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Jan Pelant; Jiří Vilímovský

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Seminar Uniform Spaces 1975-1976

Two examples of reflections

J. Pelant and J. Vilímovský

Two constructions of modifications (set preserving reflections) are presented. These constructions give examples of modifications with unbounded point-character and of quite "large" modifications not preserving Cauchy filters. Connections with Katětov-Shirota theorem are added.

Definition 1: For any uniform space X, let $\mathcal{S}(X)$ be a family of subsets of X fulfilling the following:
(*) For a mapping $f: X \longrightarrow Y$ being either uniform embedding or Cartesian projection (i.e. $X = \prod X_a$, $Y = X_a$, $f = \pi_a: X \longrightarrow Y$) and $S \in \mathcal{S}(X)$ there is $f[S] \in \mathcal{S}(Y)$.

Let further $\mathcal{R}$ be a modification in uniform spaces. We denote Mdf $(\mathcal{S}, \mathcal{R})$ the class of all uniform spaces X such that $S \in \mathcal{R}$ for all $S \in \mathcal{S}(X)$.

Proposition 1: The classes Mdf $(\mathcal{S}, \mathcal{R})$ in Definition 1 are modifications, $\mathcal{R} \subset$ Mdf $(\mathcal{S}, \mathcal{R})$

Proof: is easy verifying that Mdf $(\mathcal{S}, \mathcal{R})$ is closed under products and subspaces.

Now we turn our attention to an interesting example. At first we recall a concept of a bounded set in a uniform space (cf. [H]). A subset B of a uniform space X ill be called bounded if all uniformly continuous real valued functions on X are bounded on B. Th following proposition is essentially proved in [H] :

Proposition 2: The following properties of a subset B of a uniform space X are equivalent:
(1) B is bounded
(2) For any uniformly continuous mapping $f: X \longrightarrow E$, E being a topological vector space, $f[B]$ is bounded in E (i.e. $f[B]$ is absrobed by all neighbourhoods of zero in E).

(3) For each uniform vicinity V of the diagonal of X there is a positive integer n such that $B \times B \subset V^n$.

(4) For each $\mathcal{U}$ uniform cover of X there is a point $x \in X$ and a positive integer n such that $st^n(x, \mathcal{U})$ (the n-th $\mathcal{U}$-star of x) contains B.

A uniform space X will be called absolutely bounded if X is bounded in itself. Obviously, the families $\mathcal{B}(X)$ of all bounded subsets and $\mathcal{AB}(X)$ of all absolutely bounded s sets of X satisfy the condition (*) from Definition 1. O can immediately see that for any $A \in \mathcal{AB}(X)$, $B \subset A$ implies $B \in \mathcal{B}(X)$. There is the converse question whether each bo ded subset of X can be embedded into some absolutely boun ed subset of X . The answer is negative as shows the foll wing

Example 1: Let M be an infinite set. $[M]^{=\omega_0}$ denote the family of all infinite countable subsets of M. Take a countable disjoint family $\{S_n\} \subset [M]^{=\omega_0}$ and a maximal almost disjoint system $\mathcal{P} \subset [M]^{=\omega_0}$ such that $\{S_n\} \subset \mathcal{P}$. Take a partition $\{\mathcal{P}_n \mid n \in \omega\}$ of $\mathcal{P}$ such that $S_n \in \mathcal{P}_n$. For each $S \in \mathcal{P}$, we add a new point $\{S\}$ and if $S \in \mathcal{P}_n$ we join $\{S\}$ by intervals of length n exactly with points contained in S (i.e. a hedgehog with thorns of length n is built above each $\{S\}$), for $S \in \mathcal{P}_n$, H(S) denotes the corresponding hedgehog. Put $X = M \cup \bigcup\{H(S) \mid S \in \mathcal{P}\}$. We take on X a uniformity $\mathcal{U}$ inductively generated by $\{H(S) \mid S \in \mathcal{P}\}$. M is bounded w.r.t. $(X, \mathcal{U})$ as each infinite countable subset of M intersects some member of $\mathcal{P}$ in an infinite set.

If Q is a bounded set containing M, then there is n_0 and a uniform cover $\mathcal{T}$ of X such that $st^{n_0}(\{S\}, \mathcal{T})$ does n intersect Q for any $S \in \mathcal{P}_n$, $n \geq n_0$. Now consider the family $\{S_n \mid n \in \omega\}$ and we see that there is an unbounded uniformly continuous real-valued mapping defined on Q (special properties of $\mathcal{P}$ and $\{S_n\}$ are employed), hence M is not contained in any absolutely bounded set.

However every uniform space can be embedded into an absolutely bounded space because one can check that every injective uniform space is absolutely bounded.

Taking for $\mathcal{S}(X)$ in Definition 1 the family of bounded subsets of X and for $\mathcal{R}$ all precompact spaces, we obtain so called BP-spaces (see ([D]). Example 1 shows that this class is distinct from the class Mdf ($\mathcal{AB}$,Precompact)

Definition 2: Let $\mathcal{H}$ be a family of subsets of X. We define order of $\mathcal{H}$ by ord $\mathcal{H}$ = sup $\{$card $\mathcal{G} \mid \mathcal{G} \subset \mathcal{H}$, $\cap \mathcal{G} \neq \emptyset\}$
Let $(X,\mathcal{U})$ be a uniform space. A point-character pc $(X,\mathcal{U})$ is defined as the least infinite cardinal number α such that there is a base of $\mathcal{U}$ whose all members are of order less than α. In studying modifications preserving Cauchy filters (see [P_3]) there appeared an importance of modifications with unbounded point-character. We are going to show that point-character of BP-spaces is not bounded by any cardinal, nevertheless, BP-modification does not preserve Cauchy filters.

We need the following

Lemma: Let X be a uniform space. $X \in$ BP iff no infinite countable discrete subset of X is bounded.

Proof: The only question arises which implication is more trivial.

Construction: Let $(X,\mathcal{U})$ be a uniform space. Let $\mathcal{T}$ be a uniform cover of X. $\{T_n \mid n \in \omega_0\}$ is a disjoint subfamily of $\mathcal{T}$. For $\{\mathcal{V}_n \mid n \in \omega_0\}$ we define a cover

$$(**)\ [\{\mathcal{V}_n\},\{T_n\}] = (\mathcal{T} - \{T_n \mid n \in \omega_0\}) \cup \bigcup \{\mathcal{V}_n \wedge T_n \mid n \in \omega_0\}$$

The set of all covers of the form (**) represents a base of a uniformity. Denote this uniformity by $Z(\mathcal{U})$, a uniform space $(X,Z(\mathcal{U}))$ will be denoted by Z(X). Lemma and Proposidition 2 imply that Z(X) is BP for each uniform space X. (The fact that Z(X) and X have the same set of all countable discrete sets is helpful.)

Consider now $\ell_\infty(\alpha)$ where α is an infinite cardinal number. It is shown in [P_2] that pc $(\ell_\infty(\alpha)) \geq \alpha$. $Z(\ell_\infty(\alpha))$ differs from $\ell_\infty(\alpha)$ only on countable discrete families, so it is not hard to see that the point-character of $Z(\ell_\infty(\alpha))$ is large.

Definition 3: For $\mathcal{R}$ a modification in uniform spaces with the reflector r we define a class Γ_r of uniform spaces in the following way: $X \in \Gamma_r$ iff $r\hat{X}$ is complete where $\hat{X}$ stands for the completion of X.
One can easily prove the following

Proposition 3: (1) Γ_r is always a modification
(2) $X \in \Gamma_r$ iff rX has the same Cauchy filters as X;
(3) if $\mathcal{R} \subset \mathcal{S}$ with reflectors r, s then $\Gamma_r \subset \Gamma_s$
For $\mathcal{R}$ closed under completions, it holds
(4) $\mathcal{R} \subset \Gamma_r$
(5) $X \in \Gamma_r$ iff $r\hat{X} = \widehat{rX}$

Proof: (1) Let X_a be from Γ_r, $\widehat{\prod X_a} = \prod \hat{X_a}$, $r \prod$ is finer than $\prod r\widehat{X_a}$ complete, hence $r\,\widehat{\prod X_a}$ is complete. Analogously for subspaces. Statements (2) - (5) are self-evident.

Remark: Let c be a modification into the spaces projectively generated by all uniformly continuous real valued functions.
The class Γ_c is then the field of validity of Katětov-Shirota theorem which asserts that all topologically fine uniform spaces having its basis consisting of covers of non-measurable cardinality are contained in Γ_c. J. Pachl [P$_1$] generalized this result to the class of all sub-inversion closed spaces with basis of nonmeasurable covers. One can see from Proposition 3 that making products and their subspaces we do not get out of Γ_c.

Theorem 1: $\Gamma_{BP} = BP$.

Proof: The inclusion $\supset$ follows from Proposition 3 because BP is evidently closed under completions.
Conversely, take X complete such that BP (X) is again complete. For B a bounded subset of X, B is bounded also in BP (X) because boundedness depends only on uniformly continuous functions (see Proposition 2) hence B is relatively compact in BP (X) and so B is precompact in X.

Corollary: Γ_c is contained in BP.
The following theorem shows that there is not a significant connection between a combinatorial complexity of a uniform space and Cauchy filters.

Theorem 2: Two following assertions are equivalent:
(1) There exists an Ulam measurable cardinal
(2) Γ_r has bounded point character iff $(\mathcal{R},r)$ is the precompact modification.

Proof: (1) $\Rightarrow$ (2)
Obviously, if r is a precompact modification then $\Gamma_r =$ = Precomp.
Let r be distinct from the precompact modification. Then a countable uniformly discrete space N lies in $\mathcal{R}$ and so each countable uniform partition of any X is a uniform cover of rX. Take a metric space (M,ϱ) of point-character α. Define a base of a metric uniformity $\mathcal{U}$ on $M\times N$ by

$$\mathcal{B}_n = \{\{\langle m,j\rangle\} \mid m\in M, j\leq n\}\cup\{B^{\varrho}_{\frac{1}{n}}(m)\times\{j\} \mid m\in M, j>n\}$$

Clearly, point- haracter of $\{M\times N,\mathcal{U}\}$ is α and $r(M\times N,\mathcal{U})$ and $(M\times N\ \mathcal{U})$ have the same collection of Cauchy filters because each countable partition of $M\times$ is uniformly locally uniform and all cardinals are non-measurable.

(2) $\Rightarrow$ (1)
If there is an Ulam measurable cardinal then Γ_{p^1} (p^1 is a separable modification) contains no uniform space of po nt-character greater than some measurable cardinal.

Concluding remarks: Mdf $(\mathcal{G},\mathcal{R})$ in Definition 1 is a generalization of the concept $K*r$, here K is a clas of uniformspaces, r a modification; these classe are studied in [V] ($K*r$ is a class of all spaces with th property that each uniformly continuous map from a spac Y .n K into X factorizes through rY.)

There are many other pos ible gene alizations of Definition 1, e.g. we may put ome conditions on covers, so can form a class of all uniform spaces whose poin -fi it covers are refinable by σ-disc ete covers etc.

The problem whether Γ_r can be the class of all uni form spaces for some nonidentical modification r is equivalent to that one concerning modifications preserving Cauchy filters studied in our seminar (e.g. see [P_3]).

References:

[D] Deiabes A. Espaces uniformes et espaces de mesures, These, L'Université Claude-Barnard, Lyon.

[H] Hejcman J. Boundedness in uniform spaces and topol gical groups, Czech. Math. J. 9(84) 1959

[P_1] Pachl J. Katětov Shirota Theorem in uniform spaces (to appear)

[P_2] Pelant J. Point character of uniformities and compl teness, this volume

[P_3] Pelant J. Reflections not preserving completeness, Seminar Uniform Spaces 1973-74, directed by Z. Fro

[V] Vilímovský J. Generations of coreflections in categ ries, Comment. Math. Univ. Carolinae 14,2(1973)