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In 1973, J. Pelant and J. Reiterman wrote a paper concerning atoms in uniformities [PR]. It seems that there is a little investigation only in this direction till yet, though the problems in this area are worthy to be studied: Any result on uniform atoms tells something about the lattice of all uniformities, moreover, it tells something about the properties of points in the Čech-Stone compactification of discrete space, too.

The aim of the present paper is to construct, assuming continuum hypothesis, some ultrafilters on ω in order to obtain examples of uniform atoms whose nature is essentially dissimilar to those ones described in [PR]. We also give a proof of non-published theorem due to J. Pelant, which shows some properties of non-0-dimensional atoms on ω. It remains an open questions whether such atoms exist at all.

0. General background. Consider the lattice of all uniformities on the set A, the order given by the uniform continuity of an identity mapping: $\mathcal{U} \prec \mathcal{V}$ iff $id_A : \langle A, \mathcal{U} \rangle \longrightarrow \langle A, \mathcal{V} \rangle$ is uniformly continuous. Atoms in this lattice will be called uniform atoms.

Let us give a brief review of the main results from [PR].

Let q be a filter on the set X, denote by σ_q a uniformity on X defined as follows: A cover $\{U_i : i \in I\}$ belongs to σ_q iff there is some $i \in I$ with $U_i \in q$.

(a) Proposition. $\langle \omega, \sigma_q \rangle$ is a uniform atom if and only if q is a selective ultrafilter on ω.

The uniformity will be called proximally discrete if the induced proximity is discrete.

(b) Proposition. For any proximally discrete atom there is an ultrafilter q such that $\mathcal{U} \prec \sigma_q$.

If $\mathcal{U}$ is a uniformity on X, denote by $N_{\mathcal{U}}$ the family of all subsets $B \subset X$ such that $\langle B, \mathcal{U}/B \rangle$ is not uniformly discrete. For the proof of (b) one must check that if $\mathcal{U}$ is a proximally discrete atom, then $N_{\mathcal{U}}$ is an ultrafilter.

Suppose $\{\langle X_i, \mathcal{U}_i \rangle : i \in I\}$ be a family of uniform spaces with X_i pairwise disjoint, let q be an ultrafilter on I. The family of covers

$$\{\{x\} \mid x \in \bigcup X_i\} \cup \bigcup \{\mathcal{P}_i : i \in V\}$$

($V \in q$, $\mathcal{P}_i$ is a uniform cover of $\langle X_i, \mathcal{U}_i \rangle$) form a base of a uniformity $\mathcal{U}$ on $\bigcup X_i$ which will be called an ultraproduct of uniformities $\mathcal{U}_i$, and denoted by $\mathcal{U} = \sum_q \mathcal{U}_i$.

(c) Proposition. If $\mathcal{U}_i$ are atoms, so is $\sum_q \mathcal{U}_i$.

(d) Proposition. No atom $\mathcal{U}$ which is an ultraproduct of atoms is of the form $\mathcal{O}_q$.

(e) Proposition. Each atom on ω has a basis consisting of point-finite covers (see also [V]).

For the details and proofs, see [PR].

1. Up to now, we have no example of a proximally discrete atom on ω other than $\mathcal{O}_q$ with a selective ultrafilter q, and ultraproducts of such $\mathcal{O}_q$´s. There are known, of course, other atoms. Let q be a non-selective ultrafilter on ω whose type is minimal in Rudin-Frolík´s order (abbr. RF-minimal, see [R] or [CN]), according to (a), $\mathcal{O}_q$ is not an atom. Let $\mathcal{A} \prec \mathcal{O}_q$ be an atom and we are to show that $\mathcal{A}$ is not an ultraproduct of atoms. Suppose $\mathcal{A} = \sum_r \mathcal{A}_i$, then the equality $q = N_{\mathcal{A}} = \sum_r N_{\mathcal{A}_i}$ contradicts the assumption that q is RF-minimal.

Thus, we shall construct several ultrafilters q and study $\mathcal{O}_q$ and atoms below it. Let us start with the simplest one:

2. Theorem. Assume [CH]. Then there exists a P-point q on ω such that there is precisely one atom $\mathcal{A} \prec \mathcal{O}_q$,

$\mathcal{A} \neq \mathcal{O}_q$.

Proof. Let $\mathcal{R} = \{R_n : n < \omega\}$ be partition of ω such that all R_n are finite and $\sup |R_n| = \omega$. Let A be a subset of ω. We shall call A to be $\mathcal{R}$-unbounded, if $\sup \{|R_n \cap M| . n < \omega\} = \omega$.

Let $\mathcal{B}$ be a famly of all point-finite covers of ω, let $\mathcal{G} = \mathcal{B} \cup \mathcal{P}(\omega)$. Since $|\mathcal{G}| = 2^{\omega}$, assuming [CH], we may well-order it in the manner $\mathcal{G} = \{g_\alpha : \alpha < \omega^+\}$.

I. The construction of q goes by transfinite induction. For each $\alpha < \omega^+$ we shall define a filter base $\mathcal{F}_\alpha$ such that:

(i) $\mathcal{F}_0 = \{\bigcup \{R_i : i > n\} : n < \omega\}$;

(ii) if $\alpha < \omega^+$, then $|\mathcal{F}_\alpha| = \omega$;

(iii) if $\alpha < \beta < \omega^+$, then $\mathcal{F}_\alpha \subset \mathcal{F}_\beta$;

(iv) if $\alpha < \omega^+$ and if $F \in \mathcal{F}_\alpha$, then F is $\mathcal{R}$-unbounded;

(v) if $\alpha < \omega^+$ is a limit ordinal, then $\mathcal{F}_\alpha \supset \supset \bigcup_{\beta < \alpha} \mathcal{F}_\beta \cup \{H\}$ with $|H - F| < \omega$ for each $F \in \bigcup_{\beta < \alpha} \mathcal{F}_\beta$;

(vi) if $\beta = \alpha + 1$, $g_\alpha = M \in \mathcal{P}(\omega)$, then either $M \in \mathcal{F}_{\alpha+1}$ or $(\omega - M) \in \mathcal{F}_{\alpha+1}$;

(vii) if $\beta = \alpha + 1$, $g_\alpha = \mathcal{C} \in \mathcal{B}$, then there exists an $F \in \mathcal{F}_{\alpha+1}$ such that either $|F \cap C| \leq 1$ for every $C \in \mathcal{C}$ or there exists a sequence $\{x_n\}$ with $x_n \in R_n$ and $st(x_n, \mathcal{C}) \cap R_n \supset F \cap R_n$ for every $n < \omega$.

0-th step is precisely described in (i).

Suppose $\alpha < \omega^+$ to be limit. α is countable, all $\mathcal{F}_\beta$'s with $\beta < \alpha$ are countable, thus $\bigcup_{\beta < \alpha} \mathcal{F}_\beta = \{F_n : : n < \omega\}$.

By an induction over ω, let us find a sequence $\{n_j : j < \omega\}$ such that $n_0 < n_1 < n_2 < \ldots$ and $|F_0 \cap F_1 \cap \ldots \cap F_j \cap R_{n_j}| > j$ for all $j < \omega$. Indeed, if $n_0, n_1, \ldots, n_{k-1}$ be defined, then by (iv) the set

$F_0 \cap F_1 \cap \dots \cap F_k$ is $\mathcal{R}$-unbounded, thus there is some $n_k > n_{k-1}$ with $|F_0 \cap F_1 \cap \dots \cap F_k \cap R_{n_k}| > k$.

Let $H = (F_0 \cap R_{n_0}) \cup (F_0 \cap F_1 \cap R_{n_1}) \cup \dots \cup$
$\cup (F_0 \cap F_1 \cap \dots \cap F_k \cap R_{n_k}) \cup \dots$, and let
$\mathcal{F}_\alpha = \{F \cap H : F \in \bigcup_{\beta<\alpha} \mathcal{F}_\beta\} \cup \bigcup_{\beta<\alpha} \mathcal{F}_\beta$.

$\mathcal{F}_\alpha$ is obviously countable filter base. Let $F_k \in \in \bigcup_{\beta<\alpha} \mathcal{F}_\beta$. Then $H \cap F_k$ is $\mathcal{R}$-unbounded, because $|H \cap F_k \cap R_{n_m}| > m$ whenever $m > k$.

The set $H - F_k$ is finite, since R_n is finite for every $n < \omega$ and since $H - F_k \subset (F_0 \cap R_{n_0}) \cup (F_0 \cap F_1 \cap R_{n_1}) \cup \cup \dots \cup (F_0 \cap F_1 \cap \dots \cap F_{k-1} \cap R_{n_{k-1}}) \subset R_{n_0} \cup R_{n_1} \cup \dots \cup R_{n_{k-1}}$.

Let $\beta = \alpha + 1$, suppose $g_\alpha = M \subset \omega$. Then either $M \cap F$ is $\mathcal{R}$-unbounded for all $F \in \mathcal{F}_\alpha$ or $(\omega - M) \cap F$ is $\mathcal{R}$-unbounded for all $F \in \mathcal{F}_\alpha$. (Suppose the contrary: There are $F, F' \in \mathcal{F}_\alpha$ and r, s natural such that $|F \cap M \cap R_n| \leq \leq r$ for each $n < \omega$, $|F' \cap (\omega - M) \cap R_n| \leq s$ for each $n < \omega$. Then $|F \cap F' \cap R_n| \leq r + s$ for each $n < \omega$, a contradiction, since $\mathcal{F}_\alpha$ satisfies (iv).) In the first case, define $\mathcal{F}_{\alpha+1} = \mathcal{F}_\alpha \cup \{M \cap F : F \in \mathcal{F}_\alpha\}$, in the second, $\mathcal{F}_{\alpha+1} = \mathcal{F}_\alpha \cup \{(\omega - M) \cap F : F \in \mathcal{F}_\alpha\}$.

Let $\beta = \alpha + 1$, suppose $g_\alpha = \quad \in \mathcal{B}$ For every sequence $\{x_n\}$ such that $x_n \in R_n$ denote by $S\{x_n\}$ the set
$\bigcup \{st(x_n, \mathcal{C}) \cap R_n : n < \omega\}$.
Two cases are possible:

a) There exists a sequence $\{x_n\}$, $x_n \in R_n$, such that $S\{x_n\} \cap F$ is $\mathcal{R}$-unbounded for every $F \in \mathcal{F}_\alpha$. In this case, let $\mathcal{F}_{\alpha+1} = \mathcal{F}_\alpha \cup \{F \cap S\{x_n\} : F \in \mathcal{F}_\alpha\}$.

b) There is no such sequence. In this case, we shall proceed by induction as follows: $\mathcal{F}_\alpha$ is countable, $\mathcal{F}_\alpha = \{F_j : j < \omega\}$. Let us define natural numbers j_k, n_k and finite subsets H_k of ω such that $j_0 = 0$ and for every $k < \omega$ the following holds:

(1) $n_k < n_{k+1}$, $j_k < j_{k+1}$, $H_k \subset H_{k+1}$,

(2) $| H_k \cap R_{n_k} \cap F_o \cap F_1 \cap \dots \cap F_{j_k} | > k$ and

(3) for every $y \in H_k$, $st(y, \mathcal{C}) \cap H_k = \{y\}$.

F_o is $\mathcal{R}$-unbounded, thus there exists some n_o with $| F_o \cap R_{n_o} | > 0$; pick a point $y \in F_o \cap R_{n_o}$ and define $H_o = \{y\}$.

Let n_k, j_k, H_k be defined. The cover $\mathcal{C}$ is point-finite and H_k is finite, thus the set $\mathcal{C}_k = \{C \in \mathcal{C} : C \cap H_k \neq \emptyset\}$ is finite. For every $C \in \mathcal{C}_k$, let $\{x_n^C\}$ be some sequence chosen as follows: If $R_n \cap H_k \cap C \neq \emptyset$, then, according to (3), this intersection contains precisely one point which will be denoted by x_n^C. If $R_n \cap H_k \cap C$ is empty, but $R_n \cap C$ is non-empty, pick x_b^C from $R_n \cap C$ arbitrarily. Finally, if $R_n \cap C = \emptyset$, let $x_n^C = \text{Min } R_n$. Fix one such sequence $\{x_n^C\}$ for every $C \in \mathcal{C}_k$; as we assume that a) fails, there is some j_{k+1} such that $j_{k+1} > j_k$ and $F_{j_{k+1}} \cap \bigcup \{S\{x_n^C\} : C \in \mathcal{C}_k\}$ is not $\mathcal{R}$-unbounded. Denote by G_{k+1} the set

$$\bigcap \{F_j : 0 \leq j \leq j_{k+1}\} - \bigcup \{S\{x_n^C\} : C \in \mathcal{C}_k\}.$$

The set G_{k+1} is obviously $\mathcal{R}$-unbounded and there exists natural $n_{k+1} > n_k$ such that $G_{k+1} \cap R_{n_{k+1}}$ contains $k + 1$ distinct points $y_o, y_1, \dots, y_k$ with $y_p \notin st(y_q, \mathcal{C})$ if $p \neq q$. (If not, then for every $n > n_k$ there are $y_o(n)$, $y_1(n), \dots, y_{k-1}(n)$ such that $\bigcup \{st(y_p(n), \mathcal{C}) : 0 \leq p \leq k - 1\} \supset G_{k+1} \cap R_n$. Define $x_n^p = y_p(n)$ for $0 \leq p \leq k - 1$ and $n > n_k$, $x_n^p = \text{Min } R_n$ for $0 \leq p \leq k - 1$ and $n \leq n_k$. Then the finite union $\bigcup \{S\{x_n^p\} : 0 \leq p \leq k - 1\} \cup \bigcup \{S\{x_n^C\} : C \in \mathcal{C}_k\}$ covers some member of $\mathcal{F}_\alpha$, namely $F_o \cap F_1 \cap \dots \cap F_{j_k} \cap (\bigcup \{R_i : i > n_k\})$, which contradicts the assumption that a) does not take place.) Let $H_{k+1} = H_k \cup \{y_o, y_1, \dots, y_k\}$. Obviously, (1), (2) and (3) are satisfied for H_k.

Let $F = \bigcup \{H_k : k < \omega\}$, let $\mathcal{F}_{\alpha+1} = \mathcal{F}_\alpha \cup \{F \cap F' : F' \in \mathcal{F}_\alpha\}$.

In both cases, the filter base $\mathcal{F}_{\alpha+1}$ is well-defined:

$\mathcal{R}$-unboundedness of its members is clear in a) and a consequence of (1) and (2) in b), $\mathcal{F}_{\alpha+1}$ satisfies (vii), too: The added set equals to some $S\{x_n\}$ in the case a) and meets every $C \in \mathcal{C}$ in at most one point in the case b), as can be deduced from (3).

Now, having the whole induction verified, it remains to define $q = \bigcup\{\mathcal{F}_\alpha : \alpha < \omega^+\}$. The filter q is an ultrafilter because of (vi), it is a P-point by (v) and it cannot be selective, since its members are $\mathcal{R}$-unbounded by (iv).

II. Let $\mathcal{A}$ be a uniformity whose base consists of all covers $\mathcal{R} \wedge \mathcal{P}$ ($= \{R \cap P : R \in \mathcal{R}, P \in \mathcal{P}\}$), where $\mathcal{P}$ is a uniform cover from $\mathcal{O}_q$. The following facts are clear: $\mathcal{A}$ is not uniformly discrete (all members of q are $\mathcal{R}$-unbounded), $\mathcal{A} \neq \mathcal{O}_q$ (no member of $\mathcal{R}$ belongs to q) and $\mathcal{A} \prec \mathcal{O}_q$.

$\mathcal{A}$ is the unique uniform atom below $\mathcal{O}_q$: Let $\mathcal{A}'$ be a proximally discrete atom, let $\mathcal{C}'$ be a uniform cover belonging to $\mathcal{A}'$, let $\mathcal{C}$ be a point-finite cover from $\mathcal{A}'$, which star-refines $\mathcal{C}'$ (the existence of such $\mathcal{C}$ is implied by Proposition ( )). Since $\mathcal{C}$ belongs to $\mathcal{B}$, $\mathcal{C} = g_\alpha$ for a suitable $\alpha < \omega^+$. We know that $\mathcal{F}_{\alpha+1} \subset q$ and that there is some $F \in \mathcal{F}_{\alpha+1}$ satisfying the condition (vii) from the induction. Let $\mathcal{P}_0$ be a cover consisting of F and of all one-point subsets of ω. Supposing $\mathcal{A}' \prec \mathcal{O}_q$, we obtain $\mathcal{P}_0 \wedge \mathcal{C} \in \mathcal{A}'$, so it cannot happen that $|F \cap C| \leq 1$ for every $C \in \mathcal{C}$, because then $\mathcal{P}_0 \wedge \mathcal{C} = \{\{x\} : x \in \omega\}$, which is impossible - $\mathcal{A}'$ is not uniformly discrete. Thus there is a sequence $\{x_n\}$, $x_n \in R_n$, with $st(x_n, \mathcal{C}) \supset F \cap R_n$; in other words, $\mathcal{P}_0 \wedge \mathcal{R}$ refines $\mathcal{C}'$.

We have shown that every atom $\mathcal{A}' \prec \mathcal{O}_q$ is uniformly coarser than $\mathcal{A}$, thus $\mathcal{A}' = \mathcal{A}$, which completes the proof.

3. It may be instructive to analyse the proof of Theorem 2. We needed to construct a non-selective ultrafilter, hence the starting point with some partition $\mathcal{R}$ where the non-selectivity should appear, was necessary. It was sufficient to assume that $|R_n| < \omega$ and $\sup\{|R_n| : n<\omega\} = \omega$, since we were looking for a P-point. (Such partitions will be called admissible in the rest of the paper.) There were three essential steps in the proof: verifying of (v), (vi) and of (vii). Only the property (vii) was crucial for uniform properties of the desired atom, (vi) was necessary to obtain an ultrafilter, (v) implied that the future ultrafilter would be a P-point. We wanted all members of q to be $\mathcal{R}$-unbounded - let us say, we wanted all members of q to have some property $\mathbb{P}$. The property "M is $\mathcal{R}$-unbounded" was, moreover, of very special kind: There were, in fact, a countable collection $\{\mathbb{P}(k)\}$ of properties of finite subsets of ω, namely "$|M| > k$", and it was sufficient to verify, whether $M \cap R_n$ has $\mathbb{P}(k)$ for arbitrary k and some $n < \omega$, depending on k.

Now, let us return to (vii) from the proof of Theorem 2. It was, in fact, a collection of ω^+ properties $\mathbb{S}_\alpha$ of subsets of ω (each was described using some point-finite cover), and we wanted to satisfy this: "For every $\alpha < \omega^+$ there is at least one $F \in q$ which has $\mathbb{S}_\alpha$

In order to avoid unnecessary repeating of some steps given in the proof of Theorem 2 we shall prove the following lemma, which is some kind of recipe, how to obtain ultrafilters.

4. Lemma. Let $\mathcal{R} = \{R_n : n < \omega\}$ be an admissible partition of ω. Suppose that for every $k < \omega$ there is a property $\mathbb{P}(k)$ of finite subsets of ω such that

(0) there exist a $k < \omega$ such that $\emptyset$ has not $\mathbb{P}(k)$;

(i) if $M \in \mathcal{P}_{fin}(\omega)$ satisfies $\mathbb{P}(k)$ for som $k > 0$, then M satisfies $\mathbb{P}(k-1)$;

(ii) for every $k < \omega$ and every $n_0 < \omega$ there is some $n > n_0$ such that R_n has $\mathbb{P}(k)$;

(iii) there exists a mapping $f \in \omega^\omega$ which satisfies:

given $k < \omega$ and $M, M' \in \mathcal{P}_{fin}(\omega)$, if $M \cup M'$ has $\mathbb{P}(f(k))$, then either M or M' has $\mathbb{P}(k)$;

(iv) for every $k < \omega$ and every $M, Q \in \mathcal{P}_{fin}(\omega)$, if M has $\mathbb{P}(k)$ and $M \subset Q$, then Q has $\mathbb{P}(k)$.

Let the property $\mathbb{P}$ of subsets of ω be defined by the rule

(v) M has $\mathbb{P}$ iff for every $k < \omega$ and every $n_0 <$ there is an $n > n_0$ such that $M \cap R_n$ has $\mathbb{P}(k)$.

Then the following holds:

A. If $\mathcal{F}$ is a filter on ω with a countable base, if $M \in \mathcal{P}(\omega)$ and if every $F \in \mathcal{F}$ has $\mathbb{P}$, then there exists a filter $\mathcal{G}$ with countable base, $\mathcal{G} \supset \mathcal{F}$, all members of $\mathcal{G}$ have $\mathbb{P}$ and either $M \in \mathcal{G}$ or $(\omega - M) \in \in \mathcal{G}$.

B. If $\mathcal{F}$ is a filter on ω with a countable base and if every $F \in \mathcal{F}$ has $\mathbb{P}$, then there exists a subset M of ω such that $M \cap F$ has $\mathbb{P}$ and $|M - F| < \omega$, for each $F \in \mathcal{F}$.

C. Let $\{ \mathfrak{S}_\alpha : \alpha < 2 \}$ be a collection of properties of subsets of ω. Suppose that for every filter $\mathcal{F}$ with countable base consisting of sets with $\mathbb{P}$ and for every $\alpha < 2^\omega$ there exists an $M_\alpha \subset \omega$ such that M_α has $\mathfrak{S}_\alpha$ and $M_\alpha \cap F$ has $\mathbb{P}$ for every $F \in \mathcal{F}$.

Then, assuming [CH], there exists a P-point q such that $U \in q$ has $\mathbb{P}$ and for every $\alpha < 2^\omega$ there is a set $U_\alpha \in q$ satisfying $\mathfrak{S}_\alpha$. If, moreover, each M with $\mathbb{P}$ is $\mathcal{R}$-unbounded, then q is not selective.

Proof. A. Let $\mathcal{G}_1$ ($\mathcal{G}_2$, resp.) be a filter generated by $\mathcal{F} \cup \{ M \}$ ($\mathcal{F} \cup \{ (\omega - M) \}$, respectively). By the method of contradiction, let us suppose that neither $\mathcal{G}_1$ nor $\mathcal{G}_2$ has the desired properties. Then there exist $F_1, F_2 \in \mathcal{F}$ and natural numbers k_1, k_2, n_1, n_2 such that

$F_1 \cap M \cap R_n$ has not $\mathbb{P}(k_1)$ whenever $n > n_1$ and $_2 \cap (\omega - M) \cap R_n$ has not $\mathbb{P}(k_2)$, for $n > n_2$ (a consequence of v)). Let $n_o = \max(n_1, n_2)$, $k_o = \max(k_1, k_2)$. $F_1 \cap F_2$ belongs to $\mathcal{F}$, according to (v), there exists an $n > n_o$ such that $F_1 \cap F_2 \cap R_n$ has $\mathbb{P}(f(k_o))$. By (iii), either $F_1 \cap F_2 \cap M \cap R_n$ has $\mathbb{P}(k_o)$ or $F_1 \cap F_2 \cap (\omega - M) \cap R_n$ has $\mathbb{P}(k_o)$, thus by (iv) and (i) either $F_1 \cap M \cap R_n$ has $\mathbb{P}(k_1)$ or $F_2 \cap (\omega - M) \cap \cap R_n$ has $\mathbb{P}(k_2)$, a contradiction.

B. Let $\{F_j : j < \omega\}$ be a base of $\mathcal{F}$; we may assume that $F_o \supset F_1 \supset F_2 \supset \dots$. The proof goes by an obvious induction:

F_o has $\mathbb{P}$; by (v) there is some $n_o < \omega$ such that $F_o \cap R_{n_o}$ has $\mathbb{P}(0)$.

F_1 has $\mathbb{P}$; by (v) there is some $n_1 > n_o$ such that $F_1 \cap R_{n_1}$ has $\mathbb{P}(1)$.

Let $n_o < n_1 < n_2 < \dots < n_k$ be defined. F_{k+1} has $\mathbb{P}$, thus, applying (v) once more, there is some $n_{k+1} > n_k$ such that F_{k+1} has $\mathbb{P}(k+1)$.

Let $M = \bigcup \{F_i \cap R_{n_i} : i < \omega\}$. The set $M \cap F_i \cap R_{n_k}$ satisfies $\mathbb{P}(k)$ whenever $k \geq i$, thus $M \cap F_i$ satisfies $\mathbb{P}$, and $M - F_i \subset R_{n_o} \cup R_{n_1} \cup \dots \cup R_{n_{i-1}}$, thus $M - F_i$ is finite.

C. The proof of C. is a mere copy of the proof of Theorem 2 and may be left to the reader. Use A., B. and the assumptions of C. for inductive steps, 0´th step is guaranteed by (ii).

5. Theorem. Assume [CH], let L be a natural number. Then there exists a P-point q on ω such that there are precisely L distinct uniform atoms below σ_q.

Proof. The special cases L = 0, L = 1 have already been shown (Proposition (a), Theorem 2). The proof for $1 < L < \omega$ is divided into four sections. At first, the notation used throughout this proof will be given. Then the assumptions of Lemma 4 will be verified with help of two combinatorial statements. Finally, it will be shown that the ultrafilter q constructed by Lemma 4 has all the desired properties.

I. Let $A_1, A_2, \ldots, A_L$ be finite and pairwise disjoin sets, $|A_1| = |A_2| = \ldots = |A_L| = n$, where $n < \omega$. The set $A_1 \times A_2 \times \ldots \times A_L$ will be called an L-cube. If $A_1 \times A_2 \times \ldots \times A_L$ is an L-cube and if $B_i \subset A_i$ for $i = 1,2,\ldots,L$, $|B_1| = |B_2| = \ldots = |B_L|$, the cube $B_1 \times B_2 \times \ldots \times B_L$ will be called a subcube of $A_1 \times A_2 \times \ldots \times A_L$. If no special emphasis on the coordinate sets will be needed, we shall use for an L-cube a notation $Q(n^L)$, where n is the cardinalit of coordinate set; or $Q(n^{L-1}) \times A_L$, if A_L is the only coordinate set we are interested in. Similarly, if $Q(n^L) = A_1 \times A_2 \times \ldots \times A_L$ and if $a \in A_L$, then the set $A_1 \times A_2 \times \ldots \times A_{L-1} \times \{a\}$ will be often denoted as $Q(n^{L-1}) \times \{a\}$ and called to be an a-th square; if $Q(k^{L-1})$ is a subcube $Q(n^{L-1})$, then $Q(k^{L-1}) \times \{a\}$ will be called a subsquare o $Q(n^{L-1}) \times \{a\}$.

Given $L < \omega$, $L \geq 1$, and a countably infinite pairwise disjoint family $\{Q(n_i^L): i < \omega\}$ of L-cubes with $\sup n_i = \omega$, we may identify the set ω with $\bigcup \{Q(n_i^L): i < \omega\}$ and thus we have a partition $\mathcal{R} = \{Q(n_i^L): i < \omega\}$ of ω consisting of L-cubes; in accordance with the notation of Lemma 4, we shall also write $\mathcal{R} = \{R_n: n < \omega\}$. Clearly $\mathcal{R}$ is an admissible partition.

Given a partition $\mathcal{R}$ of ω into L-cubes, let $\mathcal{Q}$ b a family of L-cubes defined as follows: $Q \in \mathcal{Q}$ iff Q is a subcube of some $R_n \in \mathcal{R}$. Thus the family $\mathcal{Q}$ is completely determined by $\mathcal{R}$.

Let $\mathcal{R}$ be an admissible partition of ω into L-cu let $i \in \{1,2,\ldots,L\}$. For an $R_n \in \mathcal{R}$, $R_n = A_1 \times A_2 \times \ldots \times A_L$, and for $\langle a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_L \rangle \in A_1 \times A_2 \ldots \times A_{i-1} \times A_{i+1} \times \ldots \times A_L$, let $T = \{\langle a_1, a_2, \ldots, a_{i-1}, t, a_{i+1}, \ldots, a_L \rangle \in R_n: \in A_i\}$. Define $\mathcal{T}_i$ to be a family o all such T's with n and $\langle a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_L \rangle$

variable. Obviously each $\mathcal{T}_i$ is a subpartition of $\mathcal{R}$, $\mathcal{T}_i \wedge \mathcal{T}_j = \{\{x\} : x \in \omega\}$ whenever $i \neq j$ and the partitions $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_L$ are uniquely determined by $\mathcal{R}$. Further, if $T \in \mathcal{T}_L$, if $A_1 \times \ldots \times A_L = Q(n^L) \in \mathcal{R}$ and if $T \subset (n^L)$, then $|T \cap Q(n^{L-1}) \times \{a\}| = 1$ for each $a \in A_L$. If $i \neq L$, then there are n^{L-2} members T of $\mathcal{T}_i$ which are contained in $Q(n^{L-1}) \times \{a\}$.

Let $\mathcal{C}$ be a cover of ω, let $x \in \omega$, let $M \subset \omega$. Denote by $st^2(x,\mathcal{C})$ the set $st(st(x,\mathcal{C}),\mathcal{C})$ and call a set M to be $\mathcal{C}$-discrete if for each $x \in M$, $M \cap st(x,\mathcal{C}) = \{x\}$.

In order to apply Lemma 4, let $L > 1$ be a natural number, let $\mathcal{R}$ be an admissible partition of into L-cubes, let $k < \omega$.

The set $M \in \mathcal{P}_{fin}(\omega)$ has $\mathbb{P}(k)$ iff there is some $Q(k^L) \in \mathcal{Q}$ contained in M.

Let $\mathcal{C}$ be a point-finite cover of ω, let $M \subset \omega$. The set M has $\mathfrak{S}_{\mathcal{C}}$ iff either M is $\mathcal{C}$-discrete or there exists an $i \in \{1,2,\ldots,L\}$ and a sequence $\{x_T : T \in \mathcal{T}_i\}$, where $x_T \in T$, such that $st^2(x_T,\mathcal{C}) \supset M \cap T$ for each $T \in \mathcal{T}_i$.

II. It is clear that the family of properties $\{\mathbb{P}(k) : k < \omega\}$ satisfies (0),(i),(ii) and (iv) from Lemma 4. To prove (iii), we need to find a mapping $f \in \omega^\omega$ with the desired properties. The existence of such a mapping follows immediately from the following combinatorial statement (take the value 2 for m):

(*) For each $L \geq 1$ there exists a mapping $f_L : \omega \times \omega \to \omega$ such that every subset $X \subset Q(f_L(k,m))^L)$ with $|X| \geq (f_L(k,m))^L \cdot \frac{1}{m}$ contains some subcube $Q(k^L)$.

Though this statement is a well-known combinatorial result (see e.g. Erdös-Spencer's book [ES], Theorem 12.2 and Corollary 12.5), it will not do any harm to prove i here.

Induction: $f_1(k,m) = km + 1$ is clearly better than a satisfactory mapping for $L = 1$. Suppose f_L suit the statement and $f_L(k,m) > km$.

Let $n = f_L(k,2m)$, $p = 2^{n^L}$ and define $f_{L+1}(k,m) = np = N$. Obviously $f_{L+1}(k,m) > km$. Suppose $X \subset Q(N^{L+1}) = A_1 \times A_2 \times \dots \times A_{L+1}$, $|X| \geq \frac{N^{L+1}}{m}$. Let $\mathcal{B}$ be a partition of $A_1 \times A_2 \times \dots \times A_L$ into p^L pairwise disjoint subcubes $Q_i(n^L)$, we have $Q(N^{L+1}) = \bigcup \{ Q_i(n^L) \times A_{L+1} : i = 1,2,\dots,p^L \}$ such that

$$|X \cap Q_{i_0}(n^L) \times A_{L+1}| \geq N \cdot m^L .$$

Let $X_a = \{ \langle a_1, a_2, \dots, a_L \rangle \in Q(N^L) : \langle a_1, a_2, \dots, a_L, a \rangle \in X \cap Q_{i_0}(n^L) \times \{a\} \}$. Obviously $X_a \subset Q_{i_0}(n^L)$ and $X \cap Q_{i_0}(n^L) \times A_{L+1} = \bigcup \{ X_a : a \in A_{L+1} \}$.

Define $D = \{ a \in A_{L+1} : |X_a| \geq \frac{m^L}{2m} \}$, from the estim·tion

$$|X \cap Q_{i_0}(n^L) \times A_{L+1}| \leq |D|\, n^L + |A_{L+1} - D| \cdot \frac{m^L}{2m} \leq |D|\, n^L + \frac{N \cdot m^L}{2m}$$

follows that $|D| \geq \frac{N}{2m}$, thus $|D| \geq \frac{pn}{2m} > pk$, because $n = f_L(k,2m) > 2km$. Since $p\ (= 2^{n^L})$ i[s] the cardinality of the power set of $Q_i(n^L)$, there must be a subset B of D, $|B| = k$ such that $X_b = X_{b'}$ whenever $b, b' \in B$. But every X_b with $b \in B$ is of cardinality at least $\frac{m^L}{2m}$ and $X_b \subset Q_{i_0}(n^L)$, thus by the induction hypothesis there exists some cube $Q(k^L) \subset X_b$. Consequently $Q(k^L) \times B$ is the $(L+1)$-cube contained in X.

III. The verifying of the assumptions of C. from Lemma 4 needs further combinatorial proposition:

(**) Let $L \geq 1$ be a natural number. Then there exists a mapping $g \in \omega^\omega$ with the following property:

If $\mathcal{C}$ is a cover of ω, if $\mathcal{R}$ is an admissible partition of ω into L-cubes, if $\mathcal{Q}$ and $\{ \mathcal{T}_i : i = 1,2,.$

$\dots, L\}$ are defined by the rules given in I and if $n \geq$ $\geq g(k)$, then each cube $Q(n^L) \in \mathcal{Q}$ contains a subcube $Q(k^L)$ which is either $\mathcal{C}$-discrete or contained in $\bigcup \{ st^2(x_T, \mathcal{C}) \cap T : T \in \mathcal{T}_i , T \cap Q(n^L) \neq \emptyset \}$ for some $i \in$ $\in \{1,2,\dots,L\}$ and some suitable choice of $x_T \in T$.

The proof goes by an induction. For $L = 1$ we have the case from Theorem 2, $\mathcal{T}_1 = \mathcal{R}$ and obviously the function $g(k) = k^2$ will suffice for an arbitrary $\mathcal{C}$.

Assume the statement $(**)$ holds for $L \geq 1$. If $\mathcal{R}$ is an admissible partition of ω into $(L + 1)$.cubes, then each cube $Q(n\ ) \times A_{L+1} \in \mathcal{R}$ is a disjoint union of squares $Q(n^L) \times \{a\}$ with $a \in A_{L+1}$, let $\mathcal{R}'$ be a collection of all those squares. We may consider $\mathcal{R}'$ as a partition of ω into L-cubes; if $\mathcal{T}'_1, \mathcal{T}'_2, \dots, \mathcal{T}'_L$ and $\mathcal{Q}'$ are the corresponding partitions and subcubes, then $\mathcal{T}'_i = \mathcal{T}_i$ for $i =$ $= 1,2,\dots,L$ and $|T \cap Q'| \leq 1$ whenever $T \in \mathcal{T}_{L+1}$, $Q' \in \mathcal{Q}'$. Let g' be the function from $(**)$ for L, let f_{L+1} be the function from $(*)$.

For every $k < \omega$, let us define by the finite recursion:

$u_0 = k;$

$v_i = f_{L+1}(u_i, 2);$

$w_i = f_{L+1}(k, 16 v_i^L);$

$u_{i+1} = g'(w_i);$

$g(k) = u_{(L+1)k}.$

For $N = g(k)$ we must prove that the cube $Q(N^{L+1})$ contains some subcube $Q(k^{L+1})$ with the desired properties. Let us write $Q(N^{L+1}) = A_1 \times A_2 \times \dots \times A_{L+1}$; by an induction down we shall define for $i = (L + 1)k, (L + 1)k - 1, \dots$ $\dots, 2, 1$ natural numbers n_i, $n_i \geq u_{i-1}$, distinct members a_i of A_{L+1} and cubes $Q(n_i^{L+1}) = A_{1,i} \times A_{2,i} \times \dots \times A_{L+1,i}$ such that $Q(n_i^L) \times \{a_i\}$ is a subsquare of $Q(N^L) \times \{a_i\}$, $Q(n_{i-1}^{L+1})$ is a subcube of $Q(n_i^{L+1})$ and $A_{L+1,i} \cup \{a_{i+1}, a_{i-2}, \dots$

$\dots, a_{(L+1)k}\} = \emptyset$.

Let for $i + 1 \leq (L+1)k$ the cube $Q(n_{i+1}^{L+1}) = A_{1,i+1} \times \times A_{2,i+1} \times \dots \times A_{L+1,i+1}$ and the points $a_{i+1}, a_{i+2}, \dots, a_{(L+1)k}$ were defined; pick some a_i from $A_{L+1,i+1}$ other than a_{i+1} and consider the square $Q(n_{i+1}^{L}) \times \{a_i\}$ (if $i = (L+1)k$, pick arbitrarily $a_{(L+1)k} \in A_{L+1}$ and consid[er] the square $Q(N^L) \times \{a_{(L+1)k}\}$).

Since $Q(n_{i+1}^{L}) \times \{a_i\} \in \mathcal{Q}'$ and since $n_{i+1} \geq u_i = g'(w_i)$, we may assume that there is a subsquare $Q(w_i^L) \times \{a_i\}$ such that

1) either there is some $j \in \{1, 2, \dots, L\}$ and a sequence $\{x_T : T \in \mathcal{T}_j\}$, $x_T \in T$, such that
$Q(w_i^L) \times \{a_i\} \subset \bigcup \{st^2(x_T, \mathcal{C}) \cap T : T \in \mathcal{T}_j,$
$T \cap Q(n_{i+1}^{L}) \times \{a_i\} \neq \emptyset\}$,

2) or the square $Q(w_i^L) \times \{a_i\}$ is $\mathcal{C}$-discrete.

Let $Q(w_i^L) = B_{1,i} \times B_{2,i} \times \dots \times B_{L,i}$. Choose some sub[set] $B_{L+1,i} \subset A_{L+1,i+1}$ such that $B_{L+1,i}$ contains a_i, $|B_{L+1,i}| = w_i$ and $B_{L+1,i} \cap \{a_{i+1}, a_{i+2}, \dots, a_{(L+1)k}\} = \emptyset$. Then $Q(w_i^{L+1}) = Q(w_i^L) \times B_{L+1,i}$ is obviously a subcube of $Q(n_{i+1}^{L+1})$.

If the case 1) takes place, we are done when defin[ing] $n_i = w_i$ and $A_{j,i} = B_{j,i}$ for $j = 1, 2, \dots, L+1$.

The case 2) is little more complicated. Let $\mathcal{S}$ be [the] partition of the cube $Q(w_i^{L+1})$ consisting of all non-voi[d] $T \cap Q(w_i^{L+1})$ with $T \in \mathcal{T}_{L+1}$. As mentioned in I, each $S \in [\mathcal{S}]$ meets $Q(w_i^L) \times \{a_i\}$ in precisely one point, thus we may label the members of $\mathcal{S}$ in the manner $\mathcal{S} = \{S_y : y \in Q(w_i^L) \times \{a_i\}\}$.

There are two possibilities:

2a) There exists a subcube $Q(v_i^L) \subset Q(w_i^L)$ such that

$$|st\ (Q(v_i^L) \times \{a_i\}, \mathcal{C}) \cap (Q(v_i^L) \times B_{L+1,i})| < \frac{v_i \cdot w_i}{2} .$$

Then there must be a subset $D_{L+1,i} \subset B_{L+1,i}$ with $|D_{L+1,i}| = v_i$ and

$$|st\ (Q(v_i^L) \times \{a_i\}, \mathcal{C}) \cap (Q(v_i^L) \times D_{L+1,i})| < \frac{v_i^{L+1}}{2} .$$

Then, since $v_i = f_{L+1}(u_i, 2)$, there exists a subcube $Q(u_i^{L+1})$ of $Q(v_i^L) \times D_{L+1,i}$, which does not intersect the set $st\ (Q(v_i^L) \times \{a_i\}, \mathcal{C})$.

Suppose $Q(u_i^{L+1}) = A_{1,i} \times A_{2,i} \times \ldots \times A_{L,i} \times A'_{L+1,i}$; the set $A'_{L+1,i}$ does not contain from trivial reason the point a_i. Pick an arbitrary $a \in A'_{L+1,i}$ and define $n_i = u_i$, $A_{L+1,i} = \{a_i\} \cup A'_{L+1,i} - \{a\}$. It remains to write $Q(n_i^{L+1}) = A_{1,i} \times A_{2,i} \times \ldots \times A_{L+1,i}$.

2b) For every subcube $Q(v_i^L) \subset Q(w_i^L)$ the inequality

$$|st\ (Q(w_i^L) \times \{a_i\}, \mathcal{C}) \cap (Q(v_i^L) \times B_{L+1,i})| \geq \frac{v_i^L \cdot w_i}{2}$$

holds. Fix for a moment one such subcube. There exists a set $M \subset Q(v_i^L) \times \{a_i\}$, $|M| \geq \frac{v_i^L}{4}$ and points $x_y \in S_y$ for $y \in M$ such that $|st^2(x_y, \mathcal{C}) \cap S_y \geq \frac{w_i}{4v_i^L}$ (to see this, it suffices to take $M = \{y \in Q(v_i^L) \times \{a_i\} :$ $: |st\ (Q(v_i^L) \times \{a_i\}, \mathcal{C}) \cap S_y| \geq \frac{w_i}{4}\}$: This set M must be of cardinality at least $\frac{v_i^L}{4}$, because

$st\ (Q(v_i^L) \times \{a_i\}, \mathcal{C}) \cap (Q(v_i^L) \times B_{L+1,i}) =$
$= \bigcup \{st\ (Q(v_i^L) \times \{a_i\}, \mathcal{C}) \cap S_y : y \in Q(v_i^L) \times \{a_i\}\}$, but for each $y \in M$ the set $st\ (Q(v_i^L) \times \{a_i\}, \mathcal{C}) \cap S_y$ equals to the union of at most v_i sets $st\ (z, \mathcal{C}) \cap S_y$ where z varies through $Q(v_i^L) \times \{a_i\}$, thus for at least one z_y the inequality $st\ (z_y, \mathcal{C}) \cap S_y \geq \frac{w_i}{4v_i^L}$ takes place. Pick an

x_y from st $(z_y, \mathscr{C}) \cap S_y$, obviously $st^2(x_y, \mathscr{C}) \cap S_y \supset \supset st(z_y, \mathscr{C}) \cap S_y)$.

Since this result holds for an arbitrary subcube $Q(v_i^L) \subset Q(w_i^L)$, we may conclude that it is possible to choose a point $x_S \in S$ with $|st^2(x_S, \mathscr{C}) \cap S| \geq \frac{w_i}{4v_i^L}$ for at least $\frac{w_i^L}{4}$ members S of $\mathscr{S}$, thus there exist a set of points $\{x_S \in S: S \in \mathscr{S}\}$ with

$$|\bigcup \{st^2(x_S, \mathscr{C}) \cap S: S \in \mathscr{S}\}| \geq \frac{w_i^L}{4} \cdot \frac{w_i}{4v_i^L} = \frac{w_i^{L+1}}{16v_i^L}$$

Now, notice that $w_i = f_{L+1}(k, 16v_i^L)$: there must be a cube $Q(k^{L+1})$ contained in $\bigcup \{st^2(x_S, \mathscr{C}) \cap S: S \in \mathscr{S}\}$. Since $\mathscr{S}$ was a relativization of $\mathscr{T}_{L+1}$, we have obtained that if 2b) will occur, then the statement holds for L + 1 and we may stop with the induction from (L +1)k to 1.

Suppose that the only possible cases during the whole induction from (L + 1)k to 1 were those indicated in 1) and 2a). In the final step the cube $Q(n_1^L) = A_{1,1} \times A_{2,1} \times \ldots \ldots \times A_{L+1,1}$ was obtained, $n_1 \geq u_0 = k$. Choose some subsets $B_i \subset A_{i,1}$ for $i = 1,2,\ldots,L$ with $|B_i| = k$. The set $a_1, a_2, \ldots \ldots, a_{(L+1)k}$ can be divided into L + 1 subsets $M_1, M_2, \ldots \ldots, M_{L+1}$: A point a_i belongs to M_{L+1} if $Q(n_i^L) \times \{a_i\}$ is $\mathscr{C}$-discrete, a_i belongs to M_j $(j \in \{1,2,\ldots,L\})$ if there is a sequence $\{x_T: T \in \mathscr{T}_j\}$ with $x_T \in T$ and $\bigcup \{st^2(x_T, \mathscr{C}) \cap T: T \in \mathscr{T}_j$ and $T \cap Q(n_{i+1}^L) \times \{a_i\} \neq \emptyset$ contains a square $Q(n_i^L) \times \{a_i\}$.

Since $|\bigcup \{M_i: i = 1,2,\ldots,L+1\}| \geq (L+1)k$, there is some i_0 with $|M_{i_0}| \geq k$. Let $B_{L+1} \subset M_{i_0}$, $|B_{L+1}| = k$, and define $W(k^{L+1}) = B_1 \times B_2 \times \ldots \times B_{L+1}$. Now, if $i_0 \leq L$, it is clear that

$$Q(k^{L+1}) \subset \bigcup \{st^2(x_T, T) \cap T: T \in \mathscr{T}_{i_0}, T \cap Q(N^{L+1}) \neq \emptyset\}$$

for some choice $x_T \in T$, if $i_o = L + 1$, then $Q(k^{L+1})$ is $\mathcal{C}$-discrete, because $Q(k^L) \times \{a_i\}$ is $\mathcal{C}$-discrete and $st\,(Q(k^L) \times \{a_i\}, \mathcal{C}) \cap Q(k^L) \times \{a_j\} = \emptyset$ for any $i \neq j$, a_i, a_j M_{L+1} as a consequence of the fact that
$st\,(Q(n_i^L) \times \{a_i\}, \mathcal{C}) \cap Q(n_i^L) \times \{a_j\} = \emptyset$ for $j > i$.

The statement $(**)$ is proved and we are able to verify the assumptions of C. from Lemma

Let $\mathcal{F}$ be a filter on ω with a countable base, suppose that each $F \in \mathcal{F}$ has $\mathbb{P}$, let $\mathcal{C}$ be a point-finite cover of ω.

If there exists an $i \in \{1,2,\dots,L\}$ and a sequence $\{x_T: T \in \mathcal{T}_i\}$ with $x_T \in T$ such that $\bigcup \{st^2(x_T, \ ) \cap T: : T \in \mathcal{T}_i\} \cap F$ has $\mathbb{P}$ for each $F \in \mathcal{F}$, it suffices to write $M = \bigcup \{st^2(x_T, \mathcal{C}) \cap T: T \in \mathcal{T}_i\}$.

If no such i exists, then there is some $\tilde{F} \in \mathcal{F}$ and natural $m < \omega$ such that for every $i \in \{1,2,\dots,L\}$, for every sequence $\{x_T: T \in \mathcal{T}_i\}$ with $x_T \in T$ and for every $n < \omega$ the set $\bigcup \{st^2(x_T, \mathcal{C}) \cap T: T \in \mathcal{T}_i\} \cap \tilde{F} \cap R_n$ has not $\mathbb{P}(m)$. We are to find a $\mathcal{C}$-discrete set M such that $M \cap F$ has $\mathbb{P}$ for every $F \in \mathcal{F}$. Suppose $\{F_j: j < \varepsilon\}$ be the base of $\mathcal{F}$ and $\tilde{F} \supset F_0 \supset F_1 \supset F_2 \supset \dots$.

Induction: F_0 has $\mathbb{P}$, so there is a system $\{Q_0(k_i^L): i < \omega\} \subset \mathcal{Q}$ and a sequence $\{\ (0,i): i < \omega\}$ of natural numbers such that $F_0 \cap R_{n(0,i)} \supset Q_0(k_i^L)$, $\sup k_i = \omega$ and $n(0,i) \neq n(0,i')$ whenever $i \neq i'$. Let $i_0 < \omega$ be such a natural number that $k_{i_0} = g(m)$, where g is a function from $(**)$. Then there is a $\mathcal{C}$-discrete set $X_0 \subset Q_0((g(m))^L)$ which contains some cube $Q(m^L)$, other possibilities being excluded by the assumption $F_0 \subset \tilde{F}$. Set $n_0 = n(0,i_0)$, $M_0 = X_0$.

Suppose $n_0 < n_1 < n_2 < \dots < n_{p-1}$ and $M_0 \subset M_1 \subset \dots \subset M_{p-1}$ be defined with M_{p-1} finite and $\mathcal{C}$-discrete, $M_\ell \cap F_\ell \cap R_{n_\ell}$ having $\mathbb{P}(m + \ell)$ for $\ell = 1,2,\dots,p-1$. The set M_{p-1} is finite, the cover $\mathcal{C}$ is point-finite, thus $F_p \cap \cap st(M_{p-1}, \mathcal{C})$ cannot have $\mathbb{P}$ - the idea is the same as in the proof of Theorem 2. Thus $G_p = F_p - st\ (M_{p-1}, \mathscr{C})$ has $\mathbb{P}$; it follows that there exist a system $\{Q_p(k_i^L): i < < \omega\} \subset \mathcal{Q}$ and a sequence $\{n(p,i): i < \omega\}$ of natural numbers such that $G_p \cap R_{n(p,i)} \supset Q_p(k_i^L)$, $\sup k_i = \omega$ and $n(p,i) \neq n(p,i')$ whenever $i \neq i'$. Let i_p be a natural number such that $n(p,i_o) > n_{p-1}$ and $k_{i_p} \geq g(m+p)$. Using $(**)$, we can find a $\mathscr{C}$-discrete set $X_p \subset Q_p(k_{i_p}^L)$ which contains a cube $Q((m = p)^L)$. Let $n_p = n(p,i_p)$, $M_p = = M_{p-1} \cup X_p$. The set X_p is contained in $R_{n_p} \cap F_p$ and the set M_p is $\mathscr{C}$-discrete: X_p is $\mathscr{C}$-discrete and $X_p \subset \omega - - st\ (M_{p-1}, \mathscr{C})$.

It remains to define $M = \bigcup\{M_p: p < \omega\}$. M is $\mathscr{C}$-discrete and $M \cap F$ has $\mathbb{P}$ for each $F \in \mathscr{F}$.

IV. We have verified that the properties $\mathbb{P}$ and are good enough to use them in Lemma 4. Let q be the ultrafilter from C. of that Lemma, it is a P-point and it is not selective.

Let $\mathcal{A}_i$ be a uniformity with a base

$$\{\mathcal{T}_i \wedge \mathcal{P} : \mathcal{P} \text{ is a uniform cover of } \mathcal{O}_q\}.$$

Clearly each $\mathcal{A}_i$ is uniformly non-discrete and $\mathcal{A}_i \neq \mathcal{A}_j$ whenever $i \neq j$ because $\mathcal{T}_i \wedge \mathcal{T}_j = \{\{x\} : x \in \omega\}$. Thus we have L distinct uniformities below $\mathcal{O}_q$ and it remains to prove that each of them is an atom and that there is no other atom belov $\mathcal{O}_q$. Indeed, it will suffice to show that any uniformity $\mathcal{U}$ below $\mathcal{O}_q$ is either coarser than some $\mathcal{A}_i$ or uniformly discrete.

To this end, let $\mathcal{U}$ be a uniformity below $\mathcal{O}_q$: If there is an $i \in \{1,2,\dots,L\}$ such that every uniform cover $\mathcal{E}$ of $\mathcal{U}$ belongs to $\mathcal{A}_i$, then $\mathcal{U} \prec \mathcal{A}_i$, so suppose the contrary. We can find then uniform covers $\mathcal{E}_1$, $\mathcal{E}_2, \dots, \mathcal{E}_L \in \mathcal{U}$ with $\mathcal{E}_i \notin \mathcal{A}_i$ for $i = 1,2,\dots,L$. Let $\mathscr{C}$ and $\mathscr{D}$ be uniform covers from $\mathcal{U}$, $\mathscr{C}$ point-finite and suppose that $\mathscr{C} \overset{*}{\prec} \mathscr{D} \overset{*}{\prec} \mathcal{E}_1 \wedge \mathcal{E}_2 \wedge \dots \wedge \mathcal{E}_L$.

According to Lemma 4 there exists a set $M \in q$ having $\mathfrak{S}_{\mathcal{C}}$, let $\mathcal{P}_M$ be the cover $\{M\} \cup \{\{x\} : x \in \omega\}$. Since $\mathcal{P}_M \in \sigma_q$ and $\mathcal{U}$ is finer than σ_q, $\mathcal{P}_M \in \mathcal{U}$.

If there exist an $i \in \{1,2,\ldots,L\}$ and a sequence $\{x_T : T \in \mathcal{T}_i\}$ with $x_T \in T$ such that $st^2(x_T, \mathcal{C}) \cap T \supset M \cap T$, then the cover $\mathcal{P}_M \wedge \mathcal{T}_i$ would refine $\mathcal{E}_1 \wedge \mathcal{E}_2 \wedge \ldots \wedge \ldots \wedge \mathcal{E}_L$ which is impossible by the choice of $\mathcal{E}_i$. Thus M is $\mathcal{C}$-discrete, consequently, $\mathcal{P}_M \wedge \mathcal{C} = \{\{x\} : x \in \in \omega\}$ and the uniformity $\mathcal{U}$ is uniformly discrete.

The proof of Theorem 5 is complete.

Let us consider the atoms constructed in the previous proofs from another point of view. We gave some examples of P-points in $\beta(\omega)$ ω which indicate that there is a classification of types in $\beta(\omega)$ ω completely different and in some sense finer than the classifications obtained with the topological approach. Let us say that an ultrafilter q belongs to W_α (α cardinal) if there is precisely α distinct uniform atoms below σ_q. We have proved that (under [CH]) $W_\alpha \neq \emptyset$ for $\alpha < \omega$, we are able to prove that $W \neq \emptyset$, too (Theorem 7); an open question is non-voidness of W_α for $\omega \leq \alpha < 2$. Another question arises if we realize that all the described atoms were made simply by adding one partition to σ_q: is this a general principle, how to make atoms? As may be expected this is not true, not even in the case of P-points, this result is stated in Theorem 6. What may be surprising is the fact that one needs only countably many partitions to obtain an atom below this special σ_q.

6. Theorem. Assume [CH]. Then there exists a P-point q on ω such that for each partition $\mathcal{R}$ of ω a uniformity with a base $\{\mathcal{R} \wedge \mathcal{P} : \mathcal{P} \in \sigma_q\}$ is never a uniform atom on ω. Moreover, there exists precisely one atom below σ_q.

Proof. Let $\mathcal{R} = \{R_n : n < \omega\}$ be a partition of such that $|R_n| = n^n$. Then we can easily construct a sequence $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \dots$ of partitions of ω such that $\mathcal{R} \succ \mathcal{S}_1 \succ \mathcal{S}_2 \succ \dots$ and, if $S \in \mathcal{S}_m$ and $S \subset R_n$, then $|S| = \mathrm{Max}\,(1, n^{n-m})$.

Similarly as in Theorem 5, we want to use Lemma 4. The properties $\mathbb{P}(k)$ are defined as follows: A finite set $M \subset \omega$ has $\mathbb{P}(0)$ iff $M \neq \emptyset$, and M has $\mathbb{P}(k)$ iff $|\{S(1) \in \mathcal{S}_1 : |S(2) \in \mathcal{S}_2 : S(2) \subset S(1)\ \&\ |\{S(3) \in \mathcal{S}_3 : S(3) \subset S(2)\ \&\ \dots\ \&\ |\{S(k) \in \mathcal{S}_k : S(k) \subset S(k-1)\ \&\ \&\ |S(k) \cap M| > k\}| > k \dots \}| > k\}| > k\}| > k$.
(Thus M has $\mathbb{P}(1)$ iff $|\{S(1) \in \mathcal{S}_1 \quad |S(1) \cap M| > 1\}| > 1$, M has $\mathbb{P}(2)$ iff $|\{S(1) \in \mathcal{S}_1 : |\{S(2) \in \mathcal{S}_2 : S(2) \subset S(1)\ \&\ |S(2) \cap M| > 2\}| > 2\}| > 2$, and so on.)

All the conditions (0),...,(iv) from Lemma 4 are satisfied. We shall show the validity of (iii) only and leave the rest to the reader.

Let $f(k) = 2k = 1$, suppose M has $\mathbb{P}(f(k))$, $M = M_1 \cup M_2$. Since M has $\mathbb{P}(f(k))$, there are some members from $\mathcal{S}_{2k+1}$ which meet M in at least $2k + 2$ points, so there are some $S(k)$'s from $\mathcal{S}_k$ with $|M \cap S(k)| > 2k + 1$, because $\mathcal{S}_{2k+1}$ refines $\mathcal{S}_k$. Denote by

$$\mathcal{S}_k(0) = \{S(k) \in \mathcal{S}_k : |M \cap S(k)| > 2k + 1\},$$

$$\mathcal{S}_k(i) = \{S(k) \in \mathcal{S}_k : |M_i \cap S(k)| > k\},\ i = 1,2.$$

Suppose $\mathcal{S}_{\ell+1}(i)$, $i = 0,1,2$, be defined for $1 < \ell + 1 \leq k$, then

$$\mathcal{S}_\ell(0) = \{S(\ell) \in \mathcal{S}_\ell : |\{S(\ell + 1) \in \mathcal{S}_{\ell+1}(0) : S(\ell + 1) \subset S(\ell)\}| > 2k + 1\};$$

$$\mathcal{S}_\ell(i) = \{S(\ell) \in \mathcal{S}_\ell : |\{S(\ell + 1) \in \mathcal{S}_{\ell+1}(i) : S(\ell + 1) \subset S(\ell)\}| > k\},\ i = 1,2.$$

The inclusion $\mathcal{S}_\ell(0) \subset \mathcal{S}_\ell(1) \cup \mathcal{S}_\ell(2)$ holds for each $\ell = 1,2,\dots,k$: This is obvious for $\ell = k$ and for $\ell < k$ we obtain the result simply by a finite induction down using the fact that M has $\mathbb{P}(2k+1)$. Thus $\mathcal{S}_1(0) \subset \mathcal{S}_1(1) \cup \mathcal{S}_1(2)$ and, since M has $\mathbb{P}(2k+1)$, $|\mathcal{S}_1(0)| > 2k+1$ and we conclude that e.g. $|\mathcal{S}_1(1)| > k$. But then it can be quickly deduced from the definition of $\mathcal{S}_1(1)$ that M_1 has $\mathbb{P}(k)$.

Let $\mathbb{P}$ be the property from Lemma 4 (where we use $\mathcal{R}$ as an admissible partition). Let $\mathcal{C}$ be a point-finite cover of ω, let $\mathfrak{S}_{\mathcal{C}}$ be the following property of a set $M \subset \omega$

"The set M is either $\mathcal{C}$-discrete or there exist some $m < \omega$ and a sequence of points $\{x_S \in S: S \in \mathcal{S}_m\}$ such th t $st^2(x_S, \mathcal{C}) \cap S \supset M \cap S$ for every $S \in \mathcal{S}_m$."

We have to verify the assumptions of C. from Lemma 4. To this end, let $\mathcal{F}$ be a filter with a countable base $\{F_j: j < \omega\}$ consisting of sets with $\mathbb{P}$ and assume that $F_0 \supset F_1 \supset F_2 \supset F_3 \supset \dots$.

If there exist and $m < \omega$ and a sequence $\{x_S \in S: S \in \mathcal{S}_m\}$ with $\bigcup\{st^2(x_S, \mathcal{C}) \cap S: S \in \mathcal{S}_m\} \cap F_j$ having $\mathbb{P}$ for each $j < \omega$, then we may define $M = \bigcup\{st^2(x_S, \mathcal{C}) \cap S: S \in \mathcal{S}_m\}$ and the assumptions of C. from Lemma 4 are satisfied for this M.

So, suppose the opposite: No such $m < \omega$ and no such sequence $\{x_S\}$ exists. We must construct a $\mathcal{C}$-discrete set M such that $M \cap F_j$ has $\mathbb{P}$ for each $j < \omega$.

Induction: Let $j_0 = 0$, pick arbitrarily a point $y \in F_0$, let $X_0 = M_0 = \{y\}$ and let n_0 be such a natural number that $y_0 \in R_{n_0}$.

Let $k < \omega$ and suppose that the natural numbers $n_0 < n_1 < \dots < n_{k-1}$, $j_0, j_1, \dots, j_{k-1}$ and finite sets $M_0, M_1, \dots, M_{k-1}$, $X_0, X_1, \dots, X_{k-1}$ be defined such that M_{k-1} is $\mathcal{C}$-discrete, $M_i = M_{i-1} \cup X_i$, $X_i \subset F_{j_i} \cap F_i \cap R_{n_i}$ and X_i have $\mathbb{P}(i)$ for $i = 0, 1, \dots, k-1$.

By the hypothesis, there is some natural j_k such that $\bigcup\{st^2(x_S, \mathcal{C}) \cap S: S \in \mathcal{S}_k\} \cap F_{j_k}$ has not $\mathbb{P}$ for each choice $x_S \in S$. Thus we may assume that ther is some $l_k < \omega$

and natural $N > n_{k-1}$ such that for every $n > N$ and for every sequence $\{x_S \in S: S \in \mathcal{S}_k\}$ the set $\bigcup \{st^2(x_S, \mathcal{C}) \cap$ $: S \in \mathcal{S}_k\} \cap F_{j_k} \cap R_n$ has not $\mathbb{P}(\ell_k)$. The set M_{k-1} is finite and $\mathcal{C}$ is point-finite; similarly as in previous proofs we conclude that the set $G_k = F_k \cap F_{j_k} - st(M_{k-1}, \mathcal{C}$ has $\mathbb{P}$.

By the repeated use of (iii) from Lemma 4 we can fin a natural h such that if a finite set Q has $\mathbb{P}(h)$, if $\mathcal{Y}$ is a family of cardinality $(k+1)^{k+1}$ and if $\bigcup \mathcal{Y} = Q$, then at least one $Y \in \mathcal{Y}$ has $\mathbb{P}(\mathrm{Max}(k, \ell_k))$.

Since the set G_k has $\mathbb{P}$, there is some $n_k > N$ such that $G_k \cap R_n$ has $\mathbb{P}(f(h))$ (the function f was defined above). Let $\mathcal{S}'_k$ be a family of all $S \in \mathcal{S}_k$, $S \subset R_{n_k}$ such that there exists a set $D_S \subset S \cap G_k$ with a property that $st^2(x, \mathcal{C}) \cap D_S = \{x\}$ whenever $x \in D_S$ and of cardinality $|D_S| = (k+1)^{k+1}$. Let $L = \bigcup \{S \in \mathcal{S}_k: S \subset R_{n_k}$, $S \notin \mathcal{S}'_k\} \cap G_k$. The set L cannot have $\mathbb{P}(h)$: Notice tha we can choose a set $D_S \subset S \cap G_k$ for each $S \notin \mathcal{S}'_k$ such that $st^2(D_S, \mathcal{C}) \supset S \cap G_k$, $|D_S| \leq (k+1)^{k+1}$. Thus it is possibl to find one $x_S \in D_S$ for each $S \in \mathcal{S}_k - \mathcal{S}'_k$ such that the set $\{st^2(x_S, \mathcal{C}) \cap S: S \notin \mathcal{S}'_k\} \cap G_k$ has $\mathbb{P}(l_k)$, as a consequence of the definition of h and $\mathcal{S}'_k$.

Thus the set $R_{n_k} \cap G_k - L$ has $\mathbb{P}(h)$, and one can find the following families of sets:

$\{S_{i_1}: 1 \leq i_1 \leq k+1\} \subset \mathcal{S}_1$, $S_{i_1} \subset R_{n_k}$ for $1 \leq i_1 \leq k+1$;

$\{S_{i_1 i_2}: 1 \leq i_1, i_2 \leq k+1\} \subset \mathcal{S}_2$, $S_{i_1 i_2} \subset S_{i_1}$ for $1 \leq i_1$, $i_2 \leq k+1$;

$\vdots$

$\{S_{i_1 i_2 \ldots i_{k-1}}: 1 \leq i_1, i_2, \ldots, i_{k-1} \leq k+1\} \subset \mathcal{S}_{k-1}$,

$S_{i_1 i_2 \ldots i_{k-2} i_{k-1}} \subset S_{i_1 i_2 \ldots i_{k-1}}$ for $1 \leq i_1, i_2, \ldots, i_{k-2}$

$i_{k-1} \leq k + 1$;

$\{ S_{i_1 i_2 \dots i_k} : 1 \leq i_1, i\ , \dots, i_k \leq k + 1 \} \subset \mathcal{S}'_k$,

$S_{i_1 i_2 \dots i_{k-1} i_k} \subset S_{i_1 i_2 \dots i_{k-1}}$ and $| S_{i_1 i_2 \dots i_k} \cap G_k | > h$

for $1 \leq i_1, i_2, \dots, i_k \leq k + 1$.

Since $S_{i_1 i_2 \dots i_k} \in \mathcal{S}'_k$, let $D_{i_1 i_2 \dots i_k}$ be a subset of $S_{i_1 i_2 \dots i_k} \cap G_k$ which satisfies $st^2(x, \mathcal{C}) \cap D_{i_1 i_2 \dots i_k} = \{x\}$ for each $x \in D_{i_1 i_2 \dots i_k}$ and $| D_{i_1 i_2 \dots i_k} | = (k + 1)^{k+1}$.

Let $I = \{ z : 1 \leq z \leq (k + 1)^k \}$ be some well-ordering of the set of indices $I = \{ \langle i_1, i_2, \dots, i_k \rangle : 1 \leq i_1, i_2, \dots$ $\dots, i_k \leq k + 1 \}$. By an induction we may efine for each $z \in \{1, 2, \dots, (k + 1)^k\}$ a finite $\mathcal{C}$-discrete set $E_z \subset D_z$ such that $| E_z | = k + 1$ and $st(E_z, \mathcal{C}) \cap E_w = \emptyset$ for $z \neq w$. If E_w have been defined for $1 \leq w < z \leq (k + 1)^k$, then $| \bigcup \{ E_w : 1 \leq w < z \} | \leq (k + 1)^{k+1} - (k + 1)$, and from the observation $st^2(x, \mathcal{C}) \cap D_z \supset st(y, \mathcal{C}) \cap D_z$ (whenever $x \in D_z \cap$ $\cap\, st(y, \mathcal{C})$ and $y \in \bigcup \{ E_w : 1 \leq w < z \}$) together with the fact that $st^2(x, \mathcal{C}) \cap D_z = \{x\}$ (for $x \in D_z$) follows that there is a subset $E_z \subset D_z - st\,(\bigcup \{ E_w : 1 \leq w < z \}, \mathcal{C})$ with cardinality $k + 1$.

Let us define $X_k = \bigcup \{ E_z : 1 \leq z \leq k + 1)^k \}$ Clearly X_k is $\mathcal{C}$-discrete, $X_k \subset G_k \subset F_{j_k} \cap F_k \cap R_{n_k}$, $st\,(M_{k-1}, \mathcal{C}) \cap$ $\cap X_k = \emptyset$, thus $M_k = M_{k-1} \cup X_k$ is $\mathcal{C}$-discrete, and X_k has $\mathbb{P}(k)$.

It follows that the set $M = \bigcup \{ M_k : k < \omega \}$ is the set needed in Lemma 4, C.

Now, use Lemma 4: We have a P-point q and we must prove that there is at most one atom below σ_q and that his atom cannot be obtained by adding one artition to a uniformity σ_q. The uniformity $\mathcal{A}$ on ω will be th uniformity whose base consists of all $\mathcal{P} \wedge \mathcal{S}_n$ with $\mathcal{P} \in \sigma_q$, $n < \omega$. By the definition of th prop rty $\mathbb{P}$, every set belonging to q is $\mathcal{S}_n$-unbound d for each $n < \omega$, thus $\mathcal{A}$ is not the discrete uniformity; obviously $\mathcal{A} \prec \mathcal{O}_q$, $\mathcal{A} \neq \mathcal{O}_q$.

We shall show that $\mathcal{A}$ is the unique atom below $\mathcal{O}_q$. Let $\mathcal{U}$ be an arbitrary uniformity below $\mathcal{O}_q$, suppose $\mathcal{U}$ not to be uniformly discrete. Let $\mathcal{E}$ be a $\mathcal{U}$-uniform cover. We can find covers $\mathcal{C}$, $\mathcal{D} \in \mathcal{U}$ such that $\mathcal{C} \overset{*}{\prec} \mathcal{D} \overset{*}{\prec} \mathcal{E}$ with $\mathcal{C}$ point-finite.

If M is the set of q satisfying $\mathfrak{S}_{\mathcal{C}}$, then M cannot be $\mathcal{C}$-discrete, for this together with $\mathcal{P}_M \in \mathcal{U}$ (where $\mathcal{P}_M = \{\{x\} : x \in \omega\} \cup \{M\} \in \mathcal{O}_q$) implies that $\mathcal{U}$ is a discrete uniformity on ω. But if M is not $\mathcal{C}$-discrete, then there exists an $m < \omega$ and a sequence $\{x_S \in S : S \in \mathcal{S}_m\}$ such that $st^2(x_S, \mathcal{C}) \cap S \supset M \cap S$ for each $S \in \mathcal{S}_m$, in other words $\mathcal{P}_M \cap \mathcal{S}_m$ refines $\mathcal{E}$, thus $\mathcal{E} \in \mathcal{A}$, which shows that $\mathcal{A}$ is a unique atom below $\mathcal{O}_q$.

Finally, suppose that there is some partition $\mathcal{T}$ of ω such that $\{\mathcal{T} \wedge \mathcal{P} : \mathcal{P} \in \mathcal{O}_q\}$ is a base for $\mathcal{A}$. Let M be the set from q having $\mathfrak{S}_{\mathcal{T}}$, then the cover $\mathcal{P}_M \wedge \mathcal{S}_m$ refines $\mathcal{T}$ for some suitable $m < \omega$. Consider the cover $\mathcal{P}_M \wedge \mathcal{S}_{m+1}$. Assume that there is some $F \in q$ such that $\mathcal{T} \wedge \mathcal{P}_F$ refines $\mathcal{P}_M \wedge \mathcal{S}_{m+1}$, thus $\mathcal{P}_M \wedge \mathcal{P}_F \wedge \mathcal{S}_m$ refines $\mathcal{P}_M \wedge \mathcal{S}_{m+1}$. But this contradicts the condition that $F \cap M$ has $\mathbb{P}$: Consider the set $R_n \in \mathcal{R}$ such that $R_n \cap M \cap F$ has $\mathbb{P}(m+1)$. There is a set $S \in \mathcal{S}_m$, $S \subset R_n$, and a point $x \in R_n \cap S \cap M \cap F$ such that $st^2(x, \mathcal{P}_M \wedge \mathcal{P}_F \wedge \mathcal{S}_m) \cap S = M \cap F \cap S$ intersects at least $m+1$ members of $\mathcal{S}_{m+1}$. This contradiction completes the proof.

We have promised to show an example of an ultrafilter q such that there are $2^{\mathfrak{c}}$ atoms below $\mathcal{O}_q$. It is possible to arrange the proof of it in such a way that the q obtained will be a P-point, but it seems better to describe the main idea of the construction on the simpler case of non-minimal (in RF-order) point of $\beta(\omega) - \omega$.

By Theorem 5, for each $L < \omega$ there exists an ultrafilter q_L with precisely L distinct uniform atoms below σ_{q_L}.

The proof of Theorem 5 gives slightly more than stated in the theorem: each q_L is an ultrafilter defined on a union K_L of a disjoint family $\mathcal{R}_L$ of L-cubes and every $F \in q_L$ contains arbitrarily large L-subcubes of cubes from $\mathcal{R}_L$. Let p be an arbitrary free ultrafilter on a set $\omega = \{ L: L < \omega \}$, let $q = \sum_p q_L$ be defined on $K = \bigcup \{ K_L: L < \omega \}$, the union is, of course, disjoint. We claim that q is the desired ultrafilter.

Fix L for a moment. For every set $Y \subset \{ 1,2,\dots,L \}$ we may define a partition $\mathcal{T}_{L,Y}$ of K_L as follows: $T \in \mathcal{T}_{L,Y}$ iff there is an $A_1 \times A_2 \times \dots \times A_L \in \mathcal{R}_L$ and a point $\langle y_1,y_2,\dots,y_L \rangle \in A_1 \times A_2 \times \dots \times A_L$ such that $T = \{ \langle z_1,z_2,\dots,z_L \rangle \in A_1 \times A_2 \times \dots \times A_L: z_i = y_i$ for each

$i \in \{ 1,2,\dots,L \} - Y \}$.

Thus $\mathcal{T}_{L,\{i\}} = \mathcal{T}_i$ in the notation used in the proof of Theorem 5. Obviously $\mathcal{T}_{L,Y} \wedge \mathcal{T}_{L,Z}$ is a discrete cover $\{\{ x \} : x \in K_L \}$ if and only if $Y \cap Z = \emptyset$.

Consider the set $X = \{ \langle L,i \rangle : L < \omega, i = 1,2,\dots,L \}$. For $Z \subset X$, let $Z_L = \{ i: \langle L,i \rangle \in Z \}$. Having a partition $\mathcal{T}_{L,Z_L}$ of K_L, we may define a partition $\mathcal{T}_Z$ of K simply as $\bigcup \{ \mathcal{T}_{L,Z_L}: L < \omega \}$. It is self-evident that

(+) the uniformity with a subbase $\{ \mathcal{T}_Z \} \cup \sigma_q$ is uniformly discrete if and only if $\{ L: Z_L = \emptyset \} \in p$.

Finally, let $\mathcal{F}$ be a filter on X such that $F \in \mathcal{F}$ iff there is some $P \in p$ with $F \supset \{ \langle L,i \rangle : i = 1,2,\dots,L \}$ whenever $L \in P$. Let $\mathcal{M}$ be the family of all ultrafilters on X containing $\mathcal{F}$, obviously $|\mathcal{M}| = 2$. Thus, according to (+), for $t \in \mathcal{M}$, the family $\{ \mathcal{T}_Z: Z \in t \}$ of partitions of K together with σ_q is contained in some atom $\mathcal{A}_t$.

We have a sufficiently large family of atoms on K and we need to prove that $\mathcal{A}_t \neq \mathcal{A}_{t'}$ for $t \neq t'$. But this is clear since for $Z \in t$, $Z' \in t'$ with $Z \cap Z' = \emptyset$ the common refinement of $\mathcal{T}_Z \wedge \mathcal{T}_{Z'}$ is the discrete cover $\{\{x\} : x \in K\}$.

Thus we have proved

7. Theorem. Assume [CH]. Then there exists an ultrafilter q on ω such that there is $2^{\quad}$ distinct uniform atoms below σ_q.

8. Remarks and problems. a) It is possible to strengthen Theorem 7 by the condition that q is a P-poin The proof is similar to the proof of Theorem 5, one star with some partition of into cubes $Q(n^L)$ where both n and L are increasing. The choice of n (depending on L) needs some care, but this is the only difficult step - the rest is a mere amalgamation of methods used in the proofs of Theorems 5 and 7.

b) Theorem 5 can be sharpened to this form: Under [CH], there is a P-point p on ω such that there are only L + 1 uniformities below σ_p, the uniformly discrete one and L atoms. If $\mathcal{R}$ is a family of L-cubes used in the proof of Theorem 5, we can map $\bigcup \mathcal{R}$ onto ω suc that the image of the q from Theorem 5 will be the desired p. To describe the mapping f, visualize each $Q(n^L) \in$ as $\{1,2,\dots,n\}^L \subset \omega^L$; let f_n be a mapping which split together any two points $\langle a_1, a_2, \dots, a_n \rangle$ and $\langle b_1, b_2, \dots, b_n \rangle$ from $\{1,2,\dots,n\}^L$ such that $b_i - a_i = 1$ for all $i \leq n$. Then f is the canonical mapping from $\bigcup \mathcal{R}$ onto the disjoint union $\sum f_n [Q(n^L)]$.

c) I do not know whether the results obtained wit [CH] will remain valid under any other set-theoretic assumption which implies the existence of P-points.

d) Up to now, all atoms described in this paper were 0-dimensional, i.e. their base was a family of partitions. It is an open question whether there exists a no

O-dimensional uniform atom; this problem seems to be pretty hard. We also do not know an answer to this, perhaps easier, question: If $UNIF_0$ is the lattice of all O-dimensional uniformities on ω , if $\mathcal{A}$ is an atom on ω and if the uniformity $\mathcal{B}$ has a base of all $\mathcal{A}$ -uniform partitions, is then $\mathcal{B}$ necessarily an atom in $UNIF_0$?

e) Th following is the purely set-theoretic probl m concerning the properties of RF-order of $\beta(\omega) - \omega$. Suppose that $t = \sum_{p} q_n$ for some ultrafilters t, p, q_1, q_2,... . Is it true then that $t = \sum_{p'} q_n'$ for some p', where all ultrafilters $'_n$ are RF-minimal? The motivation for this problem is hidden inthis - maybe too general - question: Suppose one knows everything about atoms below σ_q for an arbitrary RF-minimal q. What are the consequences of this knowledge for atoms below σ_p with p non-minimal?

f) Maybe there are several readers satisfying the following two conditions: They are - in spite of reading the present paper till here - fresh enough to solve some of our prob ems, and they believe that at least one non-O-dimensional atom on exists. Those readers are precisely those ones who need the following description of non-O-dimensional atom,due to J. Pelant.
Recall that a component of a cover $\mathcal{V}$ is the smallest non-empty set $X \subset \bigcup \mathcal{V}$ such that $st(X,\mathcal{V}) = X$, or, equivalently, if $x, y \in X$, then there is a finite sequence $C_1, C_2, \dots, C_n$ of members of $\mathcal{V}$ such that $x \in C_1$, $y \in C_n$ and $C_i \cap C_{i+1} \neq \emptyset$ for $i = 1,2,\dots,n$ 1.

9. Theorem. (Pelant) If $\mathcal{A}$ is a non-O-dimensional uniform atom on ω , then $\mathcal{A}$ has a base consisting of point-finite covers with finite components.

Proof. The first step is to prove that there is a base $\mathcal{B}$ for $\mathcal{A}$ such that each $\mathcal{C} \in \mathcal{B}$ is point-finite and each $C \in \mathcal{C}$ is finite.

Suppose $\mathcal{U}$ to be an arbitrary $\mathcal{A}$ -uniform cover. According to Proposition (e) we may assume that there is a couple $\mathcal{V}$, $\mathcal{W}$ of point-finite covers from $\mathcal{A}$ such that $\mathcal{W} \overset{*}{<} \mathcal{V} \overset{*}{<} \mathcal{U}$. By indexing members of $\mathcal{V}$ we obtain $\mathcal{V} = \{V_\alpha : \alpha < \lambda\}$ for some ordinal number λ. Then the family $\{R_\alpha : \alpha < \lambda\}$, where $R_\alpha = \{x : st(x, \mathcal{W}) \subset \subset V_\alpha$ & $st(x, \mathcal{W}) \not\subset V_\beta$ for $\beta < \alpha$, is a partition of ω. Clearly $\{R_\alpha\}$ refines $\mathcal{U}$, and by the assumption that $\mathcal{A}$ is non-0-dimensional there exists a couple $\mathcal{V}$ $\mathcal{W}$ such that $\mathcal{W} \overset{*}{<} \mathcal{V} \overset{*}{<} \mathcal{U}$ and $\{R_\alpha\} \notin \mathcal{A}$. But $\mathcal{A}$ is an atom, so there is some point-finite $\mathcal{Z} \in \mathcal{A}$ such that $st(x, \mathcal{Z}) \cap R_\alpha = \{x\}$ whenever $x \in R_\alpha$ and $\alpha < \lambda$.

The cover $\mathcal{W} \wedge \mathcal{Z}$ is point-finite and the set $W \cap$ $(W \in \mathcal{W}, Z \in \mathcal{Z})$ is always finite: If $x \in W$ and if $y_\alpha \in W \cap \cap R_\alpha$, then $x \in st(y_\alpha, \mathcal{W})$; by point-finiteness of $\mathcal{V}$, W meets only finitely many R_α. Since Z meets every R_α in one point at most, $W \cap Z$ is finite.

Now we are ready to prove the theorem. Let $\mathcal{U} \in \mathcal{A}$, let $\mathcal{V}$ be a point-finite star-refinement of $\mathcal{U}$, let every $V \in \mathcal{V}$ be finite. Pick a point x_C from each component C of $\mathcal{V}$ and define by induction $M_0 = \{x_C : C$ is a component of $\mathcal{V}\}$, $M_n = st(M_{n-1}, \mathcal{V})$, $H_n = M_n - M_{n-1}$ for $1 \le n < \omega$. Notice that $st(x, \mathcal{V}) \cap H_n \neq \emptyset$ implies $x \in \in H_{n-1} \cup H_n \cup H_{n+1}$. Let $G_1 = \bigcup \{H_{3n+1} : n < \omega\}$, $i = = 0,1,2$. If $N_\mathcal{A}$ is the ultrafilter of all uniformly non discrete subspaces of $\langle \omega, \mathcal{A} \rangle$ (see [PR], p. 76), then one G_i, say G_0, belongs to $N_\mathcal{A}$.
Since the cover $\mathcal{W} = \{\{x\} : x \in \omega\} \cup \{G_0 \cap V : V \in \mathcal{V}\}$ obviously belongs to $\mathcal{A}$ and $\mathcal{W} \overset{*}{<} \mathcal{U}$, we need only to prove that $\mathcal{W}$ has finite components. To see this, realize that $H_n \cap C$ is finite for each $n < \omega$ as a consequence of point-finiteness of $\mathcal{V}$ and of finiteness of its members, and use the fact that each component D of $\mathcal{W}$ is either one-point or contained in some $H_{3n} \cap C$, where C is a component of $\mathcal{V}$.

G. Choquet [CH] has defined an ultrafilter q on ω

to be rar , if for ev ry partition $\mathcal{R}$ of ω nto finite sets th re is a set $F \in q$ uch th t $|F \cap R| \leq 1$ whenever $R \in \mathcal{R}$. Henc the theor m h s imm diate

1C. Corollary. Suppose q to be a rar ultrafilt r on ω . Then every atom below σ_q is 0-dimensional.

R e f e r n c e s

[Č] E. Čech: Topological spaces. 2nd ed., Academia, Prague 1966.

[Ch] G. Choquet: Deux classes rem rquables d ultrafiltres sur N. Bull. Sc. Math., 2´ éri 92(1968), 143-153.

[CN] W.W. Comfort, S. Negreponti : The theory of ultrafilters. Springer Verlag, Berlin 1974.

[ES] P. Erdös, J. Spencer: Probabilistic methods in combinatorics. Akadémiai Kiado, Budapest 1974.

[F] Z. Frolík: Sums of ultrafilters. Bull. Amer. Math. Soc. 73(1967), 87-91.

[PR] J. Pelant, J. Reiterman: Atoms in uniformities. Seminar uniform spaces 1973-1974, Matematický ústav ČSAV, Prague 1974, pp. 73-81.

[R] M.E. Rudin: Partial orders on the types in βN. Trans. Amer. Math. Soc. 155, 2(1971), 353-362.

[V] G. Vidossich: Uniform spaces of countable type. Proc. Amer. Math. Soc. 25(1970), 551-553.