial\Omega_2$ are piecewise C^1. Then* (1.20) *has only one solution.*

To prove Theorem 1.8 suppose that there are two solutions u and v. Then there exists a map $f : \mathbb{R} \to \mathbb{R}$ with $v(x) = f(u(x))$, and f is locally Lipschitz continuous, so both u and v are Lipschitz continuous. Moreover $f'(u(x)) = 1$ in $\overline{\Omega_1 \subset \Omega_2}$, because of Theorem 1.4. But according to Lemma 1.7 every level line runs through $\overline{\Omega_1 \cup \Omega_2}$, i.e. $f' = 1$ on almost all levels of u. Therefore $v = u$. $\square$

Remark 1.9. One can give sufficient conditions for the assumptions on level lines of Theorem 1.8, see [KSW]. They are in particular satisfied for regular polygons.

For every fixed λ we have now a unique solution u_λ to the relaxed problem (1.20), and we can maximize

$$\int_\Omega \{g(|\nabla u_\lambda|) - u_\lambda\}\, dx + C_0\lambda$$

with respect to λ. I denote the function u_λ associated with the maximizing λ by u. If $|\nabla u| \notin (\tau_1, \tau_2)$ a.e. in Ω, as is the case for a circular disc only, then we have a solution to the unrelaxed problem (1.17). If, however, H has positive measure, then u can be approximated by a function u_ε with $|\nabla u_\varepsilon| \notin (\tau_1, \tau_2)$ as depicted in Figure 1.2. The thin layers in H in which $|\nabla u^\varepsilon| \geq \tau_2$ are then filled with material μ_1, those with $|\nabla u^\varepsilon| \leq \tau_1$ with material μ_2 and their common boundary consists of many circular arcs with identical curvature c_1.

Second Problem

Consider the classical torsion problem

$$-\Delta u = 1 \quad \text{in } \Omega \tag{2.1}$$

$$u = 0 \quad \text{on } \partial\Omega \tag{2.2}$$

for a given domain $\Omega \subset \mathbb{R}^2$. Can one predict those points $x \in \overline{\Omega}$, where $|\nabla u(x)|$ attains its maximum simply by looking at the shape of Ω? Those points mark the onset of plasticity and I call them points of maximal stress. This question was raised by Saint Venant in his classical treatise [SV, p. 444] from 1856, and it was answered in the positive for domains, the boundary of which is described in polar coordinates by $(r/r_0)^2 - a(r/r_0)^4 \cos 4\varphi = 1 - a$. There, Saint Venant writes

"Les points dangereux sont donc, comme dans l'ellipse et le rectangle, les points du contour les plus rapprochés de l'axe de torsion, ou les extrémités des petits diamètres."

In fact, it is easy to see (although it was not shown until 1930 by Pólya [Po]) that the point of maximal stress must lie on the boundary.

Lemma 2.1. *If u solves (2.1), (2.2), then $|\nabla u(x)|$ attains its maximum over $\overline{\Omega}$ on $\partial\Omega$.*

For the proof we differentiate $|\nabla u(x)|^2$ and show that it satisfies the differential inequality

$$\Delta(|\nabla u|^2) = 2 \sum_{i,j=1}^{2} \frac{\partial^2 u}{\partial x_i \partial x_j} + 2\nabla u \nabla(\Delta u) \geq 0$$

Now Lemma 2.1 follows from the maximum principle. □

It is not so easy to see, where on $\partial\Omega$ the points of maximal stress must be located. Take an ellipse for instance. There they are located on the short axes, a result why "may be startling to many" according to [TT, Vol. 1, Part II, §710]. On the other hand J. Boussinesq gave a heuristic explanation for this in [B, p. 200]. Imagine going down from the maximal point of u in the center of the ellipse to various boundary points. Going along the short axis will require more slope than following the long axis. This reasoning is suggestive as long as level sets of u are convex, and the convexity of level sets, given convex Ω, was not shown until 1971, see [ML]. In any case these results have lead people to believe in the general conjecture

Conjecture 2.2. For a doubly symmetric domain Ω, $|\nabla u|$ attains its maximum on the intersection of $\partial\Omega$ and the largest inscribed circle.

If Ω is not convex, this conjecture is false, as can be seen from an I-beam or a domain like the one in Figure 2.1, which was found by Saint Venant in 1859 and "rediscovered" in 1900 in [Fi].

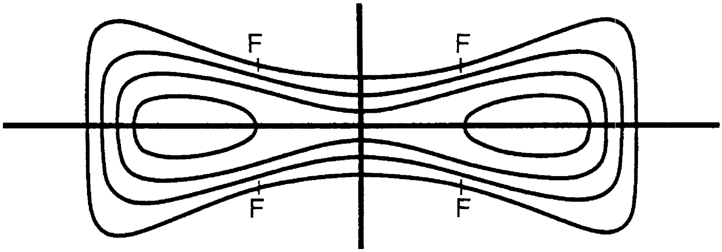

Figure 2.1 Saint Venant's rail

Under some additional assumptions on the geometry of Ω, I was able to prove the following result in 1985 [K].

Theorem 2.3. *Suppose that u solves (2.1), (2.2), and that $\Omega \subset \mathbb{R}^2$ is convex and symmetric with respect to x_1 and x_2. Suppose in addition that*

(2.3) $\partial\Omega$ *is of class* $C^{3,\alpha}$,

(2.4) *the curvature of* $\partial\Omega \cap \{x_1 > 0, x_2 > 0\}$ *is nondecreasing in* x_1.

Then $\max_{x\in\overline{\Omega}} |\nabla u(x)|$ *is attained only at those points* $(x_1, x_2) \in \partial\Omega$ *which have minimal distance to the origin. Furthermore, unless* $\partial\Omega$ *is a circle, there are precisely two points of maximal stress, namely the points of intersection of* $\partial\Omega$ *with the* x_2*-axis.*

The idea of proof is fairly simple. Let s denote arclength of $\partial\Omega$, increasing as we approach the x_2-axis along $\partial\Omega \cap \{x_1 > 0, x_2 > 0\}$. We want to show

$$\frac{\partial}{\partial s}|\nabla u|^2 \geq 0, \tag{2.5}$$

or equivalently

$$\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial n}\right) = \frac{\partial}{\partial n} \cdot \frac{\partial u}{\partial s} + \kappa\frac{\partial u}{\partial s} \leq 0 \quad \text{along } \partial\Omega. \tag{2.6}$$

But $\partial u/\partial s = 0$ along $\partial\Omega$, and to evaluate $\frac{\partial}{\partial n}\left(\frac{\partial u}{\partial s}\right)$ at a fixed point $x_0 \in \partial\Omega$, we fix $\overrightarrow{t}$ to be tangent to $\partial\Omega$ at x_0 and pointing in s-direction and obtain

$$\frac{\partial}{\partial n}\left(\frac{\partial u}{\partial s}\right) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\frac{\partial u}{\partial t}(x_0) - \frac{\partial u}{\partial t}(x_0 - \varepsilon n)\right) \leq 0 \tag{2.7}$$

or

$$\frac{\partial u}{\partial t}(x_0 - \varepsilon n) \geq 0 \tag{2.8}$$

as the desired inequality. But (2.8) follows from a well-known result of Gidas, Ni and Nirenberg, provided the lower cap that the normal to x_0 cuts off from Ω can be reflected across this normal into Ω. Now a finer analysis shows that (2.3), (2.4) lead to the reflection property (incidentally, there is a typographical error in [K,p. 200 line 2], the inequality sign has to be reversed) and that $|\nabla u|$ increases strictly unless $\partial\Omega$ is a circle. □

Remark 2.4. Of course there is an extension of Theorem 2.3 to quasilinear elliptic equations. See [K] for details.

Remark 2.5. Can one drop the assumptions (2.3) and (2.4), which were caused by the method of proof? This question has a negative answer. In fact G. Sweers has shown in [S1] that Conjecture 2.2 is not true for the barrel-shaped domain in Figure 2.2 or a nearby domain.

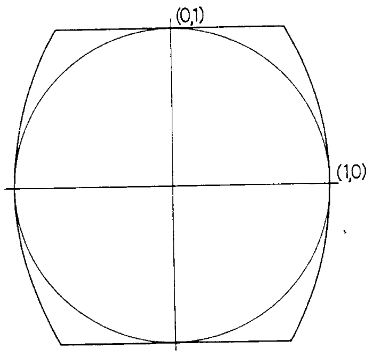

Figure 2.2 The barrel

In fact, as Ramaswamy showed in [R], the max of $|\nabla u|$ is attained on the horizontal, but not on the vertical axis. The proof of Theorem 2.3 can be easily extended to regular polygons, though.

In 1989 there was another and independent attempt [Ko] to prove Conjecture 2.2. Unfortunately it contained an error, as was later pointed out in [S2]. In fact, G. Sweers found yet another counterexample to Conjecture 2.2, a rhombus with rounded corners as in [K, Fig. 1(b)].

Third Problem

Imagine a threedimensional ball flying through a liquid. In 1685 I. Newton showed that the resistance of such a ball is half the resistance of a cylinder of same diameter, if flying in axial direction. In his words [N]:

"If in a rare medium, consisting of equal particles freely disposed at equal distances from each other, a globe and a cylinder described on equal diameter move with equal velocities in the direction of the axis of the cylinder, then the resistance of the globe will be half as great as that of the cylinder. ... I reckon that this proposition will be not without application in the building of ships."

How did Newton come up with such a statement? Let us think of the fluid as a rare gas, consisting of many free particles with large mean free paths. Suppose that these particles do not collide with each other, but that they interact with the ball or cylinder through at most one perfectly elastic collision. Other effect, such as friction, turbulence etc. are neglected in Newton's model. If the part of the body, which is exposed to such collisions, can be described by a function $u\colon \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ and if the fluid flows vertically downward, the portion of the momentum which a particle transfers to the body upon impact at $\big(x, u(x)\big)$ can be described by $\sin\alpha = (1 + |\nabla u|^2)^{-1/2}$.

The horizontal component of this portion will be balanced for rotational bodies by a corresponding momentum at $\big(-x, u(-x)\big)$. The vertical component, however, is of magnitude $(1 + |\nabla u|^2)^{-1}$, see Figure 3.1.

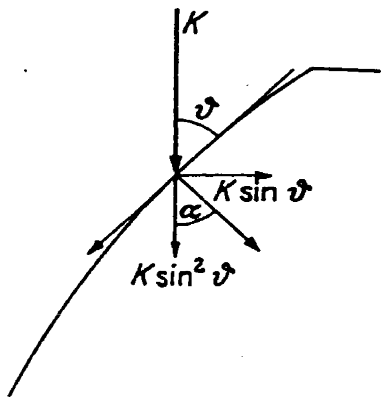

Figure 3.1 The sine-square pressure law

Therefore the total resistance of the body can be measured by

$$R(u) = \int\limits_{\Omega} \frac{1}{1 + |\nabla u(x)|^2} \, dx. \tag{3.1}$$

Now we calculate the resistance of a circular cylinder of radius 1 to be π. To calculate the resistance of a ball, we evaluate (3.1) at $u(x) = \sqrt{1 - |x|^2}$.

Since $\partial u / \partial r = -r(1 - r^2)^{-\frac{1}{2}}$, we obtain

$$R(u) = 2\pi \int_0^1 (1 - r^2) r \, dr = 2\pi \Big[\frac{r^2}{2} - \frac{r^4}{4} \Big]_0^1 = \pi/2,$$

i.e. a confirmation of I. Newton's observation that a ball has half the resistance of a cylinder.

Incidentally, Newton's considerations were illustrated by the drawing in Figure 3.2, and they have been extensively studied by D.T. Whiteside.

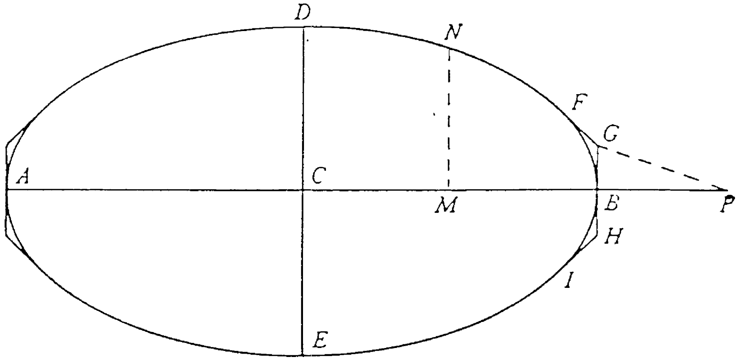

Figure 3.2 Newton's drawing

Another body of same resistance as the ball is the cone $u(r) = 1 - r$. Until recently [BK, BFK] the functional (3.1) was written differently (for radial functions) as

$$\int\limits_0^M \frac{v v'^3}{1 + v'^2} \, dt, \quad v(0) = 0, v(M) = R. \tag{3.2}$$

If M denotes the maximum of u and $0 \leq u \leq M$, then we can set $v = u^{-1}(M - t)$ and perform a simple change of variables. To do this we have to assume, that u or v are monotone.

So if we want to minimize (3.1) or the resistance of a body described by a function $u(x)$, we have to specify the class of admissible functions for u. Since $R(2u) < R(u)$ for nonconstant u, any class of admissible functions should be bounded in $L^\infty(\Omega)$. Moreover we want the body to have the property that

$$\text{every particle interacts at least once with the body.} \tag{3.3}$$

A sufficient criterion (but not a necessary one) is the convexity of the body or concavity of admissible functions. We set

$$C_M := \{v \in W^{1,\infty}_{\mathrm{loc}}(\Omega) \mid 0 \le v \le M, v \text{ concave}\}$$

and study the problem

$$\min_{v \in C_M} R(v). \tag{3.4}$$

Since R is bounded from below there exists a minimizing sequence $\{u_n\} \subset C_M$. Does it converge? If we write

$$R(v) = \int_\Omega f(|\nabla v(x)|)\, dx, \tag{3.5}$$

then standard problems from the calculus of variations have an integrand f which is convex and coercive, e.g. $f(s) = (1+s^2)^{-1}$ is neither coercive nor bounded, and so we cannot expect compactness of a minimizing sequence from the structure of f alone. This disadvantage is made up for by the set C_M, as was noticed by Marcellini.

Lemma 3.1. *If $\{u_n\} \subset C_M$, then $\{u_n\}$ has a subsequence, still denoted by $\{u_n\}$ which converges strongly in $W^{1,p}_{\mathrm{loc}}(\Omega)$ to a limit u.*

The proof of Lemma 3.1 can be found in [M] and [BK], but let me outline the idea. In a first step it is shown by geometric considerations that the pointwise estimate

$$|\nabla u_n(x)| \le \frac{2M}{\mathrm{dist}(x, \partial\Omega)} \tag{3.6}$$

holds for every $u_n \in C_M$, so that the sequence $\{u_n\}$ is locally uniformly Lipschitz continuous. By the Arzela–Ascoli theorem it has a uniform limit $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$, after passing to a subsequence. So $u_n \to u$ is $L^\infty_{\mathrm{loc}}(\Omega)$.

In a second step one has to show that $\nabla u_n \to \nabla u$ pointwise a.e. in Ω. This is done by taking difference quotients from the left and right, by using the concavity of u_n, and by proper limits. Convergence in $W^{1,p}_{\mathrm{loc}}$ follows from Lebesgue's dominated convergence theorem. □

Theorem 3.2 (Existence). *Problem* (3.4) *has a solution.*

Proof. In fact, let $\Omega\backslash\Omega'$ be a thin neighbourhood of $\partial\Omega$ in Ω, then Lemma 3.1 implies

$$\begin{aligned}\liminf_{n\to\infty} R(u_n) &= \liminf_{n\to\infty} \int\limits_{\Omega'} f(|\nabla u_n|)\,dx + \int\limits_{\Omega\backslash\Omega'} f(|\nabla u_n|)\,dx \\ &\geq \int\limits_{\Omega'} f(|\nabla u|)\,dx + \liminf_{n\to\infty} \int\limits_{\Omega\backslash\Omega'} f(|\nabla u_n|)\,dx \\ &= \int\limits_{\Omega} f(|\nabla u|)\,dx + \int\limits_{\Omega\backslash\Omega'} \{f(|\nabla u_n|) - f(|\nabla u|)\}\,dx \\ &\geq R(u) - 2|\Omega\backslash\Omega'|\,\|f\|_\infty.\end{aligned}$$

But since f is bounded, and since we can make $|\Omega\backslash\Omega'|$ arbitrarily small, we have

$$\liminf R(u_n) \geq R(u),$$

i.e. u is a solution of problem (3.4). □

Notice that the noncoerciveness of f has been extremely helpful in this proof, and that $W^{1,\infty}_{\text{loc}}(\Omega)$ is the natural function space for problem (3.4). Later I shall change the class C_M of admissible functions and come up with $W^{1,2}_{\text{loc}}(\Omega)$ or even $\text{BV}(\Omega)$ as natural function spaces.

Remark 3.3. Proving the existence of a solution has been a relatively easy task. But how about uniqueness? This appears to be a hard open problem. The functional $R(u)$ is not convex in u, for instance, which prohibits convexity arguments.

Another approach to proving uniqueness might be to show that without loss of generality solutions are ordered, i.e. if u and v are two solutions, then $w_1 = \min\{u, v\}$ and $w_2 0 \max\{u, v\}$ are solutions. Clearly $R(w_1) = R(w_2) = R(u) = R(v)$ holds in such a case, but unfortunately we cannot guarantee that $w_2 \in C_M$, i.e. that w_2 is admissible.

Those readers who are familiar with Newton's problem of minimal resistance will object to this Remark. If Ω is a circular disc, and if the class of admissible functions is restricted to radial functions in C_M, then u is known to be unique, and in fact in this case u has been known for centuries. Its representation is described in [BK] and its shape looks like the one in Figure 3.3.

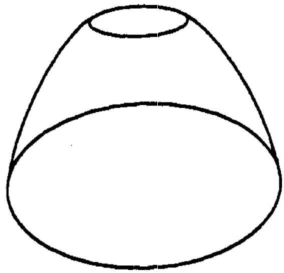

Figure 3.3 Optimal radial solution in C_M

Remark 3.4. It is remarkable that u is flat on top. In fact anybody working in fluid dynamics will immediately argue that the shape in Figure 3.3 cannot be optimal, because the body has a whole surface of stagnation points where the fluid will settle and cause frictional effects. This macroscopic point of view, although realistic for most fluids, neglects Newton's model may neglect effects from interactions among particles.

It is even more remarkable that Newton was aware of the advantage of "flatness", as can be seen from his drawing in Figure 3.2. The fact that $|\nabla u|$ is discontinuous has a simple reason: the nonconvexity of f in $R(v) = \int f(|\nabla v|)\,dx$. Along the boundary of the flat part, $|\nabla u|$ jumps from zero to at least one, and at 1 the convex lower envelope $\tilde{f}$ of f touches f again, see Figure 3.4.

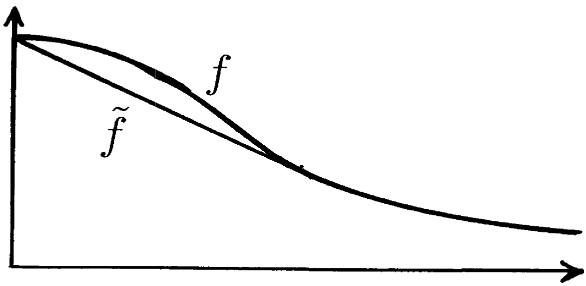

Figure 3.4 The functions f and $\tilde{f}$

Suppose that there is a set D of positive measure where $|\nabla u(x)| \in (0,1)$, the set where f differs from $\tilde{f}$. Then one can construct a function w which coincides with u outside D, and whose slope (or modulus of gradient) is 0 or 1 only. Define $w(x)$ to be the infimum of M and of all those tangent planes to the graph of u whose slope is outside $(0,1)$. It is easily seen that $w \in C_M$, provided u is, and that the equality sign holds in (3.7) below.

Moreover, an application of the coarea formula leads to the first inequality sign in (3.7). Therefore

$$R(w) = \tilde{R}(w) = \int_{\Omega} \tilde{f}(|\nabla w|)\, dx < \tilde{R}(u) < R(u), \tag{3.7}$$

a contradiction which proves

Lemma 3.5. *If u solves (3.4), then $|\nabla u(x)| \notin (0,1)$ almost everywhere in Ω.*

Remark 3.6. Note that $\tilde{f}(|\nabla u|)$ is convex in $|\nabla u|$, but not convex in u. Note also that the proof of Lemma 3.5 shows more: Minimizers of R over C_M are minimizers of $\tilde{R}$ over C_M and vice versa.

The classical questions in PDE are existence, uniqueness and regularity. To investigate regularity one looks for the Euler equation associated to (3.4).

If $u, v \in C_M$, then $(1-\varepsilon)u + \varepsilon v \in C_M$ for $0 < \varepsilon < 1$, and if u solves (3.4), then $R(u) \leq R((1-\varepsilon)u + \varepsilon v)$, so that u solves the variational inequality

$$\int_{\Omega} \frac{\nabla u \nabla (v-u)}{(1+|\nabla u|^2)^2}\, dx \leq 0 \quad \text{for any} \quad v \in C_M. \tag{3.8}$$

Now consider the set N of those points $x \in \Omega$, where $v - u$ can vary in sign, i.e. where $0 < u(x) < M$ and where the matrix of second derivatives $D^2u(x)$ is negative.

Then the Euler equation in weak form can be stated as

$$\int_{\Omega} \frac{\nabla u(x)}{(1+|\nabla u(x)|^2)^2} \nabla \varphi(x)\, dx = 0 \quad \text{for any} \quad \varphi \in C_0^\infty(N), \tag{3.9}$$

and formally, after integration by parts, it reads

$$\operatorname{div}\Big(\frac{\nabla u(x)}{(1+|\nabla u(x)|^2)^2}\Big) = 0 \quad \text{in } N, \tag{3.10}$$

or in curvilinear coordinates (n = normal, t = tangent to a level line of u)

$$(1+u_n^2)u_{tt} + (1-3u_n^2)u_{nn} = 0. \tag{3.11}$$

This equation is of elliptic-hyperbolic type. It is hyperbolic where $|\nabla u(x)| > 1/\sqrt{3}$, i.e. on the non-flat part of u. The "natural boundary condition" on $\partial\Omega \cap N$ is

$$\frac{u_n}{\left(1+|\nabla u(x)|^2\right)^2} = 0, \tag{3.12}$$

i.e. $u_n(x) = 0$ or $u_n = -\infty$ on $\partial\Omega \cap N$. But in the first case $u(x)$ must equal M, so $x \notin N$. And on $\partial\Omega \backslash (\partial\Omega \cap N)$ we can have $u = 0$ or a vanishing eigenvalue for $D^2u(x)$ as well. In summary, little seems to be known about the boundary behaviour of u except that $u(x)$ vanishes in at least one boundary point.

Remark 3.7 (On Symmetry). Figure 3.3 shows the optimal radial solution in C_M, but is it the solution of problem (3.4)? Is it the optimal solution in C_M? In other words, if Ω has symmetries, does u have symmetries? This appears to be a nontrivial open problem. There are many tricks in the calculus of variations to prove symmetry of minimizers, and many of them seem to fail due to the nonconvexity of $\tilde{R}$ in its argument u. Let me list some strategies that fail:

a) Show that u is unique, then it must be radial. This has been discussed in Remark 3.3.

b) Replace u by its spherical mean

$$\underline{u}(r) = \frac{1}{2\pi}\int_0^{2\pi} u(r,\varphi)\; d\varphi,$$

show that $u \in C_M$ and hope that $R(\underline{u}) < R(u)$. A counterexample is provided by $u(x,y) = 1 - |x|$.

c) Replace u by $u^{\#}$, its Schwarz-symmetrization as defined in the lectures of G. Talenti, show that $u \in C_M$ implies $u^{\#} \in C_M$ and hope that $R(u^{\#}) < R(u)$. Unfortunately, $u(x,y) = 1 - \min\{|x|,|y|\}$ provides a counterexample.

d) Replace $|\nabla u| = w$ by its radially increasing symmetrization $w_{\#}$ and set $w_{\#} = -\partial v/\partial r$. Then $R(v) = R(u)$ and $v \in C_M$ and one might hope that $\|v\|_\infty < \|u\|_\infty$, in which case there exists $\alpha > 1$ such that $\alpha v \in C_M$ and $R(\alpha v) < R(v) = R(u)$. G. Aronsson found out that this hope is unjustified, because $u(x,y) = \sqrt{1-y^2} - |x|$ serves as a counterexample. Incidentally, this approach leads to an interesting question for Hamilton–Jacobi–Bellmann equations. Suppose $|\nabla u(x)| = h(x)$ is given and h can vary over its equimeasurable rearrangements. For which function h is the L^∞-norm of u extremal? This question has been addressed in a recent paper

[FPV] of Ferone, Posteraro and Volpicelli, who found out that there is no general answer.

e) Replace $|\nabla u| = w$ by its spherical mean, set $\underline{w} = -\partial v/\partial r$ and show that $R(v) < R(u)$. In fact, $\tilde{R}(v) \leq \tilde{R}(u)$ follows from Jensen's inequality. Unfortunately, the counterexample from d) shows that v is not necessarily in C_M, because its L^∞-norm can exceed M.

In summary, there are many open questions on solutions to problem (3.4). When I told these questions to my friends and colleagues in Praha, they suggested to look for solutions in a different class of admissible functions. Equations of hyperbolic-elliptic type occur also in transsonic flow problems, and those equations can have multiple (nonphysical) solutions. The physically meaningful solution can be extracted by imposing an entropy condition on the class of admissible functions. In our case this condition amounts to requiring $\Delta u \leq 0$, i.e. less than concavity of u, and the notation of "entropy" has no physical meaning and was chosen for purely formal reasons of analogy. In [FN] entropy has a meaning.

A week formulation of the entropy condition is

$$-\int_\Omega \nabla u(x)\nabla\varphi(x)\,dx \geq 0 \quad \text{for every } \varphi \in H_0^1(\Omega), \tag{3.13}$$

and (3.13) suggests the use of $H^{1,2}_{\text{loc}}(\Omega)$ as the appropriate function space. Therefore, let us set

$$E_M = \{v \in H^{1,2}_{\text{loc}}(\Omega) \mid 0 \leq v \leq M \quad \text{in } \Omega,\ v \text{ satisfies (3.13)}\}$$

and investigate the problem

$$\min_{v\in E_M} R(v) \tag{3.14}$$

Fortunately, the set E_M, although larger than C_M, still has some compactness properties.

Lemma 3.8. *If $\{u_n\}$ is a sequence in E_M, then for every $\Omega' \subset\subset \Omega$ we have*

$$\|u_n\|_{H^{1,2}(\Omega')} \leq C(\Omega', M), \tag{3.15}$$

i.e. the sequence is uniformly bounded in $H^{1,2}_{\text{loc}}(\Omega)$. Moreover, for every $\alpha > 0$ there is a small set A_α of measure $|A_\alpha| < \alpha$ and a subsequence such that $\nabla u_n \to \nabla u$ strongly in $L^2_{\text{loc}}(\Omega\backslash A_\alpha)$.

In the proof of Lemma 3.8 one chooses $\varphi = \eta^2(M - u_n)$ as a test function, with η being a usual cut-off function. This leads to (3.15) and to weak convergence in $H^{1,2}_{\mathrm{loc}}(\Omega)$ or strong convergence in $L^2(\Omega)$ of a subsequence. The superharmonicity of admissible function is helpful in the proof of strong convergence in $H^{1,2}_{\mathrm{loc}}(\Omega)$ as well. Rather than give the details of proof, which can be found in [BFK], let me give a heuristic reason. Sequences in $H^{1,2}$ which are weakly but not strongly convergent show oscillatory behaviour. In particular, their second derivatives change sign. But (3.13) prohibits such sign chances. $\square$

As a consequence of Lemma 3.8 there is an existence result:

Theorem 3.9. *Problem* (3.14) *has a solution.*

The proof follows the same reasoning as the proof of Theorem 3.2. $\square$

Aside from the existence result, all the questions that were discussed for problem (3.4) remain open for problem (3.14); in particular, uniqueness, regularity and symmetry. Not even an analogue of Lemma 3.5 seems to be known for solutions of (3.14). It is interesting to note, however, that the properties of admissible functions determine the underlying function space. In fact, a first tentative description had

$$\tilde{E}_M = \{v \in \mathcal{D}'(\Omega) \mid 0 \le v \le M, -\Delta v \ge 0 \quad \text{in } \mathcal{D}'(\Omega)\}.$$

as class of admissible functions, but (3.15) justifies the use of $H^{1,2}_{\mathrm{loc}}$ as appropriate function space in the "entropy case".

Remark 3.10. Another generalization of concave functions are quasiconcave functions, i.e. functions with the property that the set $\{x \in \Omega \mid v(x) \ge c\}$ are all convex. If C_M is replaced by the class of bounded, quasiconcave functions, a companion to Lemma 3.8 states that sequences in this class of functions are uniformly bounded in $BV(\Omega)$, the space of functions of bounded variation. In this case, however, we cannot even give an existence result for minima of the original functional R. Instead even the functional $\tilde{R}$ has to be suitably modified to obtain an existence result, see [BFK] for details.

Remark 3.11. As noted earlier, concave functions have the property (3.3) that every particle interacts at most once with the body. If we take (3.3) as a characterization of admissible functions

$$P_M := \{v \in W^{1,\infty}_{\mathrm{loc}}(\Omega) \mid 0 \le v \le M, v \text{ satisfies } (3.3)\},$$

we can study the problem

$$\min_{v \in P_M} R(v), \tag{3.16}$$

If this problem has a solution, then it will lie outside C_M. In fact, instead of Newton's optimal shape for Ω a circular disk, modify the flat part of Newton's solution into a cone of opening angle exceeding $2\pi/3$, see Figure 3.5.

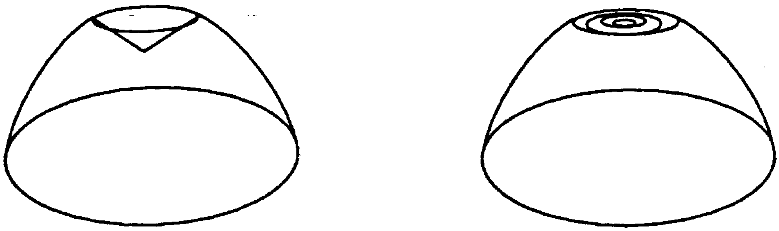

Figure 3.5 "better" profiles than Newton's

Part of this cone can be flipped up as in Figure 3.5 without changing the resistance of the profile. In fact, one can construct a sequence of oscillating solutions which tends to Newton's solution (with a rough surface on the flat part) in L^∞, but not in $W^{1,\infty}_{\mathrm{loc}}(\Omega)$. Such a development of microstructure and lack of weak lower semicontinuity is due to the nonconvexity of the functional R. At present, the investigation of problem (3.16) is still going on but there are some preliminary results.

Lemma 3.12. *Given M, the set P_M is bounded in $W^{1,\infty}_{\mathrm{loc}}(\Omega)$. In fact, for $v \in P_M$ we have the pointwise estimate*

$$|\nabla v(x)| \leq \frac{M + \sqrt{M^2 + \mathrm{dist}^2(x, \partial\Omega)}}{\mathrm{dist}(x, \partial\Omega)} \quad \textit{in } \Omega. \tag{3.17}$$

For the proof of Lemma 3.12 we note that $\nabla v(x)$ and the vertical axis span a plane, and in this plane any reflected particle path represents an upper bound for v. The rest is trigonometry and arithmetic. I refer to [BFK] for details. □

Unfortunately, unlike Lemma 3.1, Lemma 3.12 does not provide pointwise convergence of the gradients of a minimizing sequence. Therefore the existence question for (3.16) appears to be open.

Remark 3.13. Suppose that a solution of (3.16) sits on the boundary of the admissible set P_M, and thus fails to satisfy the Euler equation almost everywhere. Then a following kind of question from geometric optics comes up (at least in the radially symmetric case). What is the shape of a mirror with the property that every vertically incoming ray is reflected through the axis of rotation and leaves at the edge of the mirror? So parallel light would cause a halo to appear on the outer rim. This question has a simple answer, a polynomial of degree 2 in r, which is not differentiable at zero. If the flat part of Newton's solution is replaced by such a "mirror", the resulting profile has less resistance than the ones in Figure 3.5.

There are many more questions about Newton's problem than answers, and I shall address only one additional question. What if Ω is not a disc but a square and if u is a solution to problem (3.3), i.e. u minimizes R in C_M? To a certain extent this question can be answered by a computer. We replace C_M by a set of piecewise linear functions and try to minimize the corresponding discrete functional. But how can we tell that a modification of an approximate solution u_h^n in a single nodal point, which might lower the value of $R(u_h)$, results in a concave function u_h^{n+1}? Concavity is a nonlocal property.

For lack of time let me skip the details and be vague about answering this question. Details are written in [KS]. Concave functions have the property that their negative gradient is a monotone operator or, in other words, that their graph bends only one way. So if it bends the wrong way, the product $(\nabla u(x_k) - \nabla u(x_{k+1}))(x_k - x_{k+1})$, is positive, where x_k and x_{k+1} are centers of adjacent triangles in a triangulation of Ω. This defect is heavily penalized in [KS] by modifying the functional R; and then one can show that finite element solutions of the penalized problem converge to a concave solution of the continuous problem as the grid-size goes to zero. Similar to penalizing absence from C_M one can penalize the violation of $\Delta u \leq 0$ and obtain a convergent finite element method for problem (3.14).

A finite element solution in C_M is depicted for $M = 2$ in Figure 3.6. Here Ω is the square $[-1,1]^2$, and the plot shows $[0,1]^2$ and u on this northeast quarter square. The jumps in gradient seem to be no numerical artefacts. Hyperbolic equations can have nonsmooth solutions. Moreover, Figure 3.6 shows that u does not need to vanish everywhere on $\partial\Omega$. If you reflect the graph of u it looks like the nose of a TGV train, which I once saw at Holešovice station in Praha, and whose sight apparently caused my curiosity about the subject of minimal resistance. And as you all noted during this conference, Praha is worth visiting more than once.

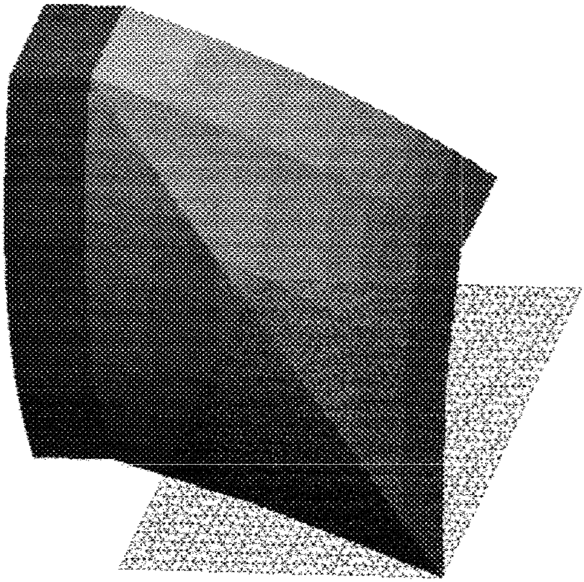

Figure 3.6 Finite element solution of (3.4) in a quarter square

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