Jan Pelant

Cardinal reflections and point-character of uniformities – counterexamples

In: Zdeněk Frolík (ed.): Seminar Uniform Spaces. , 1975. pp. 149–158.

Persistent URL: http://dml.cz/dmlcz/703125

Cardinal reflections and point-character of uniformities-counterexamples

J. Pelant

It is proved in [2] under Generalized Continuum Hypothesis that uniform covers of X of cardinality less than κ form a uniform space for any uniform space X and any infinite cardinal κ. Because of a lack of better amusements, we raised the question whether this statement depends on set-theoretical assumptions. As we knew Vidossich's theorem asserting: if X is a uniform space with σ-point-finite base and κ is any infinite cardinal, then uniform covers of X of cardinality less than κ form a uniformity, we expected that the solution of the above question could be useful for σ-point-finite base problem. It is really the case. We are going to show that there is a model of ZFC due to J.E. Baumgartner where there exists a uniform space whose uniform covers of cardinality less than ω_1 does not form a uniformity. Secondly, we show that for any cardinal κ, there is a uniform space with point-character greater than κ.

I wish to thank J.E. Baumgartner who kindly informed me about his results which I needed in the present note.

Definition: Let $(X,\mathcal{U})$ be a uniform space. A point-character $pc(X,\mathcal{U})$ is defined by $pc(X,\mathcal{U}) = \min\{\sup\{\text{card}\ U \in B \mid B \in \mathfrak{B}\ \&\ x \in U\}\ x \in X\} \mid \mathfrak{B}$ is a base of $\mathcal{U}\}$.

Definition: Let κ be an infinite cardinal. Let n be a positive integer. We define $\mathcal{K}(\kappa,n)$ as a set of all elements V of $(\exp\kappa)^n$ such that $pr_1V \supset pr_2V \supset \dots \supset pr_nV$ and $pr_nV \neq \emptyset$.

Notation: Let $n > 1$ be a positive integer. For $V \in \mathcal{K}(\kappa, n-1)$, put $\mathcal{U}(V) = \{U \in \mathcal{K}(\kappa, n) \mid pr_1 U \supset pr_1 V \supset pr_2 U \supset pr_2 V \supset \dots \supset pr_{n-1} V \supset pr_n U\}$.

The following lemma is basic for the procedure used here:

Lemma: Let κ be an uncountable cardinal. Let $n \geq 2$ be a positive integer. Let c be any mapping from $\mathcal{K}(\kappa, n)$ into κ such that $c(K) \in pr_2 K$ for any $K \in \mathcal{K}(\kappa, n)$. Let m be a regular cardinal less than κ. For any $P \subset \kappa$ of cardinality greater than m, there is $V \in \mathcal{K}(\kappa, n-1)$ such that $pr_1 V = P$ and $\operatorname{card} c(\mathcal{U}(V)) \geq m$.

Before proving Lemma, we show how the promised theorems follow from this.

Construction: Let α be an infinite cardinal. Denote $H_k = \{\frac{i}{2^k} \mid i = 0, 1, \dots, 2^k\}$ for k non-negative integer, $H = \bigcup \{H_k \mid k = 0, 1, 2, \dots\}$. Put $M(\alpha) = \{f : H \longrightarrow \exp \alpha \mid (f(h_1) \supset f(h_2)$ for any $h_1, h_2 \in H$ such that $h_1 > h_2)$ and $f(0) \neq \emptyset\}$. For $f \in M(\alpha)$, $f \restriction H_k$ is an element of $\mathcal{K}(\alpha, 2^k + 1)$ in the fact. For $V \in \mathcal{K}(\alpha, 2^k)$ we define $\tilde{V} = \{f \in M(\alpha) \mid f \restriction H_k \in \mathcal{U}(V)\}$. We define now a base of a pseudometric uniformity on $M(\alpha)$: $\mathcal{B}_i = \{\tilde{V} \mid V \in \mathcal{K}(\alpha, 2^i)\}$, $i = 0, 1, 2, \dots$.

Claim: $\mathcal{B}_{i+1} \nleq \mathcal{B}_i$ for $i = 0, 1, 2, \dots$.

Choose $f \in M(\alpha)$, take $g \in \operatorname{st}(f, \mathcal{B}_{i+1})$, then $g \in \tilde{V}$ where $V \in \mathcal{K}(\alpha, 2^i)$ such that

$$pr_j V = f\left(\frac{2j-1}{2^{i+1}}\right), \quad j = 1, 2, \dots, 2^i.$$

A uniform space just defined will be denoted by $U(\alpha)$.

Remarks. 1) $U(\alpha)$ need not be Hausdorff but $U(\alpha)$ restricted to the set $\{f \in M(\alpha) \mid \bigcap_{h>h_0} f(h) = \bigcup_{h_0>h} f(h), \forall h_0\}$ is Hausdorff and the following theorems are valid for this subspace as well.

2) Construction can be generalized: α, β are infinite cardinals, $\exp_\beta \alpha = \{A \mid A \subset \alpha, \text{ card } A < \beta\}$. $M(\alpha, \beta)$ is a set of all mappings from H into $\exp_\beta \alpha \times \exp_\beta \alpha$ such that $pr_1\, f(h_1) \supset pr_2\, f(h_1) \supset pr_1\, f(h_2) \supset pr_2\, f(h_2)$ whenever $h_1 \gneqq h_2$. Analogously, we use sequences of elements of $\exp_\beta \alpha \times \exp_\beta \alpha$ for a definition of a uniform space $U(\alpha, \beta)$. We have mentioned a space $U(\alpha, \beta)$ as any uniform space of cover-character not greater than α and point-character less than β is homeomorphic to a subspace of some product of $U(\alpha, \beta)$. Unfortunately, it is clear that cover-character of $U(\alpha, \beta)$ can be greater than α in general. Nevertheless, the cover-character of $U(\alpha, \beta)$ is not greater than if card $\exp_\beta \alpha \leq \alpha$. However, another difficulty is point-character.

Theorem 1 : Let K be an infinite regular cardinal. $U(K^+)$ has a point-character greater than or equal to K .

Suppose there exists a uniform cover $\mathcal{U} = \{U_a\}_{a \in A}$ of $U(K^+)$ such that $\mathcal{U} < \mathcal{B}_0$ and card $\{a \mid a \in A \ \&\ f \in U_a\} < K$ for any $f \in M(K^+)$. There is i such that $\mathcal{B}_i < \mathcal{U}$. Suppose A is a well-ordered set. Define a partition of $\mathcal{B}_i$, $\{R_a\}_{a \in A}$, by $R_a = \{P \in \mathcal{B}_i \mid a = \min\{b \in A \mid P \subset U_b\}\}$. Clearly, $\{R_a\}_{a \in A}$ is a uniform cover and

(1) $\quad \operatorname{card} \{a \mid a \in A \ \& \ f \in \bigcup R_a\} < \kappa$ for any $f \in M(\kappa^+)$.

For each $a \in A$ there is $V_a \in \mathcal{K}(\kappa^+, 1)$ such that $\bigcup R_a \subset \widetilde{V}_a$, it means $f(1) \supset V_a$ for any $f \in \bigcup R_a$. For $U \in \mathcal{K}(\kappa^+, 2^i)$ with $\widetilde{U} \in R_a$, define $c'(U) = \min V_a$. As $V_a \subset \mathrm{pr}_1 U$, $c'(U) \in \mathrm{pr}_1 U$. (1) implies that

$\operatorname{card} c'\{U \mid U \in \mathcal{K}(\kappa^+, 2^i) \ \& \ \widetilde{U} \ni f\} < \kappa$ for any $f \in M(\kappa^+)$.

Define now $c: \mathcal{K}(\kappa^+, 2^i + 2) \longrightarrow \kappa^+$ by $c(V_1, V_2, \ldots$ $\ldots, V_{2^i+1}, V_{2^i+2}) = c'(V_2, \ldots, V_{2^i+1})$. It follows from Lemma that there is $Q \in \mathcal{K}(\kappa^+, 2^i + 1)$ such that $\operatorname{card} c(\mathcal{U}(Q)) \geq$ $\geq \kappa$. Take $f \in M(\kappa^+)$ such that

$$\mathrm{pr}_j P = f\left(\frac{2^i + 1 - j}{2^i}\right), \qquad j = 1, 2, \ldots, 2^i + 1 .$$

Then $\operatorname{card} c'\{U \in \mathcal{K}(\kappa^+, 2^i) \mid f \in \widetilde{U}\} \geq \kappa$, which is a contradiction.

Theorem (Baumgartner): There is a model of ZFC where there is $\mathcal{A} \subset \exp \omega_1$ such that $\operatorname{card} A = \omega_1$ for each $A \in$ $\in \mathcal{A}$, $\operatorname{card} \mathcal{A} = 2^{\omega_1}$ and $\operatorname{card}(A_1 \cap A_2) < \omega_0$ for any two distinct elements of $\mathcal{A}$.

Theorem 2 : In the above model of ZFC, there is a uniform cover $\mathcal{V}$ of $U(\omega_1)$, $\operatorname{card} \mathcal{V} = \omega_1$ such that each uniform star-refinement of $\mathcal{V}$ has cardinality greater than ω_1.

Proof: For $a \in \omega_1$, put $\widetilde{a} = \{f \in M(\omega_1) \mid f(1) \ni$ $\ni a\}$. Define $\mathcal{V} = \{\widetilde{a}\}_{a \in \omega_1}$. Suppose there is $U(\omega_1)$-cover $\mathcal{U} = \{U_b\}_{b \in \omega_1}$ such that $\mathcal{U} \overset{*}{<} \mathcal{V}$. There is i such that $\mathcal{B}_i < \mathcal{U}$. Define a partition of $\mathcal{B}_i \{R_b\}_{b \in \omega_1}$ by $R_b = \{W \in \mathcal{B}_i \mid b = \min\{d \in \omega_1 \mid W \subset U_d\}\}$. Clearly,

$\{\bigcup R_b\}_{b\in\omega_1} \preceq \mathcal{V}$. Define $c': M(\omega_1) \longrightarrow \omega_1$ by $c'(f) = \min\{a \mid$ s.t $\{(f, \{\bigcup R_b\}_{b\in\omega_1}) \subset \tilde{a}\}$.

(2) According to [3], it holds $c'(\bigcup R_b) \subset \bigcap \{pr_1 V \mid \tilde{V} \in R_b\}$ for all b .

Define $c: \mathcal{K}(\omega_1, 2^i + 1) \longrightarrow \omega_1$ by $c(V) = \min\{c'(f) \mid f \in M(\omega_1)$ and $pr_j V = f\left(\frac{2^i+1-j}{2^i}\right)$ $j = 1,2,\dots, 2^i + 1\}$.

Using properties of Baumgartner's model and Lemma, we receive that there exists $\mathcal{D} \subset \mathcal{K}(\omega_1, 2^i)$ such that card $\mathcal{D} = 2^{\omega_1}$, card $c(\mathcal{U}(V)) \geq \omega_0$ for each $V \in \mathcal{D}$, card $(pr_1 V \cap \cap\, pr_1 U| < \omega_0$ for any two distinct elements of $\mathcal{D}$. It implies that (2) must fail to be true.

Remark: In the fact, properties of the model from Theorem are stronger than we need. It would be sufficient if the following statement holds: There exists $\mathcal{A} \subset \exp \omega_1$ such that card $\mathcal{A} > \omega_1$, card $A = \omega_1$ for each $A \in \mathcal{A}$ and there is a cardinal $\kappa \leq$ card $\mathcal{A}$ such that card $\bigcap \mathcal{A}' < \omega_0$ for any $\mathcal{A}' \subset \mathcal{A}$, card $\mathcal{A}' \geq \kappa$.

It is clear that we can give further counterexamples to anybody who gives us some "nice" model of ZFC.

Proof of Lemma: Suppose $n > 2$ (for $n = 2$ Lemma is obvious). Choose a mapping c like in Lemma. m is a regular cardinal less than κ . Let us assume that Lemma fails to be true. We will show that it implies a contradiction. Take $V_0 \in \mathcal{K}(\kappa, n-1)$ such that $pr_1 V_0 = P$ and card $pr_j V_0 > m$, $j = 1,\dots, n-1$.

First of all, we introduce some notation :

Suppose $W \in \mathcal{K}(K, n-1)$, $\{Y_i\}_{i=0}^{j}$ is a sequence of subsets of K , $j \leq n-1$. $W - \{Y_i\}_{i=1}^{j}$ is an element of $\mathcal{K}(K, n-1)$ such that $pr_{n-t}(W - \{Y_i\}_{i=1}^{j}) = pr_{n-t}W - \bigcup_{i=t}^{j} Y_i$, $t = 1, \dots, n-1$.

$W \nabla \{Y_i\}_{i=0}^{j} = \{X \in \mathcal{U}(W - \{Y_i\}_{i=1}^{j}) \mid pr_{n-t}X \cap Y_t = \emptyset$,

$t = 0, 1, \dots, j\}$.

M is a subset of K , W is an element of $\mathcal{K}(K, n-1)$, $j \in \{1, \dots, n-1\}$, $A(j,M,W)$ denotes the following formula (X_i and Y_i are subsets of K such that card $X_i \leq m$ and card $Y_i \leq m$) :

$$\exists X_j \; \forall Y_j \supset X_j \; \exists X_{j-1} \; \forall Y_{j-1} \supset X_{j-1} \; \exists X_{j-2} \dots \forall Y_2 \supset X_2 \; \exists X_1 \; \forall Y_1 \supset X_1 \; \exists Y_0 : \quad (W \nabla \{Y_i\}_{i=0}) - M = \emptyset .$$

A formula $\neg A(j,M,W)$ will be denoted by $B(j,M,W)$. Let us emphasize that $A(n-2, M, V_0)$ cannot be true for any M , card $M \leq m$.

Rewrite then the above formulae as follows:

$$A(j,M,W) : \quad \exists G(j) \; \forall Y_j \supset G(j) \; \exists G(\{Y_i\}_{i=j}^{j}) \; \forall Y_{j-1} \supset G(\{Y_i\}_{i=j}^{j}) \; \exists G(\{Y_i\}_{i=j-1}^{j}) \dots \forall Y_2 \supset G(\{Y_i\}_{i=3}^{j}) \; \exists G(\{Y_i\}_{i=2}^{j}) \; \forall Y_1 \supset G(\{Y_i\}_{i=2}^{j}) \; \exists G(\{Y_i\}_{i=1}^{j}) : \quad (W \nabla \{Y_i\}_{i=0}^{j}) - M = \emptyset ,$$

where $Y_0 = G(\{Y_i\}_{i=1}^{j})$.

We can suppose that G assigns to j ($\{Y_i\}_{i=k}^{j}$, resp.) the unique subset $G(j)$ of K ($G(\{Y_i\}_{i=k}^{j}$ resp.). (One can use an order structure of ordinals for a more exact definition of G .)

G will be called a corresponding choice.

$$B(j,M,W) : \forall X_j \exists F(\{X_i\}_{i=j}^{j}) \supset X_j \forall X_{j-1} \exists F(\{X_i\}_{i=j-1}^{j}) \supset$$
$$\supset X_{j-1} \forall X_{j-2} \dots \exists F(\{X_i\}_{i=2}^{j}) \supset X_2 \forall X_1 \exists F(\{X_i\}_{i=1}^{j}) \supset$$
$$\supset X_1 \quad \forall Y_0 : c(W \nabla \{Y_i\}_{i=0}^{j}) - M = \emptyset ,$$

where $Y_k = F(\{X_i\}_{i=k}^{j})$, $k = j$, $j - 1, \dots, 1$.

Again, let us suppose that F assigns to $\{X_i\}_{i=k}^{j}$, $k = 1,\dots$ $\dots, j$ the unique subset $F(\{X_i\}_{i=k}^{j})$ of K . F is called a corresponding choice.

For $X_i = \emptyset$, $i = 1,\dots, j$, $F(\emptyset)$ will denote a sequence $\{Y_k\}_{k=1}^{j}$, where $Y_k = F(\{X_i\}_{i=k}^{j})$.

We are going to define by transfinite induction the mappings $R: m \longrightarrow \{0,1,\dots, n-1\}$, $S: m \longrightarrow \{1,\dots, n-1\}$, $M: m \longrightarrow \exp K$, $V: m \longrightarrow \mathcal{K}(K, n-1)$.

V_0 is as above, $M_0 = c(\mathcal{U}(V_0))$, $R(0) = 0$, $S(0) = 1$.

If $A(1,M_0,V_0)$ holds then we define: $R(1) = 0$, $V_1 = V_0$, $M_1 =$ $= \emptyset$, $S(1) = 1$. G_1 is the corresponding choice.

If $B(1,M_0 V_0)$ holds then F_1 denotes the corresponding choice and we define $R(1) = 1$, $V_1 = V_0 - F_1(\emptyset)$, $S(1) = 1$, $M_1 =$ $= c(\mathcal{U}(V_1)) - M_0$.

Suppose that R , M , V , S are defined for all $q < p \in m$.

1) p is an isolated ordinal, $p = r + 1$

a) $R(r) > 0$: if $A(1, \bigcup\{M_q \mid q \leq r\}, V_r)$, then define $R(p) =$ $= 0$, $M_p = \emptyset$, $V_p = V_r$, $S(p) = 1$ and G_p is the corresponding choice;

if $B(1, \bigcup\{M_q \mid q \leq r\}, V_r)$ then $R(p) = 1$, $S(p) = 1$, $V_p =$ $= V_r - F_p(\emptyset)$, $M_p = c(\mathcal{U}(V_p)) - \bigcup\{M_q \mid q \leq r\}$ where F_p is the corresponding choice.

b) $R(r) = 0$

If $A(S(r)+1, \bigcup\{M_q \mid q \leq r\}, V_r)$ holds then we define $R(p) = 0$, $V_p = V_r$, $M_p = \emptyset$, $S(p) = S(r)+1$ and G_p is the corresponding choice.

If $B(S(r)+1, \bigcup\{M_q \mid q \leq r\}, V_r)$ holds then $R(p) = S(r) + {}$ ${}+ 1 = S(p)$, F_p is the corresponding choice, $V_p = V_r - F_p(\emptyset)$, $M_p = c(\mathcal{U}(V_p)) - \bigcup\{M_q \mid q \leq r\}$.

For $p \in m+1$, define $H(p) = \max\{j \mid \sup\{q \in p \,\&\, R(q) = j\} = {}$ ${}= p\}$.

2) p is a limit ordinal. Suppose that $H(p) = j$, j must be greater than 0. W_p denotes an element of $\mathcal{K}(K, n-1)$ such that $pr_j W_p = \bigcap\{pr_j V_q \mid q < p\}$ for $j = 1, \ldots, n-1$.

If $A(j, \bigcup\{M_q \mid q \in p\}, W_p)$ holds then $R(p) = 0$, $S(p) = j$, $M_p = \emptyset$, $V_p = W_p$ and G_p is the corresponding choice.

If $B(j, \bigcup\{M_q \mid q \in p\}, W_p)$ holds then $R(p) = j$, $S(p) = j$, $V_p = W_p - F_p(\emptyset)$, F_p is the corresponding choice, $M_p = {}$ ${}= c(\mathcal{U}(V_p)) - \bigcup\{M_q \mid q \in p\}$.

Let us suppose that mappings R, M, V, S are defined (and the corresponding choices as well). Let j be a positive integer which is equal to $H(m)$. Put $q_0 = \sup\{q \in m \mid R(q) > j\}$. As m is regular we have $\operatorname{card}\{p \in m \mid R(p) = j \,\&\, p > q_0\} = m$.

Let $\{p_\alpha\}_{\alpha \in m}$ be an increasing transfinite sequence such that $\{p_\alpha \mid \alpha \in m\} = \{p \in m \mid R(p) = j \,\&\, p > q_0\}$.

$X_j = \emptyset$, $Y_j^\alpha = F_{p_\alpha}(X_j)$, $Y_j = \bigcup\{Y_j^\alpha \mid \alpha \in m\}, \ldots$,

$X_k = \bigcup_{i=k+1}^{j} Y_i \cup \bigcup\{G_{p_{\alpha+1}}^{-1}(\{Y_i^\alpha\}_{i=k+1}^{j}) \mid \alpha \in m\} \cup$

$\cup\ (pr_k V_p - \bigcup\{pr_k V_{p_\alpha} \mid \alpha \in m\})$, (for $k = j-1$ replace $G_{p_{\alpha+1}}^{-1}(\{Y_i^\alpha\}_{k+1}^{j})$ by $G_{p_{\alpha+1}}^{-1}(j-1)$), $Y_k^\alpha = F_{p_\alpha}(\{X_i\}_{i=k}^{j})$,

$Y_k = \bigcup \{ Y_k^\alpha \mid \alpha \in m \}$, $k = j, j-1, \dots, 2, 1$.

Define further $Y_0^\alpha = Y_1 \cup G_{p_{\alpha+1}}^{-1}(\{Y_i^\alpha\}_{i=1}^j)$ for $\alpha \in m$.

Put $V = V_{p_0} - \{Y_i\}_{i=1}^j$. We show that card $c(\mathcal{U}(V)) \geq m$ and it will be a desired contradiction:

It holds: $(V_{p_\alpha} \nabla \{Y_i^\alpha\}_{i=0}^j) \subset \mathcal{U}(V)$ and further

$c(V_{p_\alpha} \nabla \{Y_i^\alpha\}_{i=0}^j) - \bigcup\{M_q \mid q < p_\alpha\} \neq \emptyset$

and $c(V_{p_\alpha} \nabla \{Y_i^\alpha\}_{i=0}^j \subset \bigcup\{M_q \mid q \leq p_{\alpha+1} - 1\}$.

Let us observe that $p_{\alpha+1}$ must be an isolated ordinal.

It follows immediately from these facts that card $c(\mathcal{U}(V)) \geq m$.

Remarks: 1) One can prove by the above method slightly modified that the uniform space $U(\omega_1)$ has not σ-point-finite base. More generally, if $m < \mathrm{cf}\,\beta$ then $U(\alpha, \beta)$, $\alpha^+ \geq \beta$, has no m-point-m base (a collection $\{U_a\}_{a \in A}$ of subsets of X is m-point-m iff $A = \bigcup_{b \in m} A_b$ and card $\{a \mid a \in A_b$ and $x \in U_a\} < m$ for each $b \in m$ and each $x \in X$). Outline of modification: c would be a mapping from $\mathcal{K}_\beta(\alpha, n)$ into $\alpha \times m$ such that $\mathrm{pr}_2 K \ni \mathrm{pr}_1\, c(K)$ for any $K \in \mathcal{K}_\beta(\alpha, n)$ and the formula $A(j, M, V)$ would have the form:

$$\forall b \in m \; \exists X_j \; \forall Y_j \supset X_j \dots \exists X_1 \; \forall Y_1 \supset X_1 \; \exists Y_0 :$$

$$\mathrm{pr}_1(c(V \nabla \{Y_i\}_{i=0}^j) \cap \alpha \times \{b\}) - M = \emptyset ,$$

and mappings R, S, V, M would be defined on α .

2) It follows from the precedent remark that the metric uniformity of $\ell^\infty(2^{\omega_1})$ has not σ-point-finite base.

References:

[1] Baumgartner J.E.: Almost-disjoint sets, the dense-set problem and the partition calculus, to appear in Annals of Math.Log.

[2] Kucia A.: On coverings of a uniformity, Coll.Math.27(1973), 73-74.

[3] Pelant J.: One folkloritistic lemma on cardinal reflections, **this volume, 145-147.**

[4] Vidossich G.: A note on cardinal reflections in the category of uniform spaces, Proc.A.M.S. 23(1969),55-58.