Karel Segeth; Pavel Šolín

On some a posteriori error estimation results for the method of lines

In: Jan Chleboun and Karel Segeth and Tomáš Vejchodský (eds.): Programs and Algorithms of Numerical Mathematics, Proceedings of Seminar. Prague, May 28-31, 2006. Institute of Mathematics AS CR, Prague, 2006. pp. 229–234.

Persistent URL: http://dml.cz/dmlcz/702842

ON SOME A POSTERIORI ERROR ESTIMATION RESULTS FOR THE METHOD OF LINES*

Karel Segeth, Pavel Šolín

Abstract

The paper is an attempt to present an (incomplete) historical survey of some basic results of residual type estimation procedures from the beginning of their development through contemporary results to future prospects. Recently we witness a rapidly increasing use of the *hp*-FEM which is due to the well-established theory. However, the conventional a posteriori error estimates (in the form of a single number per element) are not enough here, more complex estimates are needed, and this can be the way to obtain them.

1. Introduction

In the 1990's, the subject of a posteriori error estimation with the finite element method and adaptive solution procedures started its very rapid development. Many results for the solution of linear and nonlinear elliptic partial differential equations were reached and first results for the solution of nonlinear parabolic partial differential equations were published. A pioneering paper in this field was [2].

We present some basic results from this time period and continue to contemporary results and future prospects of this approach. A rich contemporary source of knowledge is the book [3].

Recently, we witness a rapidly increasing use of the *hp*-FEM. We are concerned with this subject in the conclusion of this paper. We also refer to some published numerical results and their accuracy.

We introduce a nonlinear parabolic model problem and its finite element solution in Sections 2 and 3 while in Section 4 we are concerned with a posteriori error estimation. We quote some adaptive grid refinement procedures and speak about further prospects in Section 5.

We apologize to all colleagues whose names and contributions to the subject were not, for the lack of space, mentioned in this paper.

2. Model problem

We introduce a nonlinear parabolic model problem. For the sake of brevity, we consider only one equation with a scalar solution u and a single 1D space variable x. All the results can be generalized to a system of parabolic equations and a d-dimensional space variable.

*This work was supported by Grant 201/04/1503 of the Czech Science Foundation and by Institutional Research Plan AV0Z10190503 of the Academy of Sciences of the Czech Republic.

Let us consider the problem

$$\frac{\partial u}{\partial t}(x,t) - \frac{\partial}{\partial x}\Big(a(u)\frac{\partial u}{\partial x}(x,t)\Big) + f(u) = 0 \quad \text{for} \quad 0 < x < 1, \quad 0 < t \le T \tag{1}$$

with the boundary conditions

$$u(0,t) = u(1,t) = 0, \quad 0 \le t \le T, \tag{2}$$

and the initial condition

$$u(x,0) = u_0(x), \quad 0 < x < 1, \tag{3}$$

where u_0 is a given function.

Let us assume

$$\begin{aligned} &0 < \mu \le a(s) \le M, \quad s \in R,\\ &|a(r) - a(s)| \le L|r-s|,\\ &|f(r) - f(s)| \le L|r-s|, \quad r,s \in R, \end{aligned}$$

where μ, M, and L are positive constants. We need some more assumptions for some of the proofs, see [9].

In the standard way we introduce the *weak solution* $u(x,t) \in H^1([0,T], H_0^1(0,1))$ of the model problem by the identity

$$\Big(\frac{\partial u}{\partial t}, v\Big) + \Big(a(u)\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}\Big) + (f(u), v) = 0 \tag{4}$$

to be satisfied for $t \in (0,T]$ by all test functions $v \in H_0^1(0,1)$ and the identity

$$\Big(a(u_0)\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}\Big) = \Big(a(u_0)\frac{\partial u_0}{\partial x}, \frac{\partial v}{\partial x}\Big) \tag{5}$$

to be satisfied for $t = 0$ also by all test functions $v \in H_0^1(0,1)$. This latter identity corresponds to the initial condition. Some other weak formulations of the initial condition are also possible. We use the symbol $(\cdot,\cdot)$ for the usual $L_2(0,1)$ inner product and $\|\cdot\|_1$ for the $H^1(0,1)$ norm.

3. Semidiscrete approximate solution

To define the finite element solution of the problem (1) to (3), we start with the space discretization (*semidiscretization*). We choose a partition

$$0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1 \tag{6}$$

of the space interval $[0,1]$ and further put

$$h_j = x_j - x_{j-1}, \quad j = 1,\dots,N, \quad \text{and} \quad h = \max_{j=1,\dots,N} h_j.$$

We use the notation

$$(v, w)_j = \int_{x_{j-1}}^{x_j} v(x)w(x)\,\mathrm{d}x$$

for the inner product restricted to the interval $[x_{j-1}, x_j]$, and similarly $\|v\|_j$ and $\|v\|_{1,j}$ for the restricted $L_2(0,1)$ and $H^1(0,1)$ norms.

On the partition (6), we construct a finite dimensional subspace

$$S_0^{N,p} = \Big\{ V \mid V \in H_0^1(0,1),\ V(x) = \sum_{j=1}^{N-1} V_{j1}\varphi_{j1}(x) + \sum_{j=1}^{N}\sum_{k=2}^{p} V_{jk}\varphi_{jk}(x) \Big\}$$

of the space $H_0^1(0,1)$.

The functions φ_{jk} are chosen to form a *hierarchic basis*. For $k = 1$, we put

$$\begin{aligned} \varphi_{j1}(x) &= (x - x_{j-1})/h_j, \quad x_{j-1} \le x < x_j, \\ &= (x_{j+1} - x)/h_{j+1}, \quad x_j \le x \le x_{j+1}, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

These functions are the well known *hat* or *chapeau functions*. For $k > 1$, we further put

$$\begin{aligned} \varphi_{jk}(x) &= \frac{\sqrt{2(2k-1)}}{h_j} \int_{x_{j-1}}^{x} \mathrm{P}_{k-1}(y)\,\mathrm{d}y, \quad x_{j-1} \le x \le x_j, \\ &= 0 \quad \text{otherwise,} \end{aligned}$$

where P_k is a *Legendre polynomial* transformed from $[-1,1]$ to $[x_{j-1}, x_j]$. These functions (primitive functions to Legendre polynomials) are called the *Lobatto polynomials* or *bubble functions*. The idea of hierarchic basis functions was first introduced in the book [11].

The principal idea of the method of lines is the space semidiscretization while the time variable remains continuous. We look for the *semidiscrete approximate solution* $\bar{U}(x,t) \in H^1([0,T], S_0^{N,p})$ in the form

$$\bar{U}(x,t) = \sum_{j=1}^{N-1} \bar{U}_{j1}(t)\varphi_{j1}(x) + \sum_{j=1}^{N}\sum_{k=2}^{p} \bar{U}_{jk}(t)\varphi_{jk}(x).$$

We require that the identities

$$\Big(\frac{\partial \bar{U}}{\partial t}, V\Big) + \Big(a(\bar{U})\frac{\partial \bar{U}}{\partial x}, \frac{\partial V}{\partial x}\Big) + (f(\bar{U}), V) = 0, \quad t \in (0,T], \quad V \in S_0^{N,p}, \tag{7}$$

$$\Big(a(u_0)\frac{\partial \bar{U}}{\partial x}, \frac{\partial V}{\partial x}\Big) = \Big(a(u_0)\frac{\partial u_0}{\partial x}, \frac{\partial V}{\partial x}\Big), \quad t = 0, \quad V \in S_0^{N,p}, \tag{8}$$

that correspond to the identities (4), (5), be satisfied. The basis functions as well as test functions are thus chosen from the same space $S_0^{N,p}$. Note that after substituting φ_{il} for the test functions $V(x)$ in (7), we obtain an initial value problem for a system of ordinary differential equations with the initial condition (8). Other initial conditions can be employed, too.

The ordinary differential system (7) with the initial condition (8) for the unknown coefficients $\bar{U}_{jk}(t)$ is then solved by standard numerical software.

4. Analysis of residual a posteriori semidiscrete error indicators

Let us denote the error of the semidiscrete solution $\bar{U}(x,t)$ by

$$e(x,t) = u(x,t) - \bar{U}(x,t).$$

We introduce the finite dimensional space

$$\hat{S}_0^{N,p+1} = \Big\{\hat{V} \mid \hat{V} \in H_0^1(0,1), \hat{V}(x) = \sum_{j=1}^{N} \hat{V}_j \varphi_{j,p+1}(x)\Big\}$$

and approximation of the error

$$\bar{E}(x,t) = \sum_{j=1}^{N} \bar{E}_j(t)\varphi_{j,p+1}(x).$$

Note that we look for approximation of the error in the finite element space of piecewise polynomials of the degree $p+1$.

Some results on the semidiscrete error for the case of linear parabolic equations and systems were given in [1], [7].

Some time later, they were generalized to the nonlinear case. If we subtract the identities (7), (8) that define the semidiscrete solution $\bar{U}$ from the identities (4), (5) that define the weak solution u we obtain for $\bar{E}(x,t) \in H^1([0,T], \hat{S}_0^{N,p+1})$ the initial value problem for the system of ordinary differential equations

$$\begin{aligned}
&\Big(\frac{\partial \bar{E}}{\partial t}, \hat{V}\Big)_j + \Big(a(\bar{U}+\bar{E})\frac{\partial \bar{E}}{\partial x}, \frac{\partial \hat{V}}{\partial x}\Big)_j \qquad (9)\\
&\quad = -(f(\bar{U}+\bar{E}), \hat{V})_j - \Big(\frac{\partial \bar{U}}{\partial t}, \hat{V}\Big)_j - \Big(a(\bar{U}+\bar{E})\frac{\partial \bar{U}}{\partial x}, \frac{\partial \hat{V}}{\partial x}\Big)_j, \ t \in (0,T], \ \hat{V} \in \hat{S}_0^{N,p+1},
\end{aligned}$$

with the initial condition

$$\Big(a(u_0)\frac{\partial \bar{E}}{\partial x}, \frac{\partial \hat{V}}{\partial x}\Big)_j = \Big(a(u_0)\frac{\partial (u_0 - \bar{U})}{\partial x}, \frac{\partial \hat{V}}{\partial x}\Big)_j, \quad t=0, \quad \hat{V} \in \hat{S}_0^{N,p+1}. \qquad (10)$$

The quantity $\bar{E}$ defined by (9), (10) is called the *nonlinear parabolic error indicator*. Note that (9), (10) is a nonlinear problem for the unknowns $\bar{E}_j(t)$. For the practical computation, these equations can be added to the system (7), (8) for finding the semidiscrete solution $\bar{U}_{jk}(t)$. Further, note that the equations of the system (9) are uncoupled.

There are some simplifications that allow for more efficient computation while, asymptotically, the error indicator is of the same quality. The *linear parabolic error indicator*, and *nonlinear* and *linear elliptic error indicator* are defined in an analogous way. The detailed description can be found in, e.g., [5] or [9]. The following theorem is proven in [9] for the nonlinear parabolic error indicator.

Theorem. *Let the weak solution $u(x,t)$ given by* (4), (5) *be smooth, let $\bar{U}(x,t)$ and $\bar{E}$ be given by* (7), (8) *and* (9), (10), *respectively. Let $\|e\|_1 \geq Ch^p$. Then*

$$\lim_{h\to 0} \frac{\|\bar{E}\|_1}{\|e\|_1} = 1.$$

The quantity $\|\bar{E}\|_1/\|e\|_1$ is called the *effectivity index*. For the linear parabolic as well as linear elliptic error indicator (but not for the nonlinear elliptic one), this theorem is proven in [9], too.

Analysis of the semidiscrete error does not include analysis of the error of solution of the corresponding system of ordinary differential equations in time. In practice, this system is solved by standard software that admits the required accuracy to be given by the user. This required accuracy is then prescribed several orders less than the total prescribed accuracy of the fully discrete solution. There are several papers concerned with the analysis of fully discrete error, see, e.g., [5], [12], [13].

5. Space h- and hp-adaptive procedures

Procedures that can adapt the space grid are very often used. They are usually based on the *principle of the equidistribution of error* that requires

$$\|e\|_{1,i} = \|e\|_{1,j}, \quad i,j = 1,\dots,N.$$

This requirement is applied to the error indicator $\bar{E}$,

$$\|\bar{E}\|_{1,i} = \|\bar{E}\|_{1,j}, \quad i,j = 1,\dots,N.$$

Several such procedures have been published, e.g. the *dynamic grid adaptation* in [1], *grading function grid adaptation* in [8], etc. We successfully tested the above introduced error indicators on these procedures.

We witness a rapidly increasing use of the hp-FEM for solving elliptic as well as parabolic problems. For this adaptive finite element method, however, the conventional error estimates (in the form of a single number per element) are not enough. There are numerous options how a higher-order element can be refined because of the interplay between h and p. Thus the estimates of higher-order derivatives of the error are required. Moreover, these hp-procedures are particularly important if the space variable is a vector. In these problems, the *reference solution* usually serves as the source of the a posteriori error estimation. Both the ideas and computational procedures of the hp-FEM are presented in, e.g., [4], [6], [10].

References

[1] S. Adjerid, J.E. Flaherty, Y.J. Wang: *A posteriori error estimation with finite element method of lines for one-dimensional parabolic system.* Numer. Math. **65**, 1993, 1–21.

[2] I. Babuška, W.C. Rheinboldt: *A-posteriori error estimates for the finite element method.* Internat. J. Numer. Methods Engrg. **12**, 1978, 1597–1615.

[3] I. Babuška, T. Strouboulis: *The finite element method and its reliability.* Oxford, Clarendon Press 2001.

[4] L. Demkowicz, W. Rachowitz, P. Devloo: *A fully automatic hp-adaptivity.* J. Sci. Comput. **17**, 2002, 117–142.

[5] P.K. Moore: *A posteriori error estimation with finite element semi- and fully discrete methods for nonlinear parabolic equations in one space dimension.* SIAM J. Numer. Anal. **31**, 1994, 149–169.

[6] P.K. Moore: *Applications of Lobatto polynomials to an adaptive finite element method: A posteriori error estimates for hp-adaptivity and grid-to-grid interpolation.* Numer. Math. **94**, 2003, 367–401.

[7] K. Segeth: *A posteriori error estimates for parabolic differential systems solved by the finite element method of lines.* Appl. Math. **39**, 1994, 415–444.

[8] K. Segeth: *Grid adjustment for parabolic systems based on a posteriori error estimates.* J. Comput. Appl. Math. **63**, 1995, 349–355.

[9] K. Segeth: *A posteriori error estimation with the finite element method of lines for a nonlinear parabolic equation in one space dimension.* Numer. Math. **83**, 1999, 455–475.

[10] P. Šolín, K. Segeth, I. Doležel: *Higher-order finite element methods.* Boca Raton, FL, Chapman & Hall/CRC 2004.

[11] B. Szabó, I. Babuška: *Finite element analysis.* New York, John Wiley & Sons 1991.

[12] T. Vejchodský: *Fully discrete error estimation by the method of lines for a nonlinear parabolic problem.* Appl. Math. **48**, 2003, 129–151.

[13] R. Verfürth: *A posteriori error estimates for nonlinear problems: $L^r(0,T;W^{1,\rho}(\Omega))$-error estimates for finite element discretizations of parabolic equations.* Numer. Methods Partial Differential Equations **14**, 1998, 487–518.