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AN ASYNCHRONOUS THREE-FIELD DOMAIN DECOMPOSITION METHOD FOR FIRST-ORDER EVOLUTION PROBLEMS

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Abstract

We present an asynchronous multi-domain time integration algorithm with a dual domain decomposition method for the initial boundary-value problems for a parabolic equation. For efficient parallel computing, we apply the three-field domain decomposition method with local Lagrange multipliers to ensure the continuity of the primary unknowns at the interface between subdomains. The implicit method for time discretization and the multi-domain spatial decomposition enable us to use different time steps (subcycling) on different parts of a computational domain, and thus efficiently capture the underlying physics with less computational effort. We illustrate the performance of the proposed multi-domain time integrator by means of a simple numerical example.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain split into a finite number of non-overlapping subdomains Ω^k $(k = 1, \dots, N_D)$. Let $\Omega = \overline{\bigcup_{k=1}^{N_D} \Omega^k}$, $\Gamma^k = \partial\Omega^k$, $\Sigma = \bigcup_{k=1}^{N_D} \Gamma^k \backslash \partial\Omega$. We introduce the bilinear form

$$((u, v))_k := \int_{\Omega^k} \sum_{|i|\le 1} \sum_{|j|\le 1} (-1)^{|i|} a_{ij}(x) D^j u \, D^i v \, \mathrm{d}x \quad \forall u, v \in H^1(\Omega^k), \tag{1}$$

where $i = (i_1, i_2)$ and $j = (j_1, j_2)$ are two-dimensional vectors, i_1, i_2, j_1, j_2 are nonnegative integers and $|i| = i_1 + i_2$ and $|j| = j_1 + j_2$. The summation in (1) means that summation should be carried out over all i and j, for which $|i| \le 1$, $|j| \le 1$ holds. We assume that the coefficient functions a_{ij} belong to $L^\infty(\Omega)$. We assume there exists a positive number ϵ (independent of v) such that $((v, v))_k \ge \epsilon \|v\|^2_{H^1(\Omega^k)}$ for every $v \in H^1_0(\Omega^k)$. Further, for every $u, v \in \prod_k H^1(\Omega^k)$ we set $((u, v)) := \sum_k ((u, v))_k$. From now on we are going to use the following notation: $V := \prod_k H^1(\Omega^k)$ and $M := \prod_k H^{-1/2}(\Gamma^k)$, $(\cdot, \cdot)$ will be the usual inner product in $L^2(\Omega)$, $(\cdot, \cdot)_k$ will be the inner product in $L^2(\Omega^k)$ and $\langle \cdot, \cdot \rangle_k$ will be the duality pairing between $H^{-1/2}(\Gamma^k)$

and $H^{1/2}(\Gamma^k)$. Finally, introduce the space $\Phi := \left\{\varphi \in L^2(\Sigma);\ \exists v \in H_0^1(\Omega),\ \varphi = v\big|_\Sigma\right\}$ equipped with the norm $\|\varphi\|_\Phi = \inf\left\{\|v\|_{H^1(\Omega)};\ v \in H_0^1(\Omega),\ v\big|_\Sigma = \varphi\right\}$ .We now consider the following two equivalent model problems. Let $T > 0$ be fixed and assume $u_0 \in H_0^1(\Omega)$, $f \in L^2(0,T;L^2(\Omega))$:

(i) find $u \in L^2(0,T;H_0^1(\Omega))$ with $\partial_t u \in L^2(0,T;L^2(\Omega))$, such that

$$(\partial_t u, v) + ((u,v)) = (f,v) \quad \forall v \in H_0^1(\Omega) \quad \text{and} \quad u(x,0) = u_0(x) \text{ in } \Omega; \tag{2}$$

(ii) find $u^k \in L^2(0,T;H^1(\Omega^k))$ with $\partial_t u^k \in L^2(0,T;L^2(\Omega^k))$, $\lambda^k \in L^2(0,T;H^{-1/2}(\Gamma^k))$ and $w \in L^2(0,T;\Phi)$, such that $u^k(x,0) = u_0(x)\big|_{\Omega^k}$ and (for $k = 1,\dots,N_D$)

$$\begin{cases} (\partial_t u^k, v^k)_k + ((u^k,v^k))_k - \langle\lambda^k, v^k\rangle_k &= (f,v^k)_k \quad \forall v^k \in H^1(\Omega^k), \\ \langle u^k, \mu^k\rangle_k &= \langle w, \mu^k\rangle_k \quad \forall \mu^k \in H^{-1/2}(\Gamma^k), \\ \sum_{k=1}^{N_D}\langle\lambda^k,\varphi\rangle_k &= 0 \qquad\quad \forall \varphi \in \Phi. \end{cases} \tag{3}$$

Let us mention that problem (3) is well suited for domain decomposition methods. By the standard linear parabolic equation theory [4], both problems (2) and (3) admit the unique solution, such that $u = u^k$ in Ω^k, $\lambda^k = \nabla u \cdot \mathbf{n}_A^k$ on $\partial\Omega^k$ and $w = u$ on Σ. To solve problem (3) numerically, we propose a new numerical scheme which is based on the subcycling algorithm using non-standard asynchronous time discretization amenable for parallel computing.

2. Asynchronous multi-domain discretization in time

Let us fix $p \in \mathbb{N}$ and let $\tau := T/p$ be a time step. Next, we introduce a substep time $\tau^k = \tau/s^k$, which is proportional to the system time step $\tau = t_{n+1} - t_n$, where s^k is the number of substeps for domain k, as shown schematically in Figure 1. Further, we introduce the backward difference quotient $\delta_{\tau^k}\phi_{n,j}^k := (\phi_{n,j}^k - \phi_{n,j-1}^k)/\tau^k$ for $n = 0,\dots,p-1$. In view of the assumed relationships between the discretization steps, the present “method of asynchronous discretization in time” consists in the following: find, successively for $n = 0,1,2,\dots,p-1$, functions $u_{n,j}^k \in H^1(\Omega^k)$, $\lambda_{n,j}^k \in H^{-1/2}(\partial\Omega^k)$ and $w_{n+1} \in \Phi$, $k = 1,\dots,N_D$, $j = 1,\dots,s^k$, as solutions of the problems

$$(\delta_{\tau^k} u_{n,j}^k, v_j^k)_k + ((u_{n,j}^k, v^k))_k - \langle\lambda_{n,j}^k, v_j^k\rangle_k = (f_{n,j}^k, v_j^k)_k \qquad \forall v_j^k \in H^1(\Omega^k), \tag{4}$$

$$\langle u_{n,j}^k, \mu_j^k\rangle_k = \langle w_{n,j}^k, \mu_j^k\rangle_k \qquad \forall \mu_j^k \in H^{-1/2}(\Gamma^k), \tag{5}$$

$$\sum_{k=1}^{N_D}\langle\lambda_{n,s^{N_D}}^k, \varphi\rangle_k = 0 \qquad\qquad \forall \varphi \in \Phi, \tag{6}$$

starting with the functions $u_{0,0}^k(x) = u_0(x)\big|_{\Omega^k} \in H^1(\Omega^k)$.

In this work, the equation of continuity of fluxes is required only at the final (system) time step, see (6). The unknown $w_{n,j}^k$ on the common interface Σ is linearly interpolated at the intermediate steps by

$$w_{n,j}^k = \left(1 - \frac{j}{s^k}\right) w_n + \left(\frac{j}{s^k}\right) w_{n+1} \quad \forall j = 1,\dots,s^k, \quad k = 1,\dots,N_D.$$

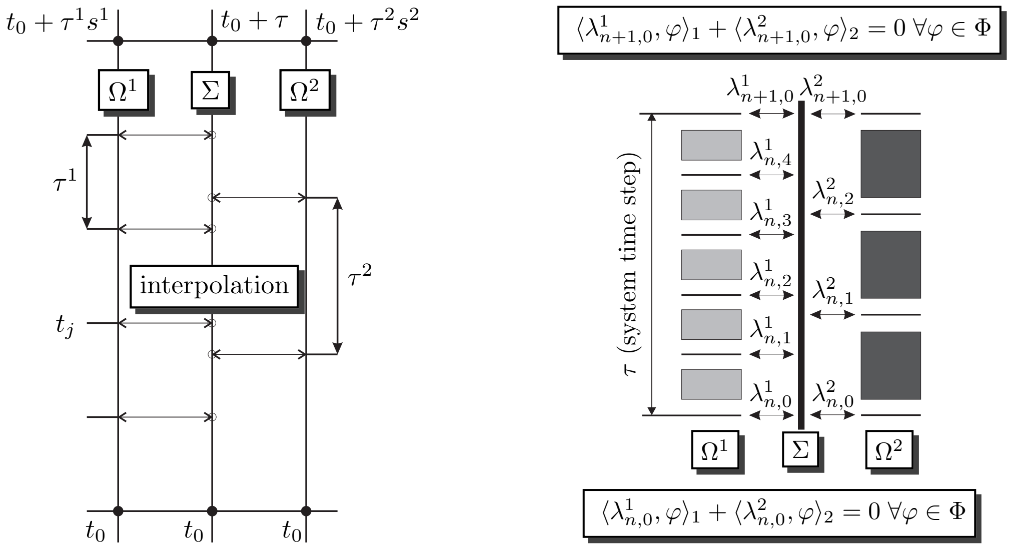

Figure 1: Substeps of the system time step. Example for $N_D = 2$, $s^1 = 5$, $s^2 = 3$.

Theorem 1. *Problem* (4)–(6) *has a unique solution.*

Proof. Without loss of generality we assume $u^k_{n,0} = 0$. First, we associate with any $\varphi \in \Phi$ a vector function

$$\widetilde{\boldsymbol{\varphi}} = (\tilde{u}^1_{n,1}, \tilde{u}^1_{n,2}, \dots, \tilde{u}^1_{n,s^1}, \tilde{u}^2_{n,1}, \tilde{u}^2_{n,2}, \dots, \tilde{u}^2_{n,s^2}, \dots, \tilde{u}^{N_D}_{n,1}, \tilde{u}^{N_D}_{n,2}, \dots, \tilde{u}^{N_D}_{n,s^{N_D}}) \in \prod_k H^1(\Omega^k)^{s^k},$$

components of which are defined as solutions of the following Dirichlet problems

$$\begin{aligned} (\delta_{\tau^k}\tilde{u}^k_{n,j}, v^k)_k + ((\tilde{u}^k_{n,j}, v^k))_k &= 0 && \forall v^k \in H^1_0(\Omega^k), \\ \langle \tilde{u}^k_{n,j}, \mu^k\rangle_k &= \langle \big(j/s^k\big)\, \varphi, \mu^k\rangle_k && \forall \mu^k \in H^{-1/2}(\Gamma^k) \end{aligned}$$

for $k = 1, \dots, N_D$, $j = 1, \dots, s^k$. Note that

$$|||\widetilde{\boldsymbol{\varphi}}|||_{\prod_k H^1(\Omega^k)^{s^k}} := \sum_k^{N_D} \sum_j^{s^k} \|\tilde{u}^k_{n,j}\|_{H^1(\Omega^k)} \le c\|\varphi\|_\Phi. \tag{7}$$

Now we assume a given function $\psi \in \Phi$ and set a vector functions

$$\boldsymbol{u} = (u^1_{n,1}, u^1_{n,2}, \dots, u^1_{n,s^1}, u^2_{n,1}, u^2_{n,2}, \dots, u^2_{n,s^2}, \dots, u^{N_D}_{n,1}, u^{N_D}_{n,2}, \dots, u^{N_D}_{n,s^{N_D}})$$

and

$$\boldsymbol{\lambda} = (\lambda^1_{n,1}, \lambda^1_{n,2}, \dots, \lambda^1_{n,s^1}, \lambda^2_{n,1}, \lambda^2_{n,2}, \dots, \lambda^2_{n,s^2}, \dots, \lambda^{N_D}_{n,1}, \lambda^{N_D}_{n,2}, \dots, \lambda^{N_D}_{n,s^{N_D}})$$

so that $\boldsymbol{u} = \widetilde{\boldsymbol{\psi}}$ and

$$(\delta_{\tau^k} u^k_{n,j}, v^k)_k + ((u^k_{n,j}, v^k))_k - \langle \lambda^k_{n,j}, v^k\rangle_k = 0 \qquad \forall v^k \in H^1(\Omega^k), \tag{8}$$

$$\langle u^k_{n,j} - \big(j/s^k\big)\, \psi, \mu^k\rangle_k = 0 \qquad \forall \mu^k \in H^{-1/2}(\Gamma^k) \tag{9}$$

for $j = 1, \dots, s^k$, $k = 1, \dots, N_D$. We now define the operator $\mathcal{S} : \Phi \to \Phi^*$ by

$$\langle \mathcal{S}(\psi), \cdot \rangle_{\Phi^*,\Phi} = \sum_{k=1}^{N_D} \langle \lambda^k_{n,s^k}, \cdot \rangle_k.$$

From (7) and (8) we easily compute (recall $\boldsymbol{u} = \widetilde{\boldsymbol{\psi}}$)

$$\langle \mathcal{S}(\psi), \varphi \rangle_{\Phi^*,\Phi} = \sum_{k=1}^{N_D} \langle \lambda^k_{n,s^k}, \varphi \rangle_k \leq \alpha \|\psi\|_\Phi \|\varphi\|_\Phi. \tag{10}$$

On the other hand, taking $\varphi = \psi$ we have, combining (8) and (9),

$$\begin{aligned} \langle \mathcal{S}(\psi), \psi \rangle_{\Phi^*,\Phi} = \sum_{k=1}^{N_D} \langle \lambda^k_{n,s^k}, \psi \rangle_k &= \sum_{k=1}^{N_D} (\delta_{\tau^k} u^k_{n,s^k}, u^k_{n,s^k})_k + \sum_{k=1}^{N_D} ((u^k_{n,s^k}, u^k_{n,s^k}))_k \\ &\geq \gamma \|\psi\|^2_\Phi. \end{aligned} \tag{11}$$

In (10) and (11), α and γ are positive constants, independent of ψ and φ. Hence, $\mathcal{S}$ is an isomorphism from Φ onto Φ^*.

We now turn back, for a moment, to (4)–(6) and consider $\breve{u}^k_{n,j} \in H^1_0(\Omega^k)$ and $\breve{\lambda}^k_{n,j} \in H^{-1/2}(\partial\Omega^k)$ as the solution of the problem

$$(\delta_{\tau^k} \breve{u}^k_{n,j}, v^k)_k + ((\breve{u}^k_{n,j}, v^k))_k - \langle \breve{\lambda}^k_{n,j}, v^k \rangle_k = (f^k_{n,j}, v^k)_k \qquad \forall v^k \in H^1(\Omega^k),$$

for $k = 1, \dots, N_D$, $j = 1, \dots, s^k$. The existence of such solutions is ensured by [1]. We now define the functional $g \in \Phi^*$ by

$$\langle g, \cdot \rangle_{\Phi^*,\Phi} = \sum_{k=1}^{N_D} \langle -\breve{\lambda}^k_{n,s^k}, \cdot \rangle_k.$$

Problem (4)–(6) can now be reduced to problem

$$\mathcal{S}(\psi) = g.$$

Now with ψ in hand, we determine $u^k_{n,j} \in H^1(\Omega^k)$ and $\lambda^k_{n,j} \in H^{-1/2}(\partial\Omega^k)$ as the solution of decoupled (independent) Dirichlet problems (8) and (9) for $k = 1, \dots, N_D$, $j = 1, \dots, s^k$. It is easy to verify, that $u^k_{n,j}$, $\lambda^k_{n,j}$ and $w^k_{n,j} = \left(\frac{j}{s^k}\right) \psi$ solve uniquely problem (4)–(6). Recall that we considered for simplicity $u^k_{n,0} = 0$. The proof is complete. □

Remark 2. *Let us explicitly mention, that $\mathcal{S}$ corresponds to the Poincaré-Steklov operator on Σ, well known in the theory of domain decomposition methods for elliptic problems, see [2, 3].*

3. Numerical example

We approximate the problem (3) in space choosing V_h, M_h and Φ_h finite dimensional subspaces of V, M and Φ and introduce $\boldsymbol{u}_h(x,t) = \mathbf{N}_u(x)\widetilde{\boldsymbol{u}}(t)$, $\boldsymbol{w}_h(x,t) = \mathbf{N}_w(x)\widetilde{\boldsymbol{w}}(t)$ and $\boldsymbol{\lambda}_h(x,t) = \mathbf{N}_{\boldsymbol{\Lambda}}(x)\widetilde{\boldsymbol{\Lambda}}(t)$, such that $\boldsymbol{u}_h(t) \in V_h$, $\boldsymbol{w}_h(t) \in \Phi_h$ and $\boldsymbol{\lambda}_h(t) \in M_h$ for all $t \in (0,T)$, respectively. Application of FEM-discretization in space leads to the following system of equations ($j = 1,\dots,s^k$, $k = 1,\dots,N_D$):

$$\begin{cases} \mathbf{M}^k \delta_{\tau^k} \boldsymbol{u}^k_{n,j} + \mathbf{K}^k \boldsymbol{u}^k_{n,j} + (\mathbf{C}^k)^T \boldsymbol{\Lambda}^k_{n,j} &= \boldsymbol{f}^k_{n,j}, \\ \mathbf{C}^k \boldsymbol{u}^k_{n,j} - \left(\frac{j}{s^k}\right) \mathbf{B}^k \boldsymbol{w}_{n+1} - \left(1 - \frac{j}{s^k}\right) \mathbf{B}^k \boldsymbol{w}_n &= \mathbf{0}, \\ \sum_{k=1}^{N_D} (\mathbf{B}^k)^T \boldsymbol{\Lambda}^k_{n,s^k} &= \mathbf{0}. \end{cases}$$

Using the common nomenclature of heat conduction, $\mathbf{M}^k$ is the capacitance matrix, $\mathbf{K}^k$ is the conductance matrix and the vector $\boldsymbol{f}^k$ represents the nodal values of the source corresponding to subdomain Ω^k. Operators $\mathbf{C}^k$ and $\mathbf{B}^k$ are the Boolean matrices extracting the interface degrees of freedom from $\boldsymbol{u}$ and the corresponding degrees of freedom from $\boldsymbol{w}$ for a particular subdomain k. The above system can be written in a matrix form, which has a block-bordered structure amenable to parallel computation. In this work, we solve for all unknowns simultaneously by a monolithic method using a direct solver. In order to briefly present the performance of the proposed algorithm, we consider a simple test problem. A square of size 1.0×1.0 is divided into two equal subdomains, and each subdomain is divided into 5×10 square elements, see Figure 2. We consider the analytical solution given by

$$u^*(x_1, x_2, t) = \sin(\pi x_1)\sin(\pi x_2)\sin(t)$$

and assume the coefficient functions as constants: $a_{ij}(x) = \delta_{ij} 10^{-4}$ in Ω. Hence, the right hand side takes the form

$$f^* = \left[\cos(t) + 2 \times 10^{-4}\pi^2 \sin(t)\right] \sin(\pi x_1)\sin(\pi x_2).$$

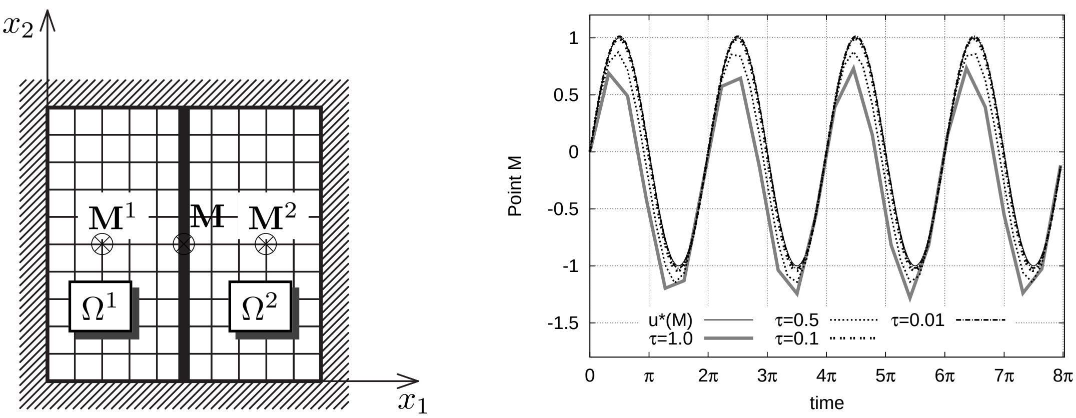

Figure 2: 2D test problem (left). Numerical results at the point M for various system time steps (right).

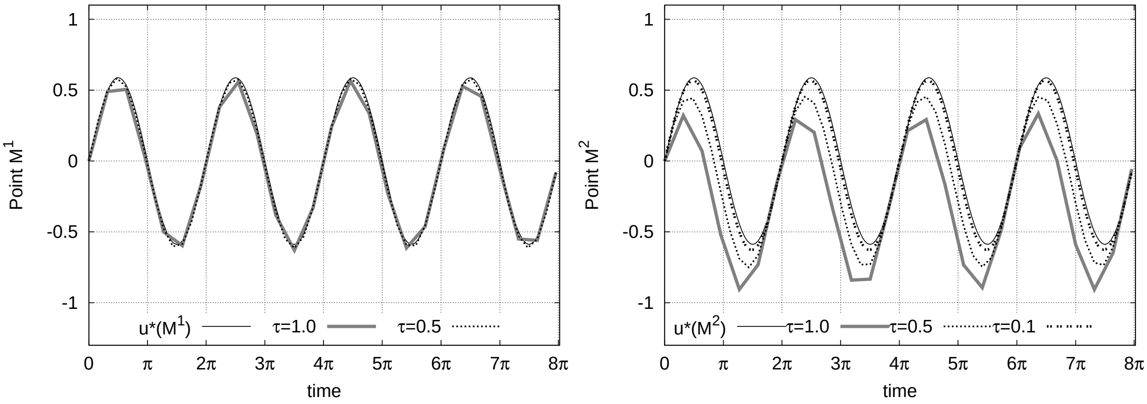

Figure 3: Results at the points M^1 (left) and M^2 (right).

In Figures 2 and 3 we have shown the results at points M, M^1 and M^2 for various system time steps τ, the ratio $s^1 : s^2 = 10 : 1$. As predicted by the theory, the numerical results are stable and match well with the analytical solution for sufficiently small system time step (approx. $\tau \approx 0.1$).

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