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The knowledge of [F_1] and [F_2] is assumed.

In this note we assume that F is a concrete covariant functor of uniform spaces into paved spaces with the following two properties:

(a) F is metrically determined, i.e. if for any X the paved space FX is projectively generated by all f: : FX ⟶ FS such that f: X ⟶ S is uniformly continuous, and S is metric, and that means that if Y is a stone in FX, then $Y = f^{-1}[Z]$ where Z is a stone in a metric space S, and f: X ⟶ S is uniformly continuous.

(b) For any metric space S the stones in $F(S\times S)$ containing the diagonal form a basis for a uniformity on the set S, denoted by mS, mS is finer than S, and

$$F(X\times S) = F(mS\times mS).$$

Examples. We have just two examples: coz and distg. For example, Ba does not satisfy Condition (b), h coz does not satisfy Condition (a).

Denote by F the refinement of U associated with the functor F, i.e.

$$F(X,Y) = \text{Paved}\ (FX,FY).$$

Theorem 1. $F_- = (F\times F)_f$ = metric- m.

Proof. I. For each space X the stones in $F(X\times X)$ which contain the diagonal form a basis for the uniform vicinities of the diagonal for some uniformity mX on the set X. Indeed, if G is such a set, then by (a) there exists a uniformly continuous mapping f: X ⟶ S, S metric, and that

$$G = (f\times f)^{-1}[H]$$

for some stone H in $F(S\times S)$, H contains the diagonal. Hence

$H' \times H'$ for some stone H' in $\mathcal{F}(S \times S)$, and hence the square of the preimage of H' is contained in G. (It is assumed that paved spaces are multiplicative and hence we really get a basis.)

II. If $f: X \longrightarrow S$, S metric, is uniformly continuous, then so is $f: mX \longrightarrow mS$. On the other hand, a part of the argument in I shows that mX is projectively generated by $f: mX \longrightarrow mS$ such that $f: X \longrightarrow S$ is uniformly continuous, and S is metric. Hence, m is metric - m. The same argument shows that $F(mX)$ is projectively generated by $f: F(mX) \longrightarrow F(S)$ with $f: X \longrightarrow S$ uniformly continuous, and S metric.

III. $F(X \times X) = F(mX \times mX)$. This follows from the fact that the relation is true for metric spaces, and from II.

IV. $f \times f: X \times X \longrightarrow Y \times Y \in F$ iff $f: mX \longrightarrow mY \in U$.

"Only if" follows immediately from III, and from the definition of m. If is yet easier from III.

V. $(F \times F)_f = m$. Indeed,

$$U(mX, Y) = (F \times F)(X, Y).$$

VI. Since m preserves F (i.e. $FX = F(mX)$), and since $m = (F \times F)_f$, necessarily $m = F_-$.

Theorem 2. If S is metric, then $F_- S = F_f S$, and hence (by Tashijan Lemma) the same is true for products of metric spaces.

References:

[F_1] Frolík Z.: Three technical tools in uniform spaces, Seminar Uniform Spaces 1973-74, directed by Z. Frolík, MÚ ČSAV Prague, pp. 3-26.

[F_2] Frolík Z.: Four functors into paved spaces, ibid., pp. 27-72.