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Superlinear equations, potential theory and weighted norm inequalities

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0 Introduction

We give a survey of some recent results on the solvability of certain superlinear differential and integral equations with minimal restrictions on the regularity of the coefficients and data, and related weighted norm inequalities. Our approach is based on harmonic analysis and functional analysis methods. We define function spaces intrinsically connected with the nonlinear problems, and use discrete models for operators involved. Our characterizations are not only sufficient but also necessary. A crucial role is played by the corresponding weighted norm inequalities, with a careful analysis of the embedding constants.

Note that we avoid using more sophisticated techniques of weighted norm inequalities and nonlinear potential theory which are not applicable directly to the solvability problems studied in this paper. However, many notions and ideas developed in that framework are used here, sometimes in a modified form. Moreover, our methods related to nonlinear equations lead naturally to new characterizations and simpler proofs for some classical multidimensional integral inequalities which involve Riesz potentials, Green's potentials, and other integral operators.

In Section 1 we discuss joint work with N. J. Kalton [KV] on the existence of positive solutions for superlinear integral equations of the type

$$u = T(u^q) + f, \qquad u \geq 0, \tag{0.1}$$

where $1 < q < \infty$, $f \geq 0$, and T is a linear integral operator with positive kernel $K(x, y)$,

$$T f(x) = \int_{\Omega} K(x, y)\, f(y)\, d\nu(y), \quad x \in \Omega,$$

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on a measure space (Ω, ν). The main tool in our study of (0.1) is its connection with weighted norm inequalities of the type

$$||T^* h||_{L^p(d\omega)} \le C\, ||h||_{L^p(d\nu)},$$

for all $h \in L^p(d\nu)$, where T^* is a formal adjoint operator, $1/p + 1/q = 1$, and $d\omega = f^q\, d\nu$. Most of our results can be proved for arbitrary inhomogeneous terms $f \ge 0$, measures ν, and integral operators T under the assumption that $\rho(x, y) = 1/K(x, y)$ is equivalent to a quasi-metric on Ω.

We consider a number of examples which can be reduced to (0.1) starting from the following one-dimensional equation of Riccati type on the half-line $\mathbb{R}_+$:

$$y'(x) = v(x)\, y^q(x) + w(x), \quad y(x) \ge 0, \quad y(0) = 0, \tag{0.2}$$

with arbitrary nonnegative coefficients v and data w. Many different characterizations of the solvability for this equation are known, as well as its connection to the Schrödinger equation and Hardy's inequality with weights. (See e.g. [Har], [To], [Mu], [KuT], [G], [Im].)

However, our primary motivation is to develop methods applicable to multidimensional problems and higher order differential operators, including the so-called q-Schrödinger equation [KV]

$$-\Delta u = \nu\, u^q + \omega, \quad u \ge 0, \tag{0.3}$$

or the multidimensional Riccati equation [HMV]

$$-\Delta u = \nu\, |\nabla u|^q + \omega, \tag{0.4}$$

where ν and ω are arbitrary nonnegative measurable functions (or measures). (See also [L], [AP], [BCa], [Ba], [BaP] for related results and alternative methods.)

More general differential equations with uniformly elliptic second order operators in place of the Laplacian, as well as higher order equations and equations with nonlocal operators can be attacked using a similar approach. In particular, we give a criterion for the solvability of the following integral equation with Riesz potentials $I_\alpha = (-\Delta)^{-\alpha/2}$, $0 < \alpha < n$:

$$u(x) = \int_{\mathbb{R}^n} \frac{[u(y)]^q}{|x-y|^{n-\alpha}}\, d\nu(y) + f(x), \quad x \in \mathbb{R}^n. \tag{0.5}$$

This completes some earlier results of [VW1].

In Sec. 2 we give a detailed proof of our main results on the solvability of equation (0.1) for the integral operator T on $(\mathbb{R}^n, \nu)$ whose kernel is defined by

$$K(x,y) = \sum_{Q \in \mathcal{D}} c_Q \, \chi_Q(x) \, \chi_Q(y). \tag{0.6}$$

Here c_Q is an arbitrary fixed sequence of nonnegative numbers, and $\mathcal{D}$ is the family of all dyadic cubes on $\mathbb{R}^n$. This *dyadic model* makes it possible to demonstrate in a clear way the main ideas of a more general construction presented in [KV], where the geometry associated with the quasi-metric $\rho = 1/K$ was extensively exploited. (Note that for the kernel defined by (0.6) the corresponding quasi-metric balls are dyadic cubes.)

We observe that dyadic operators of this type have been used previously in the literature. In particular, some of our arguments resemble the *original* proof of Th. Wolff's inequality, which plays an important role in potential theory. (See [HW], [AH], and also Sec. 4 of the present paper.) Similar dyadic models were also applied to a number of linear problems, e.g., in the theory of Triebel-Lizorkin spaces [FrJ], [V2], weighted norm inequalities [FSt], [S2], [V1], [VW2], Schrödinger equations and Toeplitz operators [Ro], Haar multipliers [NTV], etc. They are easier to investigate, and serve as a good approximation for more difficult problems of harmonic analysis, potential theory, and PDEs. Moreover, some concrete kernels which appear in applications can be reduced directly to (0.6). For instance, letting $c_Q = |Q|^{\alpha/n-1}$ we can characterize the solvability of the corresponding equation with Riesz potentials (0.5).

In Sec. 3 we are concerned with the solvability of the multidimensional Riccati equation

$$-\Delta u = |\nabla u|^q + \omega,$$

where $q > 1$, and ω is a nonnegative measurable function (or measure) on Ω. This equation is of a special interest because of its connection to the Schrödinger equation $\Delta u + \omega u = 0$ in the case $q = 2$. We also give criteria of solvability for more general semilinear equations of the type $-\Delta u = f(x, u, \nabla u) + \omega$ where $f(x, u, \nabla u) \asymp a \, |\nabla u|^{q_1} + b \, |u|^{q_2}$ ($q_1 > 1$, $q_2 > 1$). This work is joint with K. Hansson and V. Maz'ya [HMV].

In Sec. 4 we discuss characterizations of the embedding of the Sobolev space $W^{l,p}$ into $L^q(\omega)$, for an arbitrary measure ω, in the difficult "upper triangle case" $q < p$. These results are joint with C. Cascante and J. Ortega [COV].

1 Superlinear equations and weighted potential theory

Let (Ω, ν) be a measure space with σ-finite measure ν and let $L^0(\nu)$ be the space of (equivalence classes of) Borel functions on Ω. By $L^0_+(\nu)$ we denote the cone of nonnegative functions in $L^0(\nu)$. In this section we are concerned with superlinear inhomogeneous equations of the type

$$u = T(u^q) + f, \quad u \in L^0_+(\nu), \tag{1.1}$$

where $1 < q < \infty$, $f \in L^0_+(\nu)$, and T is a linear integral operator on $L^0(\nu)$ defined by

$$Tf(x) = \int_\Omega K(x,y)\, f(y)\, d\nu(y), \quad x \in \Omega, \tag{1.2}$$

where $K(x,y)$ is a positive kernel function on $\Omega \times \Omega$. (More general theory for arbitrary positive operators T which preserve $L^0_+(\nu)$ is developed in [KV].) Let us denote by $S = S_{q,K}$ the set of all $f \geq 0$ such that (1.1) has a solution $u \in L^0_+(\nu)$. We also define the space $\mathcal{Z} = \mathcal{Z}_{q,K}$ which consists of all $f \in L^0(\nu)$ such that the equation

$$u = T(u^q) + \varepsilon |f|, \quad u \in L^0_+(\nu), \tag{1.3}$$

has a solution for some $\varepsilon > 0$, i.e., $\varepsilon |f| \in S$. Under certain mild restrictions on K it can be shown that $\mathcal{Z}$ is a Banach space with norm defined by

$$||f||_{\mathcal{Z}} = \inf \{\lambda > 0 : \, \lambda^{-1} |f| \in S\}. \tag{1.4}$$

Thus S consists of $f \geq 0$ lying in the unit ball of $\mathcal{Z}$.

It is convenient to introduce a nonlinear operator $\mathcal{A}$ associated with (1.1) defined by $\mathcal{A} f = T(f^q)$ so that (1.1) may be rewritten as $u = \mathcal{A} u + f$. Note the following obvious properties of $\mathcal{A}$:

$$\mathcal{A}(\lambda f) = \lambda^q \, \mathcal{A} f, \quad [\mathcal{A}(f+g)]^{1/q} \leq (\mathcal{A} f)^{1/q} + (\mathcal{A} g)^{1/q}.$$

A crucial role in the study of the solvability problem for (1.1) is played by the fact that $\mathcal{Z}$ turns out to be invariant under $\mathcal{A}$.

A predual space to $\mathcal{Z}$ can be identified with the Banach space $\mathcal{Z}'$ of all $g \in L^0_+(\nu)$ such that

$$||g||_{\mathcal{Z}'} = pq^{p-1} \inf \left\{ \int_\Omega \frac{h^p}{(T^* h)^{p-1}} d\nu : \, h \geq |g| \right\} < \infty,$$

where $1/p+1/q=1$, and T^* is a formal adjoint operator with kernel $K(y,x)$.

In other words, the following *dual reformulation* of the solvability problem for equation (1.1) holds, which is essentially due to Baras and Pierre [BaP] (a shorter proof along with some modifications can be found in [KV]).

Theorem 1. *Let $1<q<\infty$ and let $f \in L^0_+(\nu)$. Then* (1.1) *has a solution if and only if*

$$\int_\Omega f\, g\, d\nu \le \frac{1}{pq^{p-1}} \int_\Omega \frac{g^p}{(T^*g)^{p-1}} d\nu, \quad g \ge 0.$$

However, in this paper we are interested in more explicit criteria for the solvability of (1.1). To this end we introduce the corresponding weighted norm inequalities of the type

$$||T^*h||_{L^p(d\omega)} \le C\, ||h||_{L^p(d\nu)}, \qquad h \in L^p(d\nu), \tag{1.5}$$

where $1/p+1/q=1$. More precisely, the existence of a solution to (1.1) can be expressed in terms of the best constants C_n in the weighted norm inequalities

$$||T^*h||_{L^p(f_n^q d\nu)} \le C_n\, ||h||_{L^p(d\nu)}, \qquad h \in L^p(d\nu), \tag{1.6}$$

for the iterations $f_n = \mathcal{A}^n f$, $n=0,1,2,\ldots$. Explicit estimates of these constants, based on the techniques discussed in the next section, lead to the following main result which holds for a wide class of "quasi-metric" kernels $K(x,y)$ ([KV]).

We say that $K>0$ is a *quasi-metric* kernel if K is symmetric, i.e., $K(x,y)=K(y,x)$, and there is a constant $\kappa \ge 1$ such that for all $x,y,z \in \Omega$ it follows

$$\frac{1}{K(x,y)} \le \kappa \left[\frac{1}{K(x,z)} + \frac{1}{K(z,y)}\right]. \tag{1.7}$$

Under this assumption it is natural to introduce the quasi-metric $\rho(x,y) = 1/K(x,y)$. Note however that we do not assume that $K(x,x)=\infty$ and so $\rho(x,x)>0$ is possible. We can then also define the ball of radius $r>0$, i.e.,

$$B_r(x) = \{y : \rho(x,y) \le r\}$$

but note that this set can be empty. A large class of examples is created by choosing a metric d on Ω and letting $K(x,y)=d(x,y)^{-\alpha}$ for some $\alpha>0$; this kernel defines a generalized operator of fractional integration.

Theorem 2. *Let $1 < q < \infty$ and let $f \in L^0_+(\nu)$. Suppose that T is an integral operator with positive kernel K.*

(1) *Equation* (1.1) *has a solution if $T f^q(x) < \infty$ $d\nu$-a.e. and*

$$T(T f^q)^q(x) \le C\, T f^q(x) \quad d\nu\text{-a.e.} \tag{1.8}$$

with $C = q^{-q} p^{q(1-q)}$.

(2) *Conversely, suppose that* (1.1) *has a solution and that K is a quasi-metric kernel with quasi-metric constant κ. Then* (1.8) *holds with a constant $C = C(q, \kappa)$ which depends only on q and κ.*

Remark 1. Inequality (1.8) can be rewritten as

$$\mathcal{A}^2 f \le C\, \mathcal{A} f < \infty \quad d\nu\text{-a.e.}, \tag{1.8$'$}$$

where the constant $C = q^{-q} p^{q(1-q)}$ is sharp. The sufficiency of this condition is not difficult to verify using simple iterations:

$$u_{n+1} = \mathcal{A} u_n + f, \quad n = 0, 1, 2, \ldots,$$

starting from $u_0 = 0$. It follows by induction that if (1.8$'$) holds with $C = q^{-q} p^{q(1-q)}$, then

$$u_n \le u_{n+1}, \quad \text{and} \quad f + \mathcal{A} f \le u_n \le f + p^q\, \mathcal{A} f. \tag{1.9}$$

Hence there exists a solution $u(x) = \lim_{n\to\infty} u_n(x)$ such that

$$f(x) + \mathcal{A} f(x) \le u(x) \le f + p^q\, \mathcal{A} f(x). \tag{1.10}$$

The same fact can also be derived from a well-known fixed-point theorem for lattices which goes back to Garret Birkhoff (see [Bi], [KrZ]),

Remark 2. A simpler condition

$$\mathcal{A} f \le C\, f < \infty \quad d\nu\text{-a.e.}, \tag{1.8$''$}$$

with $C = q^{-1} p^{1-q}$ is obviously sufficient, but generally not necessary for the solvability of (1.1) even for nice kernels K. However, it turns out to be necessary if the right-hand side f of (1.1) is "smooth enough" as indicated in the discussion below.

In the following theorems we concentrate on operators with quasi-metric kernels K. A number of important equivalent reformulations of (1.8) in geometric terms (via the quasi-metric $\rho(x,y) = 1/K(x,y)$), as well as in terms of the corresponding capacitary and weighted norm inequalities, are listed below. Recall that we denote by $B_r(x)$ the ball of radius r centered at x for the quasi-metric ρ. Since the kernel $K(x,y)$ is now assumed to be symmetric, we no more distinguish between T and T^*.

Theorem 3. *Let $1 < q < \infty$ and let $f \in L^0_+(\nu)$. Let $d\omega = f^q\, d\nu$. Suppose that T is an integral operator with quasi-metric kernel K. Then the following statements are equivalent.*

(1) *$f \in \mathcal{Z}$, i.e., for some $\varepsilon > 0$ there exists a solution u of the equation $u = Tu^q + \varepsilon f$.*

(2) *The inequality*

$$T(Tf^q)^q(x) \le C\, Tf^q(x) < \infty \quad d\nu\text{-a.e.}$$

holds, where C is a constant which is independent of x.

(3) *Both the "infinitesimal inequality"*

$$\sup_{a>0}\ \operatorname*{ess\,sup}_{x\in\Omega} \left(\int_0^a \frac{|B_r(x)|_\nu}{r^2}\,dr\right)^{1/q} \left(\int_a^\infty \frac{|B_r(x)|_\omega}{r^2}\,dr\right)^{1/p} < \infty \qquad (1.11)$$

and the weighted norm inequality

$$||Tg||_{L^p(d\omega)} \le C\, ||g||_{L^p(d\nu)}, \qquad g \in L^p(\nu), \qquad (1.12)$$

hold.

The weighted norm inequality (1.12) *in statement* (3) *can be replaced by the corresponding weak-type inequality*

$$||Tg||_{L^{p,\infty}(d\omega)} \le C\, ||g||_{L^p(d\nu)}, \qquad (1.13)$$

or by a testing condition of Sawyer type

$$\int_B \left[\int_B K(x,y)\, d\omega(y)\right]^q d\nu(x) \le C\, |B|_\omega, \qquad (1.14)$$

where $B = B_r(x)$ is an arbitrary quasi-metric ball.

Remark 3. We call (1.11) the infinitesimal inequality because of the method of the proof sketched in the next section. It boils down to $L^\infty(\nu)$-estimates of $(\mathcal{A}^n f)^{1/q^n}$ as $n \to \infty$ derived from L^p-estimates of the type (1.12) with iterated weights $d\omega_k = (\mathcal{A}^n f)^q\, d\nu$.

Remark 4. Note that any one of the inequalities (1.11), (1.12), (1.13), or (1.14) is generally stronger than the usual two weight Muckenhoupt condition which in this setting can be stated as

$$(|B_r(x)|_\nu)^{1/q} \, (|B_r(x)|_\omega)^{1/p} \le C\, r, \tag{1.15}$$

where the constant C is independent of $x \in \Omega$ and $r > 0$. However, generally neither of the inequalities (1.11) and (1.12) implies the other one. It can be shown using the results of [SWZ] that, for quasi-metric kernels, (1.13)⇔(1.14), but our proof of Theorem 3 is independent of this fact.

Remark 5. Theorem 3 holds true for kernels which are not necessarily symmetric. It is enough to assume that $(1/K(x,y)) \asymp \rho(x,y)$, where ρ is symmetric and satisfies the quasi-metric inequality $\rho(x,y) \le \kappa[\rho(x,z) + \rho(y,z)]$. (See also [VW2].)

The following version of Theorem 3 is important in applications to differential equations, which in many cases are reduced to integral equations whose right-hand side f has a special form. Usually f coincides with the Green potential of the data of the original differential equation and is already "smooth enough". To address this situation, we now assume that $d\omega$ is a given measure on Ω, and that $f = K\omega$, where $K\omega$ is the potential of ω defined by

$$K\omega(x) = \int_\Omega K(x,y)\, d\omega(y), \quad x \in \Omega.$$

In this case we can simplify condition (1.8), which involves second iterations $\mathcal{A}^2 f = T(Tf^q)^q$, by using only first iterations $\mathcal{A}f = Tf^q$.

Theorem 4. *Let $1 < q < \infty$, and let ν, ω be arbitrary σ-finite measures on Ω. Suppose that T is an integral operator with quasi-metric kernel K and $f = K\omega$. Then the following statements are equivalent.*

(1) $f \in \mathcal{Z}$.

(2) $T(f^q)(x) \le C\, f(x) < \infty \quad d\nu$*-a.e.*

(3) *Both the "infinitesimal inequality"* (1.11) *and the weighted norm inequality* (1.12) *hold.*

As in Theorem 3, the weighted norm inequality (1.12) in statement (3) can be replaced by the corresponding weak-type inequality (1.13), or by the testing condition (1.14).

We now discuss connections with capacitary inequalities which characterize the problems studied above in geometric terms. Let ω be a σ-finite measure on Ω. It is easy to see that the weak-type inequality (1.13) is equivalent to the capacitary condition

$$|E|_\omega \le C\,\mathrm{Cap}\,(E), \tag{1.16}$$

for all Borel sets $E \subset \Omega$; here $\mathrm{Cap}\,(E)$ is defined by

$$\mathrm{Cap}\,(E) = \inf\left\{ \int_\Omega g^p\, d\nu \,:\, g \in L^p_+(\nu),\ Tg \ge \chi_E \right\}. \tag{1.17}$$

This class of measures and its relation to embedding theorems for Sobolev spaces and spectral properties of the Schrödinger operator were studied in the pioneering work of V. Maz'ya in the early 1960's and 1970's (see [M1], [M2], [M3], [AH], and the literature cited there).

In the case of Riesz potentials $T = I_\alpha$ on $\mathbb{R}^n$ and $d\nu = dx$ it is known ([M3], [AH], [VW1]) that conditions (1.12)–(1.14) and (1.16) are equivalent to one another, and are strictly stronger than (1.11) and (1.15), which coincide with Frostman's condition

$$|B_r(x)|_\omega \le C\, r^{n-\alpha p}$$

for all Euclidean balls $B_r(x)$ of radius r. (Note that $\mathrm{Cap}\,(B_r(x)) = C(n,p,\alpha)\, r^{n-\alpha p}$ in this case.)

These facts are generalized to integral operators with quasi-metric kernels and any measure ν under the following sharp restriction [KV] (see also Theorem 9 below). Suppose that for some constant C and every $x \in \Omega$ and $a > 0$ we have

$$\int_0^a \frac{|B_r(x)|_\nu}{r^2} dr \le C a^{q-1} \int_a^\infty \frac{|B_r(x)|_\nu}{r^{1+q}} dr < \infty. \tag{1.18}$$

Roughly speaking this condition implies that the behavior of the kernel at infinity dominates the behavior locally, which eliminates the need to use the infinitesimal inequality.

Theorem 5. *Let $1 < q < \infty$, and let ν, ω be σ-finite measures on Ω. Let K be a quasi-metric kernel on Ω such that* (1.18) *holds. Then the following statements are equivalent.*

(1) *ω satisfies the weighted norm inequality* (1.12).

(2) *ω satisfies the capacitary condition* (1.16) *for all Borel sets E, or, equivalently, the weak-type inequality* (1.13) *holds.*

(3) ω *satisfies the testing condition* (1.14).

(4) $f = K\omega \in \mathcal{Z}$.

(5) *There is a constant* C *so that* $T(f)^q \leq C\, f$.

Remark 6. The quantity on the right-hand side of (1.18) also appears in the following two-sided estimate for the capacity of a ball $B = B_a(x)$:

$$\mathrm{Cap}\,(B) \asymp \left(\int_a^\infty \frac{|B_r(x)|_\nu}{r^{1+q}} dr \right)^{-p/q},$$

which is established in [KV] for a wide class of kernels K and arbitrary underlying measure ν. For Riesz potentials and $\nu \in A_\infty$ this estimate is due to D. Adams (see [AH]).

Remark 7. Hypothesis (1.18) of Theorem 5 can be replaced by the assumption that for some C and every $x \in \Omega$, $a > 0$ both of the following conditions hold:

$$\int_0^{2a} \frac{|B_r(x)|_\nu}{r^2} dt \leq C \int_0^a \frac{|B_t(x)|_\nu}{r^2} dr \tag{1.19}$$

and

$$\sup_{y \in B_a(x)} \int_0^a \frac{|B_r(y)|_\nu}{r^2} dr \leq C \int_0^a \frac{|B_r(x)|_\nu}{r^2} dr. \tag{1.20}$$

Conditions (1.19) and (1.20) essentially are assumptions that measure ν is close to being invariant for the kernel K.

We next outline connections of the general theory sketched above and some classes of superlinear differential equations. We start our discussion with the following first order ordinary differential equation:

$$y'(x) = v(x)\, y^q(x) + w(x), \quad 0 < x < a, \quad y(0) = 0, \tag{1.21}$$

where y, v and w are nonnegative locally integrable functions in $(0, a)$, $0 < a \leq \infty$, and $1 < q < \infty$. This equation is equivalent to the nonlinear integral equation

$$y(x) = \int_0^x y^q(t)\, d\nu(t) + f(x), \qquad 0 < x < a, \tag{1.22}$$

where $f(x) = \int_0^x d\omega(t)$. Here $d\nu(t) = v(t)\,dt$ and $d\omega = w(t)\,dt$. We can rewrite (1.22) in the form $y = T(y^q) + f$, where

$$Tg(x) = \int_0^x g(t)\,d\nu(t)$$

is a weighted Hardy's operator. The kernel of this integral operator is not symmetric, and the formal adjoint T^* is defined by

$$T^*h(x) = \int_x^a h(t)\,d\nu(t).$$

The following theorem characterizing the solvability of (1.21) is essentially known (cf. [Hi], [Har], [To], [G], and the literature cited there). A simple proof along the lines presented above can be found in [Im].

Theorem 6. *Let $1 < q < \infty$, and let v, w be nonnegative locally integrable functions on $(0, a)$. Let $f(x) = \int_0^x w(t)\,dt$. Then the following statements are equivalent.*

(1) *For some $\varepsilon > 0$ there is a nonnegative solution (in a weak sense) of the equation*

$$y'(x) = v(x)\,y^q(x) + \varepsilon w(x), \quad 0 < x < a, \quad y(0) = 0.$$

(2) *There is a constant C independent of $0 < x < a$ such that the inequality*

$$\int_0^x \left(\int_0^t w(\tau)\,d\tau\right)^q v(t)\,dt \le C \int_0^x w(t)\,dt < \infty$$

holds.

(3) *The weighted norm inequality for T^*,*

$$\int_0^a \left|\int_x^a h(t)\,v(t)\,dt\right|^p w(t)\,dt \le C \int_0^a |h(t)|^p\,v(t)\,dt,$$

holds, where $1/p + 1/q = 1$, and C is independent of $h \in L^p(v)$.

(4) *There is a constant C independent of $0 < x < a$ such that*

$$\left(\int_x^a v(t)\,dt\right)^{1/q} \left(\int_0^x w(t)\,dt\right)^{1/p} \le C < \infty.$$

It is easy to see that Theorem 6 remains true with obvious modifications in the case where both v and w are replaced by measures ν and ω on $(0, a)$.

We now discuss more difficult multidimensional problems. Our goal is to characterize the problem of the existence of positive solutions for the superlinear Dirichlet problem

$$\begin{cases} -\Delta u = v(x)\, u^q + w(x), \;\; u \geq 0 \quad \text{on } \Omega, \\ \quad u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.23}$$

on a regular domain $\Omega \subset \mathbb{R}^n$ for $q > 1$; here we assume that $v, w \in L^1_{\text{loc}}(\Omega)$ are arbitrary nonnegative functions. We denote by $G = G_{\Delta,\Omega}$ the Green function of the Laplacian Δ on Ω, and by Gu the Green potential

$$G\, u(x) = \int_\Omega G(x,y)\, u(y)\, dy, \quad x \in \Omega.$$

The solvability of (1.23) is understood in the sense that $u \geq 0$ satisfies the corresponding nonlinear integral equation

$$u = G\,(v\, u^q) + Gw \quad \text{a.e. on } \Omega. \tag{1.24}$$

(More general problems with uniformly elliptic differential operators L in place of the Laplacian, nonhomogeneous boundary conditions, and measures ν and ω as coefficients and data, are considered in [KV].) It follows from Remark 2 that (1.24) is solvable if

$$G[v(Gw)^q] \leq C\, Gw,$$

where $C = q^{-1} p^{1-q}$.

To show that this condition with another constant C is also necessary, we have to do some additional work, since the general results stated above are not applicable directly to (1.24). The problem is that the Green function $G(x,y)$ fails to satisfy the quasi-metric assumption

$$\frac{1}{G(x,y)} \leq C\, \left[\frac{1}{G(x,z)} + \frac{1}{G(y,z)}\right]$$

even for the simplest domains Ω, e.g. the Euclidean ball or the half-space. (Note that there is an error in the proof of this inequality in [Bas], Theorem 3.6.) A weaker version, the so-called "3G-inequality" [ChZh]

$$\frac{G(x,y)\, G(y,z)}{G(x,z)} \leq C\, (|x-y|^{2-n} + |y-z|^{2-n}),$$

is not sharp enough at the boundary of Ω for our purposes.

However, the situation can be fixed by means of a modified kernel (introduced in a different form by Linda Naïm [Na] in the theory related to Martin's kernels) which is defined by

$$K(x,y) = \frac{G(x,y)}{\delta(x)\delta(y)}, \tag{1.25}$$

where $\delta(x) = d(x, \partial\Omega)$ is the distance to the boundary. For bounded $C^{1,1}$ domains in $\mathbb{R}^n$, $n \geq 3$, it follows that the Naïm kernel $K(x,y)$ does satisfy the quasi-metric inequality

$$\frac{1}{K(x,y)} \leq \kappa \left[\frac{1}{K(x,z)} + \frac{1}{K(y,z)}\right],$$

which is stronger than the $3G$-inequality mentioned above. This quasi-metric property can be derived from the following two-sided estimate of $G(x,y)$:

$$G(x,y) \asymp \frac{\delta(x)\,\delta(y)}{|x-y|^{n-2}\,[|x-y|^2 + \delta(x)^2 + \delta(y)^2]}. \tag{1.26}$$

The preceding estimate follows from the known results of [Wi] and [Zh]. (We have recently learned that the quasi-metric inequality for $K(x,y)$ stated above was in a different but equivalent form found earlier in [Se]. Details and additional references can be found in [KV].)

To pass from the Green kernel G to its modified version K, we use the transformation $\tilde{u} = \delta^{-1}u$, and set $F = \delta^{-1}Gw$, $d\nu(y) = \delta^{1-q}\,v(y)\,dy$. Then the original integral equation (1.24) is obviously restated in the equivalent form

$$\tilde{u} = T(\tilde{u}^q) + F, \tag{1.27}$$

where

$$Th(y) = \int_\Omega K(x,y)\,h(y)\,d\nu(y),$$

and $K(x,y)$ is the Naïm kernel defined by (1.25). Note that the inhomogeneous term F has a special form

$$F(x) = \delta(x)^{-1}Gw(x) = \int_\Omega K(x,y)\,d\omega,$$

where $d\omega = \delta(y)\, w(y)\, dy$. Then Theorem 5 (see also Remark 2) gives necessary and sufficient conditions for the solvability of (1.27):

$$T(F^q) \le C\, F, \tag{1.28}$$

with $C = q^{-1} p^{1-q}$ in the sufficiency part and $C = C(\kappa, q)$ for the necessity, where κ is the quasi-metric constant of K. It remains to notice that when (1.28) is translated back from K to G by using (1.25) and the relations $u = \delta \tilde{u}$, $Gw = \delta F$, $d\nu(y) = \delta^{1-q}\, v(y)\, dy$, it gives the same condition

$$G[v(Gw)^q] \le C\, Gw. \tag{1.29}$$

In other words, (1.29) is invariant under this transformation from G to the quasi-metric kernel K.

Since estimate (1.26) is true for the Green function $G_{L,\Omega}$ of any uniformly elliptic operator L with Hölder-continuous coefficients ([Wi], [Zh]), it follows that the corresponding kernel

$$K_{L,\Omega}(x,y) = \frac{G_{L,\Omega}(x,y)}{\delta(x)\delta(y)}$$

is also quasi-metric, and hence our theory carries over to a more general equation

$$\begin{cases} -Lu = v(x)\, u^q + w(x), \;\; u \ge 0 \quad \text{on } \Omega, \\ \quad u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.30}$$

It would be of interest to determine for what other classes of operators L the kernel $K_{L,\Omega}(x,y)$, or its modification which involves an appropriate function in place of $\delta(x)$ (cf. [Na]), satisfies the quasi-metric inequality.

We sum up the preceding discussion as follows.

Theorem 7. *Suppose that $K_{L,\Omega}$ satisfies the quasi-metric condition. Then* (1.30) *is solvable if and only if* (1.29) *holds, with the usual gap in the sharp constants C.*

Another proof of the necessity of (1.29) in the case of the Laplacian $L = \Delta$ with a constant $C = p - 1$ was found later by Brézis and Cabre [BCa]. It is more direct and does not use weighted norm inequalities. On the other hand, it does not involve any geometric interpretations of (1.29), and is not applicable to nonlocal operators.

Theorem 7 remains true if both v and w are replaced by locally finite measures ν and ω on Ω. Then (1.29) should be rewritten as

$$G[(G\omega)^q d\nu] \le C\, G\omega. \tag{1.31}$$

For $v \equiv 1$, i.e., $d\nu = dx$, and ω compactly supported in Ω, a different characterization of the solvability of (1.30) was found earlier by D. Adams and Pierre [AP] in the following capacitary form:

$$|E|_\omega \le C\, \mathrm{Cap}_{2,p}(E)$$

for all compact sets $E \subset \Omega$. Here $\mathrm{Cap}_{2,p}(E)$ is a capacity associated with the Sobolev space $W^{2,p}$.

It is not obvious how to remove the restriction that ω is compactly supported and obtain a capacitary characterization which is valid up to the boundary. We do this by using the following weighted capacity defined by

$$\mathrm{Cap}\,(E) = \inf \left\{ \int_\Omega g^p\, \delta(x)^{1-p}\, dx :\ Gg(x) \ge \delta(x)\, \chi_E(x),\ g \ge 0 \right\}$$

for any $E \subset \Omega$.

Theorem 8. *Let ω be an arbitrary positive measure on a bounded domain Ω with $C^{1,1}$ boundary. Then the Dirichlet problem*

$$\begin{cases} -Lu = u^q + \omega, \quad u \ge 0 \quad \text{on } \Omega, \\ \quad u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

has a solution if and only if (with a gap in the best constants) there is a constant C so that

$$\int_E \delta(x)\, d\omega(x) \le C\, \mathrm{Cap}\,(E) \tag{1.32}$$

for all compact sets $E \subset \Omega$. Moreover, (1.32) is equivalent to the pointwise condition $G[(G\omega)^q] \le C\, G\omega$.

In the case where ω is compactly supported (1.32) reduces to the Adams-Pierre theorem since the capacity defined above is then equivalent to $\mathrm{Cap}_{2,p}(E)$.

A similar Dirichlet problem for the multidimensional Riccati's equation

$$\begin{cases} -\Delta u = v(x)\, |\nabla u|^q + w(x),\ u \ge 0 \quad \text{on } \Omega, \\ \quad u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

with $|\nabla u|$ in place of u is more complicated. Some partial results are obtained under the assumption that v is in some Muckenhoupt class. The case where $v \equiv 1$, which is of interest because of its connection to the Schrödinger equation [HMV], is treated in Sec. 3.

We complete this section by giving a characterization of the solvability problem for the nonlocal equation involving Riesz potentials on $\mathbb{R}^n$ which was mentioned in the introduction:

$$u(x) = \int_{\mathbb{R}^n} \frac{[u(y)]^q}{|x-y|^{n-\alpha}} \, d\nu(y) + f(x), \quad d\nu\text{-a.e.} \tag{1.33}$$

Here $0 < \alpha < n$, $u \geq 0$, $f \geq 0$, and ν is an arbitrary locally finite measure on $\mathbb{R}^n$. In the case $\alpha = 2$ this problem is closely related to (1.23) with $\Omega = \mathbb{R}^n$, $n \geq 3$. Note that the Riesz kernel obviously satisfies the quasi-metric assumption (with $\kappa = 1$ if $n - \alpha \leq 1$ and $\kappa = \kappa(\alpha, n)$ if $n - \alpha > 1$).

Using the notation

$$I_\alpha \nu(x) = \int_{\mathbb{R}^n} \frac{d\nu(y)}{|x-y|^{n-\alpha}},$$

we rewrite (1.33) in a more concise form

$$u = I_\alpha(u^q \, d\nu) + f, \quad d\nu\text{-a.e.} \tag{1.34}$$

We also consider a similar equation

$$u = I_\alpha(u^q \, d\nu) + \varepsilon f, \quad d\nu\text{-a.e.} \tag{1.35}$$

for small $\varepsilon > 0$. For any $E \subset \mathbb{R}^n$, we define the corresponding capacity by

$$\text{Cap}\,(E) = \inf \left\{ \int_{\mathbb{R}^n} g^p \, d\nu : I_\alpha(g d\nu)(x) \geq \chi_E(x), \; g \geq 0 \right\},$$

where as usual $1/p + 1/q = 1$. Then the following theorem is an immediate consequence of Theorems 2, 3, and 5.

Theorem 9. *Let $1 < q < \infty$ and let $0 < \alpha < n$. For a locally finite measure ν on $\mathbb{R}^n$ and $f \in L^0_+(\nu)$, we set $d\omega = f^q \, d\nu$. Then the following statements are true.*

(1) *$f \in \mathcal{Z}$, i.e., (1.35) has a solution for some $\varepsilon > 0$ if and only if the inequality*

$$I_\alpha[(I_\alpha \omega)^q \, d\nu] \leq C \, I_\alpha \omega < \infty \qquad d\nu\text{-a.e.} \tag{1.36}$$

holds. Moreover, if (1.36) *holds with* $C = p^{q(1-q)} q^{-q}$ *then* (1.34) *has a solution* u *such that* $f + I_\alpha(f^q\, d\nu) \le u \le f + p^q\, I_\alpha(f^q\, d\nu)$.

(2) $f \in \mathcal{Z}$ *if and only if both the weighted norm inequality*

$$||I_\alpha(h\, d\nu)||_{L^p(\omega)} \le C\, ||h||_{L^p(\nu)}, \quad h \in L^p(\nu), \tag{1.37}$$

and the infinitesimal inequality

$$\sup_{x\in\mathbb{R}^n,\, r>0} \left\{ \int_0^r \frac{|B_t(x)|_\nu}{t^{n-\alpha+1}}\, dt \right\}^{1/q} \left\{ \int_r^\infty \frac{|B_t(x)|_\omega}{t^{n-\alpha+1}}\, dt \right\}^{1/p} < \infty \tag{1.38}$$

hold, where $B_r(x)$ *is a Euclidean ball of radius* r *centered at* x.

(3) $f \in \mathcal{Z}$ *if and only if both the infinitesimal inequality* (1.38) *and the testing inequality*

$$\int_B [I_\alpha(\chi_B\, d\omega)]^q\, d\nu \le C\, |B|_\omega, \tag{1.39}$$

hold, where $B = B_r(x)$ *is a Euclidean ball, and* C *is independent of* B.

(4) *If* ν *satisfies the estimate*

$$\int_0^r \frac{|B_t(x)|_\nu}{t^{n-\alpha+1}}\, dt \le C\, r^{(n-\alpha)(q-1)} \int_r^\infty \frac{|B_t(x)|_\nu}{t^{(n-\alpha)q+1}}\, dt, \tag{1.40}$$

then (1.36)$\Leftrightarrow$(1.37)$\Leftrightarrow$(1.39). *Moreover,* (1.40) *is necessary in order that* (1.36)$\Leftrightarrow$(1.37).

(5) *Under the assumption* (1.40) *each of the conditions* (1.36), (1.37), *and* (1.39) *is equivalent to the capacitary condition* $|E|_\omega \le C\, \mathrm{Cap}\,(E)$ *for all compact sets* $E \subset \mathbb{R}^n$.

2 Discrete models and weighted norm inequalities

Let $\mathcal{D}$ be the family of all dyadic cubes on $\mathbb{R}^n$, and let $\{c_Q\}_{Q\in\mathcal{D}}$ be a fixed sequence of nonnegative numbers. We consider the kernel

$$K(x,y) = \sum_{Q\in\mathcal{D}} c_Q\, \chi_Q(x)\, \chi_Q(y), \quad x,\, y \in \mathbb{R}^n, \tag{2.1}$$

Note that, for $x \ne y$,

$$K(x,y) = \sum_{Q \supseteq Q(x,y)} c_Q, \tag{2.2}$$

where $Q(x,y)$ is the minimal dyadic cube containing both x and y. To avoid obvious complications, we assume that $K(x,y) > 0$, i.e., for any dyadic cube P there exists $Q \supseteq P$ such that $c_Q > 0$.

Also, just for convenience, we will impose the following additional assumptions on the sequence $\{c_Q\}$:

(i) For any dyadic cube P,

$$\sum_{Q \supseteq P} c_Q < \infty;$$

in other words, $K(x,y) < \infty$ if $x \neq y$.

(ii) For any $x \in \mathbb{R}^n$,

$$\sum_{x \in Q} c_Q = \infty,$$

i.e. $K(x,y) = \infty$ if $x = y$.

Under these assumptions the function $\rho(x,y)$ defined by $\rho(x,y) = 1/K(x,y)$ is a metric on $\mathbb{R}^n$. Moreover, it is easy to see that $\rho(x,y)$ is an ultra-metric, i.e., $\rho(x,y)$ satisfies the inequality

$$\rho(x,y) \leq \max[\rho(x,z), \rho(y,z)]. \tag{2.3}$$

Kernels of this type will be called ultra-metric. The corresponding geometry plays an important role in the sequel. In particular, it is easy to see that any ball in this metric is a dyadic cube. For applications to nonlinear equations, it is important that (2.3) makes it possible to estimate sharp constants C_n in the corresponding weighted norm inequalities (1.6).

Let ν be a locally finite measure on $\mathbb{R}^n$. The corresponding integral operator with kernel $K(x,y)$ is defined by

$$T^\nu f(x) = Tf(x) = \sum_Q c_Q \chi_Q(x) \int_Q f \, d\nu. \tag{2.4}$$

Theorem 10. *Let $1 < q < \infty$. Let $f \in L^q_+(\nu)$ and let $d\omega = f^q \, d\nu$. Suppose that T is defined by (2.4). Then the following statements are equivalent.*

(1) *$f \in \mathcal{Z}$, i.e., the equation*

$$u = T(u^q) + \varepsilon f \tag{2.5}$$

has a solution $u \in L^0_+(\nu)$ for some $\varepsilon > 0$.

(2) *There exists a constant $C > 0$ such that*

$$T[T(f^q)]^q(x) \leq C\, T(f^q)(x) < \infty \quad d\nu\text{-a.e.} \tag{2.6}$$

(3) *Both the infinitesimal inequality*

$$\sup_{P\in\mathcal{D}} \operatorname*{ess\,sup}_{x\in P} \Big[\sum_{Q\subseteq P} c_Q\, |Q|_\nu\, \chi_Q(x) \Big]^{1/q} \Big[\sum_{Q\supseteq P} c_Q\, |Q|_\omega \Big]^{1/p} < \infty, \tag{2.7}$$

and the testing inequality

$$\int_P \Big[\sum_{Q\subseteq P} c_Q |Q|_\omega \chi_Q \Big]^q d\nu \leq C\, |P|_\omega \tag{2.8}$$

hold for all dyadic cubes P.

(4) *Both the infinitesimal inequality* (2.7) *and the weighted norm inequality*

$$||Th||_{L^p(\omega)} \leq C\, ||h||_{L^p(\nu)}, \quad h \in L^p(\nu), \tag{2.9}$$

hold.

Remarks. 1. The testing inequality (2.8) is equivalent to the weak-type inequality

$$||Th||_{L^{p,\infty}(\omega)} \leq C\, ||h||_{L^p(\nu)}, \quad h \in L^p(\nu).$$

The corresponding strong-type inequality (2.9) is characterized by a pair of testing conditions, namely by (2.8) and its dual,

$$\int_P \Big[\sum_{Q\subseteq P} c_Q |Q|_\nu \chi_Q \Big]^p d\omega \leq C\, |P|_\nu. \tag{2.10}$$

These facts (cf. [VW2], [NTV]) are not used in the proof of Theorem 10 below.

2. Each of the inequalities (2.6)–(2.9) is generally stronger than the following condition of Muckenhoupt type:

$$\sup_{Q\in\mathcal{D}} c_Q\, |Q|_\nu^{1/q}\, |Q|_\omega^{1/p} < \infty. \tag{2.11}$$

Proof of Theorem 10. We show that (2)⇒(1)⇒(4)⇒(3)⇒(2). Note that the implication (4)⇒(3) is obvious since the weighted norm inequality (2.9) implies the testing inequality (2.8) by duality. The proof of the remaining implications is subdivided into the following six steps. □

Step 1. (2)⇒(1).

The sufficiency of the pointwise condition (2.6) for the solvability of (2.5) is a consequence of Theorem 2 above. Indeed, suppose that (2.6) holds with $\varepsilon = C/\left((qp^{q-1})^q\right)$. Then by letting

$$u_{n+1} = T(u_n^q) + \varepsilon f, \quad u_0 = 0,$$

where $u_n(x) \le u_{n+1}(x)$, and arguing by induction, we arrive at a solution $u(x) = \lim_{n\to\infty} u_n(x)$ of (2.5) such that

$$\varepsilon f + \varepsilon^q \, T(f^q) \le u \le \varepsilon f + p^q \, \varepsilon^q \, T(f^q).$$

(See Remark 2 after Theorem 2 above.)

Step 2. (Decomposition into "upper" and "lower" parts.)

To each dyadic cube $P \in \mathcal{D}$, we associate the "upper" and "lower" parts of the kernel $K(x,y)$ defined respectively by

$$U_P(x,y) = \sum_{Q \subseteq P} c_Q \, \chi_Q(x) \, \chi_Q(y),$$

and

$$V_P(x,y) = \sum_{Q \supseteq P} c_Q \, \chi_Q(x) \, \chi_Q(y), \quad x, y \in \mathbb{R}^n.$$

Obviously,

$$U_P h(x) \le Th(x), \quad \text{and} \quad V_P h(x) \le Th(x).$$

Proposition 1. *Let T be defined by (2.4) and let P be a fixed dyadic cube in $\mathbb{R}^n$. Suppose $h \in L^0_+(\nu)$ and $Th < \infty$ $d\nu$-a.e. Then, for every $x \in P$,*

$$Th(x) = U_P h(x) + V_P h(x) - c_P \int_P h \, d\nu. \tag{2.12}$$

Here

$$U_P h(x) = \int_{\mathbb{R}^n} U_P(x,y) \, h(y) \, d\nu(y),$$

and

$$V_P h(x) = \int_{\mathbb{R}^n} V_P(x,y)\, h(y)\, d\nu(y)$$

are respectively the "upper" and "lower" part of Th, *and* $c_P \int_P h\, d\nu$ *is the "diagonal term".*

The proof of Proposition 1 is obvious.

It is also convenient to define the "upper" and "lower" potentials of a measure μ by setting

$$U_P\mu(x) = \int_{\mathbb{R}^n} U_P(x,y)\, d\mu(y) = \sum_{Q\subseteq P} c_Q\, |Q|_\mu\, \chi_Q(x)$$

and

$$V_P\mu(x) = \int_{\mathbb{R}^n} V_P(x,y)\, d\mu(y) = \sum_{Q\supseteq P} c_Q\, |Q|_\mu\, \chi_Q(x).$$

Using the notation introduced above, we can rewrite the infinitesimal condition (2.7) in the form:

$$\sup_{P\in\mathcal{D}} \left\| [U_P\nu(x)]^{1/q}\, [V_P\omega(x)]^{1/p} \right\|_{L^\infty(\nu)} < \infty. \tag{2.13}$$

Similarly, the testing inequality (2.8) can be restated as

$$\int_P [U_P\omega(x)]^q\, d\nu \le C\, |P|_\omega, \tag{2.14}$$

for all dyadic cubes P.

Remark 1. The main advantage of using U_P and V_P is that U_P is a "self-similar" restriction of T to the cube P, while V_P is constant on P: For $x \in P$,

$$V_P\mu(x) = \sum_{Q\supseteq P} c_Q\, |Q|_\mu\, \chi_Q(x) = \sum_{Q\supseteq P} c_Q\, |Q|_\mu. \tag{2.15}$$

Step 3. (Integration by parts inequality.)

Proposition 2. *Let* T *be an operator defined by* (2.4) *and let* $1 \le p < \infty$. *Let* $h \in L^0_+(\nu)$. *Then*

$$[Th(x)]^p \le p\, T[h\,(Th)^{p-1}](x). \tag{2.16}$$

Proof of Proposition 2. We will use the following elementary inequality:

$$\left(\sum_{k=1}^{\infty} a_k\right)^p \le p \sum_{k=1}^{\infty} a_k \left(\sum_{j=k}^{\infty} a_j\right)^{p-1}, \tag{2.17}$$

where $1 \le p < \infty$, and $0 \le a_k < \infty$. By (2.17),

$$\begin{aligned}
[Th(x)]^p &= \left[\sum_{Q\in\mathcal{D}} c_Q \int_Q h(y)\, d\nu(y)\, \chi_Q(x)\right]^p \\
&\le p \sum_{Q\in\mathcal{D}} c_Q \int_Q h(y)\, d\nu(y)\, \chi_Q(x) \left[\sum_{P\supseteq Q} c_P \int_P h(y)\, d\nu(y)\, \chi_P(x)\right]^{p-1} \\
&= p \sum_{Q\in\mathcal{D}} c_Q \int_Q h(y)\, d\nu(y)\, \chi_Q(x)\, [V_Q h(x)]^{p-1}.
\end{aligned}$$

Now by (2.15), for x and $y \in Q$, it follows that $V_Q h(x) = V_Q h(y) \le Th(y)$. Hence

$$[Th(x)]^p \le p \sum_{Q\in\mathcal{D}} c_Q \int_Q [Th(y)]^{p-1}\, h(y) d\nu(y)\, \chi_Q(x) = p\, T[h\,(Th)^{p-1}](x).$$

The proof of Proposition 2 is complete. □

Step 4. (1)⇒(2.9).

To prove the necessity of the weighted norm inequality (2.9), we will need the following proposition.

Proposition 3. *Let T be an operator defined by* (2.4) *and let $T(u^q) \le u$ for some $u \in L^0_+(\nu)$. Then the weighted norm inequality*

$$\int (Th)^p\, u^q\, d\nu \le p^p \int h^p\, d\nu \tag{2.18}$$

holds for every $h \in L^p_+(\nu)$.

Proof of Proposition 3. Suppose $h \in L^p_+(\nu)$. By using Proposition 2, Fubini's theorem and Hölder's inequality, we get

$$\begin{aligned}\int (Th)^p\, u^q\, d\nu &\le p \int T[h\,(Th)^{p-1}]\, u^q\, d\nu \\ &= p \int h\,(Th)^{p-1}\, Tu^q\, d\nu \\ &\le p\, ||h||_{L^p(\nu)} \left[\int (Th)^p\, (Tu^q)^q\, d\nu \right]^{1/q} \\ &\le p\, ||h||_{L^p(\nu)} \left[\int (Th)^p\, u^q\, d\nu \right]^{1/q}.\end{aligned}$$

Hence

$$\left[\int (Th)^p\, u^q\, d\nu \right]^{1/p} \le p\, ||h||_{L^p(\nu)},$$

provided the left-hand side of the preceding inequality is finite. The last restriction is easy to remove by a standard argument, assuming first that $\{c_Q\}$ is a finitely supported sequence (see details in [VW2]). □

Corollary. *Let T be an operator defined by* (2.4) *and suppose that* $1 < q < \infty$. *If the equation $u = T(u^q) + f$ has a solution, and $d\omega = f^q\, d\nu$, then*

$$||Th||_{L^p(\omega)} \le p\, ||h||_{L^p(\nu)} \tag{2.19}$$

for every $h \in L^p_+(\nu)$.

The corollary is immediate from Proposition 3 since $u \ge T(u^q)$ and $u \ge f$. To complete the proof of Step 4, note that if the equation $u = T(u^q) + \varepsilon f$ is solvable for some $\varepsilon > 0$, then by the Corollary the inequality

$$||Th||_{L^p(\omega)} \le \frac{p}{\varepsilon^{q-1}}\, ||h||_{L^p(\nu)}$$

holds.

Step 5. (1)⇒(2.7).

The proof of the necessity of the infinitesimal inequality (2.7) is based on the following proposition.

Proposition 4. *Let T be an operator defined by* (2.4) *and let* $1 < q < \infty$. *Suppose that there exists $u \in L^0_+(\nu)$ such that $T(u^q) \le u$. Set $d\omega = u^q\, d\nu$.*

Then, for every dyadic cube P and $x \in P$, the infinitesimal inequality holds:

$$[U_P\nu(x)]^{1/p}\,[V_P\omega(x)]^{1/q} \le C < \infty \qquad d\nu\text{-a.e.} \tag{2.20}$$

where C is a constant which depends only on q. Here U_P and V_P are the "upper" and "lower" potentials defined above.

Proof of Proposition 4. Fix a dyadic cube P and $x \in P$. As in Sec. 1, it will be convenient to use the notation $\mathcal{A}u = T(u^q)$, so that $\mathcal{A}(\lambda u) = \lambda^q\,\mathcal{A}u$ for $\lambda > 0$. We also set

$$\mathcal{A}_P u = U_P(u^q) \quad \text{and} \quad \mathcal{B}_P u = V_P(u^q).$$

As was mentioned above, $\mathcal{A}_P$ is a "self-similar" restriction of $\mathcal{A}$ supported on the cube P, and $\mathcal{B}_P$ is constant on P. (See (2.15).)

Iterating the inequality $u \ge \mathcal{A}u$, we get

$$u(x) \ge \mathcal{A}u(x) \ge \ldots \ge \mathcal{A}^n u(x) \ge \mathcal{A}^{n+1}u(x) \ge \ldots .$$

Since $\mathcal{A}_P u(x) \le \mathcal{A}u(x)$, and $\mathcal{B}_P u(x) = \text{const}$ on P, we have, for $x \in P$,

$$u(x) \ge \mathcal{A}^{n+1}u(x) \ge \mathcal{A}_P^n[\mathcal{B}_P u](x) \ge \mathcal{A}_P^n\mathbf{1}(x)\,[\mathcal{B}_P u(x)]^{q^n}. \tag{2.21}$$

Here

$$\mathcal{A}_P^n\mathbf{1}(x) = U_P[U_P\ldots(U_P\mathbf{1})^q\ldots]^q(x)$$

is an iterated "upper part" of T applied to $h \equiv 1$, and

$$U_P\mathbf{1}(x) = U_P\nu(x) = \sum_{Q\subseteq P} c_Q\,|Q|_\nu\,\chi_Q(x).$$

Since $\mathcal{A}_P$ is supported on P, it follows that (2.21) actually holds $d\nu$-a.e. on $\mathbb{R}^n$.

We now estimate $\mathcal{A}_P^n\mathbf{1}(x)$ from below using Proposition 2 repeatedly with $1+q$, $1+q+q^2, \ldots, 1+q+q^2+\cdots+q^{n-1}$ in place of p. By induction, we get:

$$\begin{aligned}
\mathcal{A}_P\mathbf{1}(x) &= U_P\mathbf{1}(x);\\
\mathcal{A}_P^2\mathbf{1}(x) &= U_P(U_P\mathbf{1})^q(x) \ge \frac{1}{1+q}[U_P\mathbf{1}(x)]^{1+q};\\
\mathcal{A}_P^3\mathbf{1}(x) &= U_P(\mathcal{A}_P^2\mathbf{1})^q(x) \ge \frac{1}{(1+q)^q}U_P[U_P\mathbf{1}(x)]^{q(1+q)}\\
&\qquad\qquad \ge \frac{1}{(1+q)^q(1+q+q^2)}[U_P\mathbf{1}(x)]^{1+q+q^2};\\
&\cdots\\
\mathcal{A}_P^{n+1}\mathbf{1}(x) &= U_P(\mathcal{A}_P^n\mathbf{1})^q(x) \ge \frac{1}{\prod_{j=1}^n(1+q+\cdots+q^j)^{q^{n-j-1}}}\\
&\qquad\qquad \times [U_P\mathbf{1}(x)]^{1+q+\cdots+q^{n-1}}.
\end{aligned}$$

Combining these estimates and (2.21), we obtain

$$u(x) \ge \prod_{j=1}^n (1+q+q^2+\cdots+q^j)^{-q^{n-j-1}} \times [U_P\mathbf{1}(x)]^{1+q+q^2+\cdots+q^{n-1}}\,[V_P(u^q)(x)]^{q^n}, \quad d\nu\text{-a.e.}$$

We then raise both sides of the preceding estimate to the power $1/q^n$:

$$u(x)^{1/q^n} \ge \prod_{j=1}^n (1+q+q^2+\cdots+q^j)^{-q^{-j-1}} \times [U_P\mathbf{1}(x)]^{(1-q^{-n})/q-1}\,[V_P(u^q)(x)].$$

Letting $n\to\infty$, we get

$$1 \ge \prod_{j=1}^\infty (1+q+q^2+\cdots+q^j)^{-q^{-j-1}}\,[U_P\mathbf{1}(x)]^{\frac{1}{q-1}}\,[V_P(u^q)(x)] \qquad d\nu\text{-a.e.}$$

Since the infinite product above converges for every $q>1$, and $U_P\mathbf{1} = U_P\nu$, $V_P(u^q) = V_P\omega$ for $d\omega = u^q\,d\nu$, this estimate yields the infinitesimal inequality (2.20). The proof of Proposition 4 is complete. □

To show that (1)⇒(2.7), it remains to notice that if the equation $u = T(u^q)+\varepsilon f$ has a solution, then $u \ge T(u^q)$, and $u \ge \varepsilon f$. Hence

$\int_Q f^q\, d\nu \le \varepsilon^{-q} \int_Q u^q\, d\nu$ for every $Q \in \mathcal{D}$, and by Proposition 4 it follows that the infinitesimal inequality (2.7) (with $d\omega = f^q\, d\nu$) holds.

Step 6. (3)⇒(2)

To show that (2.7)&(2.8)⇒(2.6), we follow the argument used in [VW2] in the case of Riesz potentials. Let $d\omega = f^q\, d\nu$. The pointwise condition (2.6) can be rewritten in the form

$$\sum_{P\in\mathcal{D}} c_P\, \chi_P(x) \int_P [T\omega(y)]^q\, d\nu(y) \le C\, T\omega(x), \tag{2.22}$$

where

$$T\omega(x) = \sum_{P\in\mathcal{D}} c_P\, |P|_\omega\, \chi_P(x).$$

Using the decomposition of $T\omega$ into its "upper" and "lower" parts (Step 2), we have, for $y \in P$,

$$T\omega(y) = U_P\omega(y) + V_P\omega(y).$$

By the testing inequality (2.8),

$$\int_P [U_P\omega(y)]^q\, d\nu(y) \le C\, |P|_\omega,$$

and hence

$$\sum_{P\in\mathcal{D}} c_P\, \chi_P(x) \int_P [U_P\omega(y)]^q\, d\nu(y) \le C\, T\omega(x).$$

It remains to prove

$$\sum_{P\in\mathcal{D}} c_P\, \chi_P(x) \int_P [V_P\omega(y)]^q\, d\nu(y) \le C\, T\omega(x). \tag{2.23}$$

Since, for $y \in P$,

$$V_P(y) = \text{const} = \sum_{Q\supseteq P} c_Q\, |Q|_\omega,$$

the preceding inequality is equivalent to

$$\sum_{P\in\mathcal{D}} c_P\, |P|_\nu\, \chi_P(x) \left[\sum_{Q\supseteq P} c_Q\, |Q|_\omega\right]^q \le C\, T\omega(x). \tag{2.24}$$

From estimate (2.17), as in Step 3, we get

$$\left[\sum_{Q \supseteq P} c_Q \, |Q|_\omega \right]^q \leq q \sum_{Q \supseteq P} c_Q \, |Q|_\omega \left[\sum_{R \supseteq Q} c_R \, |R|_\omega \right]^{q-1}.$$

Using this inequality and changing the order of summation, we see that the left-hand side of (2.24) is bounded from above by

$$q \sum_{Q \in \mathcal{D}} c_Q \, |Q|_\omega \, \chi_Q(x) \left\{ \left[\sum_{P \subseteq Q} c_P \, |P|_\nu \, \chi_P(x) \right] \left[\sum_{R \supseteq Q} c_R \, |R|_\omega \right]^{q-1} \right\}.$$

Since the expression in the curly brackets is uniformly bounded by the infinitesimal inequality, this yields estimate (2.24), which completes the proof of Step 6. Thus the proof of Theorem 10 is complete.

3 Criteria of solvability for multidimensional Riccati's equations

Here we present our joint work with Kurt Hansson and Vladimir Maz'ya [HMV] on the solvability problem for the multidimensional Riccati equation

$$-\Delta u = |\nabla u|^q + \omega \quad \text{on } \Omega, \tag{3.1}$$

where $q > 1$, and ω is an arbitrary nonnegative measurable function (or measure) on a domain $\Omega \subset \mathbb{R}^n$. All solutions are understood in the usual weak sense; i.e., u is a solution of (3.1) if $u \in W^{1,q}_{\text{loc}}(\Omega)$ and

$$\int_\Omega \nabla u \cdot \nabla \phi \, dx = \int_\Omega |\nabla u|^q \, \phi \, dx + \int_\Omega \phi \, d\omega \tag{3.2}$$

for all test functions $\phi \in C_0^\infty(\Omega)$. We also consider more general superlinear equations of the type $-Lu = f(x, u, \nabla u) + \omega$ where $f(x, u, \nabla u) \asymp a(x) \, |\nabla u|^{q_1} + b(x) \, |u|^{q_2}$ ($1 < q_1 < \infty$, $1 < q_2 < \infty$), a and b are bounded positive functions, and L is a uniformly elliptic operator.

In the case $\Omega = \mathbb{R}^n$, where our results are more complete, we establish explicit necessary and sufficient conditions for the existence of global solutions, together with sharp pointwise estimates of solutions and their gradients. For bounded regular domains Ω in $\mathbb{R}^n$, similar results are obtained for the Dirichlet problem

$$\begin{cases} -\Delta u = |\nabla u|^q + \omega & \text{on } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.3}$$

in the case $q > 2$. The case $q = 2$ is intimately connected, via the substitution $v = e^u$, with the classical problem of the existence of positive solutions for the Schrödinger equation

$$\begin{cases} -\Delta v = \omega v, \quad v > 0 \quad \text{on } \Omega, \\ \quad v = \phi \quad \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

where $\phi \equiv 1$. This problem has been studied in the literature, but mostly for regular enough potentials and bounded solutions. See [ChZh] where (3.4) is studied in detail for ω in Kato's class. Some preliminary results on the solvability of (3.4) for an arbitrary nonnegative potential ω and boundary data $\phi \geq 0$ have been obtained very recently (joint work with Michael Frazier).

We start with the following criterion for the existence of global solutions for (3.1) on $\mathbb{R}^n$. Recall that the Riesz potential I_α of order α $(0 < \alpha < n)$ on $\mathbb{R}^n$ is defined by

$$I_\alpha f(x) = c(n, \alpha) \int_{\mathbb{R}^n} \frac{f(t)}{|x - t|^{n-\alpha}} \, dt, \tag{3.5}$$

where $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $\int_{|x|\geq 1} |x|^{\alpha-n} \, |f(x)| \, dx < \infty$. Similarly, for a locally finite measure ν, the Riesz potential of ν is defined by

$$I_\alpha \nu(x) = c(n, \alpha) \int_{\mathbb{R}^n} \frac{d\nu(t)}{|x - t|^{n-\alpha}} \, dt.$$

Note that $I_\alpha \nu \equiv +\infty$ unless $\int_{|x|\geq 1} |x|^{\alpha-n} \, d\nu(x) < \infty$.

Theorem 11. *Let $1 < q < \infty$, and let ω be a locally finite measure on $\Omega = \mathbb{R}^n$. Then there exist positive constants C_1 and C_2 which depend only on q and n such that the following statements hold.*

(1) *If* (3.1) *has a solution $u \in W^{1,q}_{\text{loc}}$, then $I_1\omega < \infty$ a.e. and*

$$I_1[(I_1\omega)^q](x) \leq C \, I_1\omega(x) \quad \text{a.e.} \tag{3.6}$$

with $C < C_1(q, n)$.

(2) *Conversely, if $I_1\omega < \infty$ a.e., and* (3.6) *holds with $C < C_2(q, n)$, then* (3.1) *has a solution $u \in W^{1,q}_{\text{loc}}$ such that*

$$|\nabla u(x)| \leq C \, I_1\omega(x) \quad \text{a.e.} \tag{3.7}$$

(3) *If $I_2\omega < \infty$ a.e., and* (3.6) *holds with $C < C_2(q, n)$, then there is a solution u such that*

$$I_2\omega(x) \leq u(x) \leq C(q, n) \, I_2\omega(x). \tag{3.8}$$

Remarks. 1. It follows from Theorem 9 that the pointwise condition (3.6) is equivalent to the capacitary inequality

$$|E|_\omega \le C \operatorname{Cap}_{1,p}(E),$$

for every compact $E \subset \mathbb{R}^n$, where $1/p + 1/q = 1$, and the Riesz capacity $\operatorname{Cap}_{\alpha,p}$ $(0 < \alpha < n)$ is defined by

$$\operatorname{Cap}_{\alpha,p}(E) = \inf\{||f||^p_{L^p} : I_\alpha f \ge \chi_E, \quad f \in L^p_+(\mathbb{R}^n)\}. \tag{3.9}$$

If $\alpha = k$ is an integer, then this capacity is equivalent to the following capacity associated with the homogeneous Sobolev space $L^{k,p}$ (see [M3]):

$$\operatorname{Cap}_{k,p}(E) = \inf\{||\nabla^k u||^p_{L^p} : u \ge \chi_E, \quad u \in C_0^\infty(\mathbb{R}^n)\}. \tag{3.9$'$}$$

2. It follows from the previous remark that $q = n/(n-1)$ is a critical exponent for the solvability of (3.1) on $\mathbb{R}^n$: If $1 < q \le n/(n-1)$, then $\operatorname{Cap}_{1,p}(E) = 0$ for all $E \subset \mathbb{R}^n$ (see [AH], Proposition 2.6.1); i.e., (3.1) has no global solutions on $\mathbb{R}^n$ provided $\omega \ne 0$.
3. If $1 < q < 2$, then condition $I_2\omega < \infty$ a.e. is necessary in order that (3.1) be solvable in a weak sense, and hence estimate (3.8) holds. This assertion fails for $q \ge 2$ (see examples in [HMV]).

Proof of Theorem 11. To prove statement (1), we introduce a natural space of solutions of equation (3.1) as the class of $u \in W^{1,q}_{\text{loc}}$ with finite seminorm

$$|||u||| = \sup\left\{ \left(\frac{\int_E |\nabla u|^q \, dx}{\operatorname{Cap}_{1,p}(E)} \right)^{1/q} : \operatorname{Cap}_{1,p}(E) > 0, \ E \text{ compact} \right\}. \tag{3.10}$$

□

Proposition 5. *Let $1 < q < \infty$ and let $1/p + 1/q = 1$. Let ω be a locally finite measure on $\Omega = \mathbb{R}^n$. If (3.1) has a solution $u \in W^{1,q}_{\text{loc}}$ then*

$$\int |h|^p \, |\nabla u|^q \, dx \le C \int |\nabla h|^p \, dx \tag{3.11}$$

and

$$\int |h|^p \, d\omega \le C \int |\nabla h|^p \, dx \tag{3.12}$$

for every $h \in C_0^\infty$, where C is a constant which depends only on p.

In particular, it follows from Proposition 5 and Remark 1 that all weak solutions u satisfy the inequality $|||u||| \le C(q,n)$.

Proof of Proposition 5. Without loss of generality we may assume that the test functions h in both (3.11) and (3.12) are nonnegative (see [M3]). Let u be a solution to (3.1). Then setting $\phi = h^p$ in (3.2), we get

$$\int -\Delta(h^p)\, u\, dx = \int |\nabla u|^q\, h^p\, dx + \int h^p\, d\omega. \tag{3.13}$$

Rewriting the left-hand side of the preceding equation as

$$\int \nabla u \cdot \nabla (h^p)\, dx = p \int (\nabla u \cdot \nabla h)\, h^{p-1}\, dx,$$

we have

$$p \int (\nabla u \cdot \nabla h)\, h^{p-1}\, dx = \int |\nabla u|^q\, h^p\, dx + \int h^p\, d\omega. \tag{3.14}$$

By Hölder's inequality

$$\int h^p\, d\omega \le p \left(\int |\nabla u|^q\, h^p\ dx \right)^{1/q} ||\nabla h||_{L^p}. \tag{3.15}$$

On the other hand, from (3.14) we get

$$\begin{aligned} \int |\nabla u|^q\, h^p\, dx &\le p \int (\nabla u \cdot \nabla h)\, h^{p-1}\, dx \\ &\le p\, ||\nabla h||_{L^p} \left(\int |\nabla u|^q\, h^p\ dx \right)^{1/q}. \end{aligned}$$

Since the right-hand side of the preceding inequality is finite, we obtain

$$\int |\nabla u|^q\, h^p\ dx \le p^p\, ||\nabla h||^p_{L^p}, \tag{3.16}$$

which proves (3.11). Combining (3.15) and (3.16) we get (3.12). The proof of Proposition 5 is complete. □

By the standard properties of the Riesz transforms [St], (3.12) is equivalent to the following imbedding theorem for I_1,

$$||I_1 h||_{L^p(\omega)} \le C\, ||h||_{L^p}, \quad h \in L^p,$$

which by Theorem 9 is characterized by the pointwise condition (3.6). This completes the proof of statement 1 of the theorem.

To avoid some complications at infinity, we will sketch the proof of statement (2) of Theorem 11 only for ω such that $I_2\omega < \infty$ a.e., i.e., under the additional restriction $\int_{|x|\geq 1} |x|^{\alpha-n}\, d\omega < \infty$. If this condition is violated, then a solution u to (3.1) can still be constructed provided (3.6) holds, but in this case u possibly changes sign and has some growth at infinity (see the general case and examples in [HMV]).

Since by our assumption $I_2\omega < \infty$ a.e., it follows that (3.1) is solvable if there is a (nonnegative) solution to the following integro-differential equation:

$$u = I_2(|\nabla u|^q) + I_2\omega, \tag{3.17}$$

where $I_2\omega = (-\Delta)^{-1}\omega$ is the Newtonian potential of ω. Now we construct a solution of (3.17) under the assumption (3.6). We set $u_0 = I_2\omega$ and

$$u_{k+1} = I_2(|\nabla u_k|^q) + I_2\omega, \quad k = 0, 1, 2, \ldots. \tag{3.18}$$

Proposition 6. *Suppose that u_k are defined by* (3.18). *There exists a constant $0 < C < 1$ which depends only on q and n such that if*

$$I_1[(I_1\omega)^q](x) \leq C\, I_1\omega(x) < \infty, \tag{3.19}$$

then the following inequalities hold:

$$|\nabla u_k(x)| \leq a\, I_1\omega(x), \tag{3.20}$$

and

$$|\nabla u_{k+1}(x) - \nabla u_k(x)| \leq b\, c^k\, I_1\omega(x) \tag{3.21}$$

where the constants a, b, and c depend only on q and n.

Proof of Proposition 6. We first prove (3.20), which is obvious if $k = 0$. We show by induction that

$$|\nabla u_k(x)| \leq a_k\, I_1\omega(x). \tag{3.22}$$

It follows from (3.19) that

$$|\nabla u_{k+1}(x)| = |\nabla[I_2|\nabla u_k(x)|^q] + \nabla I_2\omega(x)| \leq C(n)\, [I_1|\nabla u_k(x)|^q + I_1\omega(x)].$$

By (3.22) and (3.19),

$$I_1|\nabla u_k(x)|^q \le I_1[a_k\,(I_1\omega)]^q = a_k^q\, I_1(I_1\omega)^q \le a_k^q\, C\, I_1\omega.$$

Combining these estimates, we get

$$|\nabla u_{k+1}(x)| \le a_{k+1}\, I_1\omega(x), \tag{3.23}$$

where

$$a_{k+1} = C(n)\,(a_k^q\, C + 1).$$

It is easily seen that for $C = C(n,q)$ small enough it follows that $a = \lim_{k\to\infty} a_k < \infty$. This proves (3.20) with a constant a which depends only on n, q.

We next prove by induction that (3.21) holds. Note that $u_1 - u_0 = I_2\,|\nabla u_0|^q$, and hence

$$|\nabla u_1 - \nabla u_0| \le C(n)\, I_1|\nabla u_0(x)|^q \le C(n)\, a^q\, I_1(I_1\omega)^q.$$

Then

$$|\nabla u_1 - \nabla u_0| \le b_0\, I_1\omega, \tag{3.24}$$

where $b_0 = C(n)\, a^q\, C$ and C is the constant from (3.19).

Similarly,

$$u_{k+1} - u_k = I_2[\,|\nabla u_k|^q - |\nabla u_{k-1}|^q\,]$$

and hence

$$|\nabla u_{k+1} - \nabla u_k| \le C(n)\, I_1\,[\,|\,|\nabla u_k|^q - |\nabla u_{k-1}|^q|\,].$$

Using the inequality $|r^q - s^q| \le q\,|r-s|\,\max(r,s)^{q-1}$ with $r = |\nabla u_k|$ and $s = |\nabla u_{k-1}|$ together with (3.20) we have

$$|\,|\nabla u_k|^q - |\nabla u_{k-1}|^q| \le q\, a\, |\nabla u_k - \nabla u_{k-1}|\,(I_1\omega)^{q-1}. \tag{3.25}$$

From this we obtain

$$|\nabla u_{k+1} - \nabla u_k| \le C(n)\, q\, a\, I_1\,[\,|\nabla u_k - \nabla u_{k-1}|\,(I_1\omega)^{q-1}\,]. \tag{3.26}$$

Suppose

$$|\nabla u_k - \nabla u_{k-1}| \le b_k\, I_1\omega.$$

Then by (3.26)

$$|\nabla u_{k+1} - \nabla u_k| \le C(n,q)\, b_k\ I_1\,(I_1\omega)^q.$$

Using (3.19), we see by induction

$$|\nabla u_{k+1} - \nabla u_k| \le b_{k+1}\, I_1\omega,$$

where $b_{k+1} \le C(n,q)\, C\, b_k$ and C is a constant in (3.19). Thus

$$b_{k+1} \le [\, C(n,q)\, C\,]^{k+1}\, b_0,$$

where $b_0 = C(n)\, a^q\, C$. Choosing C in (3.19) so that $c = C(n,q)\, C < 1$, we complete the proof of Proposition 6. □

Proposition 7. *Suppose that u_k are defined by* (3.18). *Then*

$$|u_{k+1}(x) - u_k(x)| \le c\, C^k\, I_2\omega(x), \tag{3.27}$$

where the constants $c > 0$ and $0 < C < 1$ depend only on q and n.

Proof of Proposition 7. Applying (3.25) and Proposition 6, we obtain

$$\begin{aligned}|u_{k+1}(x) - u_k(x)| &= \big|\, I_2[\,|\nabla u_k|^q - |\nabla u_{k-1}|^q\,]\big| \le C\, I_1 \big|\, |\nabla u_k|^q - |\nabla u_{k-1}|^q\, \big| \\ &\le C\, I_1\, \big[|\nabla u_k - \nabla u_{k-1}|\, (I_1\omega)^{q-1}\big] \le c\, C^k\, I_2\omega(x).\end{aligned}$$

The proof of Proposition 7 is complete. □

We now complete the proof of statements (2) and (3) of Theorem 11. Let

$$u(x) = u_0(x) + \sum_{k=0}^{\infty} [u_{k+1}(x) - u_k(x)],$$

where $u_0 = I_2\omega$ and u_k are defined by (3.18). By Proposition 7,

$$|u_{k+1}(x) - u_k(x)| \le c\, C^k\, I_2\omega(x),$$

where $0 < C < 1$. Hence $u(x) = \lim_{k\to\infty} u_k(x)$ and

$$|u(x)| \le C\, I_2\omega(x) \quad \text{a.e.} \tag{3.28}$$

Moreover, by Proposition 6,

$$|\nabla u_{k+1}(x) - \nabla u_k(x)| \le b\, C^k\, I_1\omega(x)$$

and hence

$$|\nabla u(x)| \leq C\, I_1\omega(x). \tag{3.29}$$

By Theorem 9 it follows that ω satisfies the inequality $|E|_\omega \leq C\,\mathrm{Cap}_{\,1,p}(E)$, and a similar estimate is true for $d\omega_1(x) = (I_1\omega)^q\,dx$:

$$\int_E (I_1\omega)^q\,dx \leq C\,\mathrm{Cap}_{\,1,p}(E),$$

for any compact set E (see also [MV]). In particular, it follows from (3.29) that $u \in W^{1,q}_{\mathrm{loc}}$, and

$$|||u||| = \sup\left\{ \left(\frac{\int_E |\nabla u|^q\,dx}{\mathrm{Cap}_{\,1,p}(E)}\right)^{1/q} \; : \; \mathrm{Cap}_{\,1,p}(E) > 0\right\} < C(q,n).$$

Let $\phi \in C_0^\infty$ be an arbitrary test function. Since

$$\nabla u(x) = \lim_{k\to\infty} \nabla u_k(x) = \nabla u_0(x) + \sum_{k=0}^{\infty} [\nabla u_{k+1}(x) - \nabla u_k(x)]$$

a.e., we have

$$\int \nabla\phi\cdot\nabla u_k\,dx \to \int \nabla\phi\cdot\nabla u\,dx, \qquad \int \phi\,|\nabla u_k|^q\,dx \to \int \phi\,|\nabla u|^q\,dx$$

as $k \to \infty$ by the dominated convergence theorem. By (3.18)

$$\int \nabla\phi\cdot\nabla u_{k+1}\,dx = \int \phi\,|\nabla u_k|^q\,dx + \int \phi\,d\omega$$

Letting $k \to \infty$ in the preceding inequality, we obtain

$$\int \nabla\phi\cdot\nabla u\,dx = \int \phi\,|\nabla u|^q\,dx + \int \phi\,d\omega.$$

Thus $u \in W^{1,q}_{\mathrm{loc}}$ is a (weak) solution to (3.1), and the estimates (3.7) and (3.8) hold. The proof of Theorem 11 is complete.

The following corollary gives new pointwise estimates for positive solutions of the Schrödinger equation

$$-\Delta v = \omega v, \qquad v \geq 0, \tag{3.30}$$

which, as was mentioned above, is equivalent to (3.1) with $q = 2$ and $u = \log v$.

Corollary. *Suppose that ω is a locally finite positive measure on $\mathbb{R}^n$.*

(1) *If* (3.30) *has a nonnegative (weak) solution v, then $I_1\omega < \infty$ a.e. and*

$$I_1[(I_1\omega)^q](x) \le C_1\, I_1\omega(x), \tag{3.31}$$

where C_1 depends only on n.

(2) *Conversely, there exists a constant C_2 depending only on n such that if* (3.31) *holds with C_2 in place of C_1, then there is a positive solution v to* (3.30).

Moreover, the following estimates hold:

$$|\nabla \log v(x)| \le C\, I_1\omega(x), \tag{3.32}$$

$$\int_E |\nabla \log v(x)|^2\, dx \le C\, \mathrm{Cap}_{1,2}(E) \tag{3.33}$$

for all compact sets E. If in addition $I_2\omega < \infty$ a.e., then

$$I_2\omega(x) \le \log\, v \le C\, I_2\omega(x), \tag{3.34}$$

for some $C > 0$, where all constants depend only on n.

Results similar to Theorem 11 hold for more general superlinear inhomogeneous equations of the type

$$-Lu = f(x, u, \nabla u), \tag{3.35}$$

where $f(x, u, \nabla u) \asymp a(x)\, |\nabla u|^{q_1} + b(x)\, |u|^{q_2} + \omega(x)$, L is a uniformly elliptic second order operator, and q_1, $q_2 > 1$.

As above, we are interested in sharp solvability results for an arbitrary nonnegative inhomogeneous term $\omega \ne 0$. For simplicity, we consider the solvability problem on $\mathbb{R}^n$ for the equation

$$-\Delta u = a\, |\nabla u|^{q_1} + b\, |u|^{q_2} + \omega, \tag{3.36}$$

with constant coefficients $a > 0$, $b > 0$, and arbitrary measure ω. The solvability of this equation is understood in a weak sense, i.e., there exists $u \in W^{1,q_1}_{\mathrm{loc}} \cap L^{q_2}_{\mathrm{loc}}$ such that

$$\int \nabla u \cdot \nabla \phi\, dx = \int a\, |\nabla u|^{q_1}\, \phi\, dx + \int b\, |u|^{q_2}\, \phi\, dx + \int \phi\, d\omega, \tag{3.37}$$

for all $\phi \in C_0^\infty$. One can actually show that, if $b \neq 0$, there exists a *nonnegative* solution $u \in W^{1,q_1}_{\mathrm{loc}} \cap L^{q_2}_{\mathrm{loc}}$, or equivalently

$$u = I_2(a\,|\nabla u|^{q_1}) + I_2(b\,u^{q_2}) + I_2\omega + c \quad \text{a.e.}, \tag{3.38}$$

where $u \geq 0$, $c \geq 0$ and $I_2 = I_1^2$ is the Newtonian potential.

Theorem 12. *Let $1 < q_i < \infty$ and $1/p_i + 1/q_i = 1$, $i = 1, 2$. Let ω be a locally finite measure on $\mathbb{R}^n$. Then there exist positive constants C_j, $j = 1, \ldots, 6$, which depend only on q_i and n such that the following statements hold.*

(1) *If equation* (3.36) *with constant coefficients $a, b > 0$ has a solution $u \in W^{1,q_1}_{\mathrm{loc}} \cap L^{q_2}_{\mathrm{loc}}$, then*

$$I_1[(I_1\omega)^{q_1}](x) \leq \frac{C_1}{a}\, I_1\omega(x) < \infty \quad a.e., \tag{3.39}$$

and

$$I_2[(I_2\omega)^{q_2}](x) \leq \frac{C_2}{b}\, I_2\omega(x) < \infty \quad a.e. \tag{3.40}$$

(2) *Conversely, if the inequalities* (3.39) *and* (3.40) *hold with the constants C_3 and C_4 in place of C_1 and C_2, then* (3.36) *has a solution $u \in W^{1,q_1}_{\mathrm{loc}} \cap L^{q_2}_{\mathrm{loc}}$ such that the following inequalities hold:*

$$|\nabla u(x)| \leq C_5\, I_1\omega(x), \quad I_2\omega(x) \leq u(x) \leq C_6\, I_2\omega(x) \quad a.e. \tag{3.41}$$

Remarks. 1. It can be shown that any solution $u \in W^{1,q_1}_{\mathrm{loc}} \cap L^{q_2}_{\mathrm{loc}}$ of (3.36) (with constant coefficients a and b) satisfies the estimates

$$\int_E (a\,|\nabla u|^{q_1} + b\,|u|^{q_2})\, dx + |E|_\omega \leq a^{1-p_1}\, C(q_1)\, \mathrm{cap}_{1,p_1}(E),$$
$$\int_E (a\,|\nabla u|^{q_1} + b\,|u|^{q_2})\, dx + |E|_\omega \leq b^{1-p_2}\, C(q_2, n)\, \mathrm{cap}_{2,p_2}(E),$$

for all compact sets E; here $\mathrm{cap}_{\alpha,p}(\cdot)$ is the capacity of order $\alpha = 1, 2$. In particular, a nontrivial global solution to (3.36) may exist only if $n/(n-1) < q_1 < \infty$ and $n/(n-2) < q_2 < \infty$.

2. It follows from the known relations between Riesz capacities (see [AH], Theorem 5.5.1) that, for $p_1 = 2\,p_2$, the inequality

$$\mathrm{cap}_{1,p_1}(E) \leq C\, \mathrm{cap}_{2,p_2}(E)$$

holds with a constant C independent of compact sets $E \subset \mathbb{R}^n$; in this case the second term on the right hand side of (3.36) is "dominated" by the first one. In all other cases the contributions of the nonlinearities involving $|\nabla u|^{q_1}$ and $|u|^{q_2}$ are generally not comparable.

In the case of a bounded regular domain Ω, we only state the following theorem [HMV]. For any compact set $E \subset \Omega$, set

$$\mathrm{Cap}_{\,p,\Omega}(E) = \inf \{||h||^p_{W^{1,p}(\Omega)} : \, h > \chi_E, \; h \in C_0^\infty(\Omega)\}.$$

Theorem 13. *Let $1 < q < \infty$ and $1/p + 1/q = 1$. Let Ω be a bounded $C^{1,1}$ domain in $\mathbb{R}^n$. Let ω be a locally finite positive measure on Ω. Then there exist positive constants C_1 and C_2 which depend only on q, n, and Ω such that the following statements hold.*
(1) *If* (3.1) *has a solution $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$, then the inequality*

$$|E|_\omega \le C \, \mathrm{Cap}_{p,\Omega}(E), \tag{3.42}$$

holds for all compact sets $E \subset \Omega$ with a constant $C < C_1(q,n)$.
(2) *If $2 < q < \infty$ and* (3.42) *holds with $C < C_2(q,n,\Omega)$, then* (3.1) *has a solution u with zero boundary values.*
(3) *The solution u whose existence is claimed in* (2) *satisfies the inequality*

$$|\nabla u(x)| \le C \, I_1\omega(x) \quad \textit{a.e. on } \Omega, \tag{3.43}$$

with a constant which depends only on q, n, and Ω.

4 Wolff's potentials and trace inequalities

We consider the trace inequality

$$||I_\alpha f||_{L^q(\omega)} \le C \, ||f||_{L^p(dx)}, \qquad f \in L^p(\mathbb{R}^n), \tag{4.1}$$

for $1 < p < \infty$ and $1 < q < \infty$, where ω is an arbitrary measure on $\mathbb{R}^n$, and $I_\alpha f(x) = (-\Delta)^{-\alpha/2} f = c(\alpha, n) \, |x|^{\alpha - n} * f$ is the Riesz potential of order $0 < \alpha < n$. Note that in this section we change our notation and assume that p and q are generally unrelated to each other; a conjugate exponent to p will be denoted by p', so that $1/p + 1/p' = 1$.

If $\alpha = k$ is an integer, then (4.1) is equivalent to a generalized Sobolev inequality

$$\left\{ \int_{\mathbb{R}^n} |u(x)|^q \, d\omega(x) \right\}^{1/q} \le C \, \left\{ \int_{\mathbb{R}^n} |\nabla^k u(x)|^p \, dx \right\}^{1/p}, \tag{4.1$'$}$$

for $u \in C_0^\infty(\mathbb{R}^n)$. Similar inequalities are of interest for Bessel potentials $J_\alpha f = (1-\Delta)^{-\alpha/2} f$, and in particular for (inhomogeneous) Sobolev spaces $W^{k,p}$ with the norm

$$||u||_{W^{k,p}} = ||\nabla^k u||_{L^p} + ||u||_{L^p}$$

on the right-hand side of (4.1′). As was mentioned above, a systematic study of these inequalities and their applications was started by V. Maz'ya more than 30 years ago (see [M3]).

The classical case $q \geq p > 1$ of this problem is now well understood ([AH], [M3], [S3]). In the easier case $q > p$, an elegant theorem of D. Adams says that (4.1) holds if and only if the following Frostman condition is valid:

$$|B_r(x)|_\omega \leq C\, r^{(n-\alpha p)q/p}$$

for all balls $B_r(x)$ of radius r. In the diagonal case $p = q$ a complete characterization of the class of measures ω such that (4.1) holds can be given in terms of Riesz capacities defined by

$$\text{Cap}_{\,\alpha,p}(E) = \inf\left\{\int_{\mathbb{R}^n} g^p\, dx \,:\, g \in L^p_+(\mathbb{R}^n),\; I_\alpha g \geq \chi_E\right\} \tag{4.2}$$

for a Borel set $E \subset \mathbb{R}^n$. Then (4.1) holds for $p = q$ (Maz'ya, D. Adams, Dahlberg) if and only if

$$|E|_\omega \leq C\, \text{Cap}_{\,\alpha,p}(E)$$

with a constant C which is independent of E. (Another proof of of this fact which is valid for more general convolution operators was obtained by Hansson [H].) An equivalent testing condition

$$\int_B [I_\alpha(\chi_B\, d\omega)]^{p'}\, dx \leq C\, |B|_\omega,$$

where C is independent of $B = B_r(x)$, was found by Sawyer along with its two weight generalization [S3] (see also a new proof in [NTV]). Simpler proofs of these results, as well as similar estimates for Green's potentials and Poisson integrals (Carleson measure theorems) can be given using the ideas discussed above. (Cf. Theorems 5 and 9 above.)

The "upper triangle case" $q < p$ is considered to be more difficult and less studied. The following capacitary characterization of (4.1) is due to Maz'ya and Netrusov [MNe]. (An earlier version of this result can be found in [M3].) For a measure ω on $\mathbb{R}^n$ define the function ϕ by

$$\phi(t) = \inf\{\text{Cap}_{\,\alpha,p}(F) \,:\, |F|_\omega \geq t\} \tag{4.3}$$

if $0 < t \leq \omega(\mathbb{R}^n)$ and $\phi(t) = +\infty$ if $t > \omega(\mathbb{R}^n)$.

Theorem 14. *Let* $0 < q < p < \infty$ *and* $p > 1$. *Let* ω *be a locally finite measure on* $\mathbb{R}^n$. *Then* (4.1) *holds if and only if*

$$\int_0^\infty \left[\frac{t^p}{\phi(t)^q}\right]^{1/(p-q)} \frac{dt}{t} < \infty. \tag{4.4}$$

There is a non-capacitary characterization [V1] of the inequality (4.1) in the "upper triangle case", but it is very cumbersome. However, for the fractional maximal operator M_α $(0 < \alpha < n)$ defined by

$$M_\alpha f(x) = \sup_{r>0} \frac{1}{|B_r(x)|^{1-\alpha/n}} \int_{B_r(x)} |f(t)|\, dt,$$

the corresponding weighted norm inequality

$$||M_\alpha f||_{L^q(\omega)} \le C\, ||f||_{L^p(dx)}, \qquad f \in L^p(\mathbb{R}^n)$$

for $0 < q < p$, $p > 1$, can be characterized by the following condition [V1]:

$$M_{\alpha p}\, \omega = \sup_{r>0} \frac{|B_r(x)|_\omega}{r^{n-\alpha p}} \in L^{q/(p-q)}(d\omega). \tag{4.5}$$

In this section we present a new non-capacitary characterization of the embedding (4.1), which is analogous to (4.5) in a sense, established jointly with Carme Cascante and Joaquin Ortega [COV]. It is given in terms of Wolff potentials [HW] defined by

$$\mathcal{W}_{\alpha,p}\, \omega(x) = \int_0^\infty \left[\frac{|B_r(x)|_\omega}{r^{n-\alpha p}}\right]^{p'-1} \frac{dr}{r},$$

where $p' = p/(p-1)$. For fixed α and p, we will also use a brief notation $\mathcal{W}\omega$ in place of $\mathcal{W}_{\alpha,p}\, \omega$.

Wolff's potentials were studied by Adams and Meyers, and Hedberg and Wolff (see [HW], [AH]), and were used in the proof of Wolff's inequality, which is equivalent to our characterization of (4.1) in the case $q = 1$ (see the discussion below). They play an important role in potential and PDE theory, and have been used extensively in the recent study of the p-Laplacian and more general quasilinear equations [KiMa], [MaZi].

Theorem 15. *Let* $1 \le q < p < \infty$. *Let* ω *be a locally finite measure on* $\mathbb{R}^n$. *Then* (4.1) *holds if and only if*

$$\mathcal{W}_{\alpha,p}\, \omega \in L^{q(p-1)/(p-q)}(d\omega). \tag{4.6}$$

Moreover, the embedding constant C *in* (4.1) *is equivalent to the quantity* $||\mathcal{W}_{\alpha,p}\, \omega||^{1/p'}_{L^{q(p-1)/(p-q)}(d\omega)}$.

Analogous embedding theorems for Hardy-Sobolev functions in the unit ball of $\mathbb{C}^n$, where the complex geometry comes into play, are studied in [COV].

Remarks. 1. Theorem 15 remains true in the case $0 < q < 1$, which requires a different proof; it will be given elsewhere, along with more general inequalities with two weights, and other related results.
2. In the case $q = 1$ it follows by duality that (4.1) holds if and only if the energy $\mathcal{E}(\omega)$ is finite, where $\mathcal{E}(\omega) = \mathcal{E}_{\alpha,p}(\omega)$ is defined by

$$\mathcal{E}(\omega) = ||I_\alpha \omega||^{p'}_{L^{p'}(dx)}. \tag{4.7}$$

Moreover, it is easily seen that $\mathcal{E}(\omega) \asymp C^{p'}$, where C is the embedding constant in (4.1) with $q = 1$. In other words, Theorem 15 in the case $q = 1$ is equivalent to Th. Wolff's inequality (see [AH], [HW]):

$$C_1\, \mathcal{E}(\omega) \le \int_{\mathbb{R}^n} \mathcal{W}\omega(x)\, d\omega(x) \le C_2\, \mathcal{E}(\omega), \tag{4.8}$$

where the constants of equivalence depend only on p, α, and n.
3. If $\alpha \ge n/p$, then $\mathcal{W}\omega \equiv +\infty$ and $\mathcal{E}(\omega) = +\infty$ for all measures ω (see [AH]). From this it follows that (4.1) holds only if $\omega = 0$ in this case, so one may assume without loss of generality that $0 < \alpha < n/p$.
4. An analogue of Theorem 15 for Bessel potentials J_α $(0 < \alpha < +\infty)$ is stated in a similar way with a modified potential

$$\widetilde{\mathcal{W}}\omega(x) = \int_0^1 \left[\frac{|B_r(x)|_\omega}{r^{n-\alpha p}}\right]^{p'-1} \frac{dr}{r}.$$

in place of $\mathcal{W}\omega$.
5. There is a weak-type analogue of Theorem 15 for the inequality

$$||I_\alpha f||_{L^{q,\infty}(\omega)} \le C\, ||f||_{L^p(dx)}$$

in terms of the weak integrability of $\mathcal{W}$ [COV]: $\mathcal{W}_{\alpha,p}\,\omega \in L^{q(p-1)/(p-q),\infty}(d\omega)$.

In the proof of Theorem 15 sketched below it will be more convenient to work with a dyadic version of $\mathcal{W}$ also introduced in [HW]. It is defined by

$$\mathcal{W}^d\omega(x) = \sum_{Q\in\mathcal{D}} \left[\frac{|Q|_\omega}{l(Q)^{n-\alpha p}}\right]^{p'-1} \chi_Q(x),$$

where $\mathcal{D} = \{Q\}$ is the family of all dyadic cubes Q in $\mathbb{R}^n$, and $l(Q)$ is the side length of Q. Note that (4.8) holds true for $\mathcal{W}^d\omega$ as well.

We will also need a "shifted" version of $\mathcal{W}\omega$ defined for all $t \in \mathbb{R}^n$ by

$$\mathcal{W}^{d,t}\omega(x) = \sum_{Q \in \mathcal{D}_t} \left[\frac{|Q|_\omega}{l(Q)^{n-\alpha p}} \right]^{p'-1} \chi_Q(x), \tag{4.9}$$

where Q now denotes a shifted dyadic cube in the lattice $\mathcal{D}_t = \mathcal{D} + t = \{Q' + t\}_{Q' \in \mathcal{D}}$. We prove the following version of Theorem 15 which involves the "dyadic potentials" defined above.

Theorem 16. *Let $1 \le q < p < +\infty$, $\alpha > 0$, and let ω be a positive Borel measure on $\mathbb{R}^n$. Then the following statements are equivalent.*

(1) *Inequality* (4.1) *holds for all $f \in L^p(dx)$.*

(2) $\displaystyle\int_{\mathbb{R}^n} \mathcal{W}^d\omega(x)^{q(p-1)/(p-q)}\, d\omega(x) < +\infty.$

(3) $\displaystyle\sup_{t \in \mathbb{R}^n} \int_{\mathbb{R}^n} \mathcal{W}^{d,t}\omega(x)^{q(p-1)/(p-q)}\, d\omega(x) < +\infty.$

(4) *Condition* (4.6) *holds.*

Proof. We first prove (4)⇒(1). By duality (4.1) is equivalent to the inequality

$$||I_\alpha(g\, d\omega)||_{L^{p'}(dx)} = \left\{ \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{g(y)}{|x-y|^{n-\alpha}}\, d\omega(y) \right)^{p'} dx \right\}^{1/p'} \le C\, ||g||_{L^{q'}(d\omega)}$$

for all $g \in L^{q'}(d\omega)$. Without loss of generality we may assume that $g \ge 0$. Now by Wolff's inequality (4.8)

$$||I_\alpha(g\, d\omega)||^{p'}_{L^{p'}(dx)} \le C \int_{\mathbb{R}^n} \mathcal{W}(g\, d\omega)(x)\, g(x)\, d\omega(x).$$

Hence it is enough to show that

$$\int_{\mathbb{R}^n} \mathcal{W}(g\, d\omega)(x)\, g(x)\, d\omega(x) \le C\, ||g||^{p'}_{L^{q'}(d\omega)} \tag{4.10}$$

for all $g \in L^{q'}(d\omega)$, $g \ge 0$. Note that obviously

$$\mathcal{W}(g\, d\omega)(x) = \int_0^\infty \left(\frac{\int_{B_r(x)} g\, d\omega}{r^{n-\alpha p}} \right)^{p'-1} \frac{dr}{r} \le M_\omega g(x)^{p'-1}\, \mathcal{W}\omega(x),$$

where

$$M_\omega\, g(x) = \sup_{r>0} \frac{1}{|B_r(x)|_\omega} \int_{B_r(x)} |g(y)|\, d\omega(y)$$

is the centered maximal function of g with respect to ω.

Hence the above estimate, together with Hölder's inequality with exponents $r = q'/(p'-1)$, $r' = r/(r-1)$, gives

$$\begin{aligned}\int_{\mathbb{R}^n} \mathcal{W}(g\, d\omega)(x)\, g(x)\, d\omega(x) &\le \int_{\mathbb{R}^n} M_\omega g(x)^{p'-1}\, g(x) \mathcal{W}\omega(x)\, d\omega(x)\\ &\le \left(\int_{\mathbb{R}^n} M_\omega g(x)^{q'}\, d\omega(x)\right)^{1/r}\\ &\quad \times \left(\int_{\mathbb{R}^n} (g(x)\mathcal{W}\omega(x))^{r'}\, d\omega(x)\right)^{1/r'}.\end{aligned}$$

It is known that M_ω is bounded in $L^{q'}(d\omega)$ (see [Fe]), and this fact together with Hölder's inequality with exponent $\lambda = q'/r' > 1$ (since $q < p$) gives that the above is bounded by

$$C\, ||g||^{p'}_{L^{q'}(d\omega)} \left(\int_{\mathbb{R}^n} \mathcal{W}_{\alpha p}(\omega)(x)^{r'\lambda'}\, d\omega(x)\right)^{1/(r'\lambda')}.$$

Since $r'\lambda' = q(p-1)/(p-q)$, it follows that (4.9) holds. The proof of (4)⇒(1) is complete.

The same argument with $\mathcal{W}^d$ in place of $\mathcal{W}$ proves (2)⇒(1). We only have to apply Wolff's inequality in the form

$$||I_\alpha(g\, d\omega)||^{p'}_{L^{p'}(dx)} \le C \int_{\mathbb{R}^n} \mathcal{W}^d(g\, d\omega)(x)\, g(x)\, d\omega(x),$$

and also use the dyadic maximal operator

$$M^d_\omega g(x) = \sup_{Q\in\mathcal{D};\, x\in Q} \frac{1}{|Q|_\omega} \int_Q |g(y)|\, d\omega(y)$$

in place of the centered version M_ω; it is known that M^d_ω is bounded in $L^r(d\omega)$, $r > 1$ (see e.g. [S2]).

We now prove (1)⇒(2). This could be shown by using the estimates established in [V2] in the framework of discrete Littlewood-Paley spaces.

An alternative direct proof of this crucial step may be sketched as follows. By duality (4.1) is equivalent to

$$||I_\alpha(g\,d\omega)||^{p'}_{L^{p'}(dx)} \le C\,||g||^{p'}_{L^{q'}(d\omega)},$$

for all $g \in L^{q'}(d\omega)$. Let $d\nu = g\,d\omega$, $g \ge 0$. Then by (4.8) with $\mathcal{W}^d\nu$ in place of $\mathcal{W}\omega$ we have

$$\begin{aligned} ||I_\alpha(g\,d\omega)||^{p'}_{L^{p'}(dx)} &\ge C \int_{\mathbb{R}^n} \mathcal{W}^d(g\,d\omega)(x)\,g\,d\omega(x) \\ &= C \sum_Q \left(\frac{\int_Q g(x)\,d\omega(x)}{|Q|^{1-\alpha/n}} \right)^{p'} |Q|. \end{aligned}$$

Hence

$$\sum_Q \left(\frac{\int_Q g(x)\,d\omega(x)}{|Q|^{1-\alpha/n}} \right)^{p'} |Q| \le C\,||g||^{p'}_{L^{q'}(d\omega)}.$$

Let

$$c_Q = \left(\frac{|Q|_\omega}{|Q|^{1-\alpha/n}} \right)^{p'} |Q|. \tag{4.11}$$

Then the preceding inequality may be rewritten in the form

$$\sum_Q c_Q \left(\frac{\int_Q g(x)\,d\omega(x)}{|Q|_\omega} \right)^{p'} \le C\,||g||^{p'}_{L^{q'}(d\omega)}.$$

Let

$$g(x) = (M^d_\omega \phi)^{1/p'}(x) = \left(\sup_{x\in Q} \frac{1}{|Q|_\omega} \int_Q \phi(y)\,d\omega(y) \right)^{1/p'},$$

where $\phi \in L^{\frac{q'}{p'}}(\omega)$, $\phi \ge 0$. Then obviously

$$\left(\frac{\int_Q g(x)\,d\omega(x)}{|Q|_\omega} \right)^{p'} \ge \frac{\int_Q \phi(x)\,d\omega(x)}{|Q|_\omega}$$

and

$$||g||^{p'}_{L^{q'}(d\omega)} \le C\,||\phi||_{L^{q'/p'}(d\omega)}.$$

Combining these estimates, we get

$$\sum_Q c_Q \, \frac{\int_Q \phi(x)\, d\omega(x)}{|Q|_\omega} \le C\,||\phi||_{L^{q'/p'}(d\omega)},$$

for any $\phi \in L^{q'/p'}(d\omega)$. By duality this gives

$$\sum_Q \frac{c_Q}{|Q|_\omega} \chi_Q \in L^{q'/(q'-p')}(d\omega),$$

which is equivalent to $\int_{\mathbb{R}^n} \mathcal{W}^d\omega(x)^{q(p-1)/(p-q)}\, d\omega(x) < +\infty$. We have proved (1)⇔(2).

Applying the same argument as above with $\mathcal{D}_t$ in place of $\mathcal{D}$, we obtain

$$\int_{\mathbb{R}^n} \mathcal{W}^{d,t}\omega(x)^{q(p-1)/(p-q)}\, d\omega(x) \le C < +\infty$$

with a constant C independent of $t \in \mathbb{R}^n$, i.e. (1)⇔(3).

It remains to show (3)⇒(4). To handle the case of a possibly non-doubling ω we use the well-known idea of C. Fefferman and E. M. Stein [FSt] (see also [S2]) based on the averaging of $\mathcal{D}_t$ over the shifts of the dyadic lattice $\mathcal{D}$. For $R > 0$, set

$$\mathcal{W}^R\omega(x) = \int_0^R \left[\frac{\omega(B_r(x))}{r^{n-\alpha p}}\right]^{p'-1} \frac{dr}{r}.$$

We derive (4) from (3) using the estimate

$$\mathcal{W}^R\omega(x) \le C\,R^{-n} \int_{|t|\le cR} \mathcal{W}^{d,t}\omega(x)\, dt. \qquad (4.12)$$

where the constants C and c depend only on n.

Assuming that (4.12) holds, and applying Hölder's inequality with exponent $\dfrac{q(p-1)}{p-q} > 1$ together with Fubini's theorem, we have

$$\int_{\mathbb{R}^n} \mathcal{W}^R\omega(x)^{q(p-1)/(p-q)}\, d\omega(x)$$
$$\le C\,R^{-n} \int_{|t|\le cR} \int_{\mathbb{R}^n} \mathcal{W}^{d,t}\omega(x)^{q(p-1)/(p-q)}\, d\omega(x)\, dt \le C < \infty,$$

where C is independent of R. The proof of (4) is then completed by letting $R \to +\infty$ and using the monotone convergence theorem.

To prove (4.12), we need to modify the argument of Lemma 2 in [S2] as follows. The maximal functions $M_d^\omega f$ are replaced here by the corresponding potentials $\mathcal{W}^d\omega$, with ω playing the role of $|f|\, d\omega$. Then the doubling property of ω used in [S2] is irrelevant; we just make use of the trivial doubling property of Lebesgue measure. (See details in [COV].) □

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