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Jan Reiterman; Vojtěch Rödl

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A NON-ZERO DIMENSIONAL ATOM

J. Reiterman, V. Rödl

It is shown that, under the CH, there exists a non-zero dimensional uniformity which is an atom in the lattice of uniformities on a countable set.

1. Introduction

All uniformities on a given set form a complete lattice with respect to the order " $\mathcal{U} < \mathcal{V}$ iff $\mathcal{U}$ is finer than $\mathcal{V}$". The zero of the lattice is the uniformly discrete uniformity and a uniformity $\mathcal{U}$ is an atom in this lattice iff there is no $\mathcal{V}$ with $0 \lneqq \mathcal{V} \lneqq \mathcal{U}$. Papers [3],[4] present various constructions of atoms leaving open the problem of the existence of an atom which is not zero dimensional, i.e. which has no basis consisting of partitions. In the current paper, assuming the CH, we present a construction of a uniformity on a countable set such that each atom refining it is non-zero dimensional. The following three results show that a non-zero dimensional atom must be very complicated.

1.1. Proposition. a/ Each proximally non-discrete atom is zero dimensional.

b/ For each proximally discrete atom there is an ultrafilter $\mathcal{F}$ such that the atom refines the uniformity $\mathcal{U}_{\mathcal{F}}$ where $\mathcal{U}_{\mathcal{F}}$ consists of all covers $\mathcal{C}$ with $\mathcal{C} \cap \mathcal{F} \neq \emptyset$.

1.2. Proposition. Each non-zerodimensional atom on a countable set admits a uniform cover which is a partition into finite sets.

1.3. Proposition. A non-zerodimensional atom is non-distal; in particular, it is infinite dimensional / a uniformity is distal if it has a basis consisting of covers of finite order/.

For 1.1. see [3], 1.2. is due to Pelant [4] and 1.3. can be proved easily by using 1.1.

2. Embeddings of cubes

The construction of the non-zerodimensional atom is based on cubes and their embeddings. By a cube we shall mean a set of the form $\hat{n}^m = \hat{n}\times\hat{n}\times\ldots\times\hat{n}$ (m-times), where $\hat{n} = \{1,2,3,\ldots,n\}$. Elements of $\hat{n}^m$ will be identified with functions from $\hat{m}$ to $\hat{n}$. Each cube will be regarded as a metric space with the metric defined by

$$\rho(f, g) = \sum_{x=1} d(f(x), g(x))$$

where d is the 0-1 metric on $\hat{n}$. In other words, $\rho(f, g) = |\{x\in\hat{m};\ f(x) \neq g(x)\}|$.

Let $n\leq N$ and $k\leq K$. Then we say that a mapping $\psi : \hat{n}^k\to\hat{N}^K$ is an embedding if there are $a_1,\ldots,a_K$ with $\hat{K} = \{a_1,\ldots,a_K\}$ such that $\psi(f)(a_x) = \psi(g)(a_x)$ iff either $x>k$ or $x\leq k$ and $f(x)= g(x)$ for every $f, g\in\hat{n}^k$. It is clear that an embedding of cubes is always an isometry.

The following is an easy consequence of Theorem 12,2 [2].

2.1. Lemma. Let m, j be positive integers. Then there exists a positive integer $\varkappa = \varkappa_j(m)$ such that for every subset $F\subset \hat{\varkappa}^j$ with $|F|\geq \varkappa^j/2$ there exists an embedding

$$\psi : \hat{m}^j\to\hat{\varkappa}^j \text{ whose image is contained in } F .$$

Further, we shall need three lemmas on matrices.

2.2. Lemma. Let p be a positive integer and let $A = \{a_{ij}\}$ be a $k\times\ell$ matrix where $\ell \geq (p-1)^{2^k} + 1$. Then there exist $j_1, j_2,\ldots,j_p\leq\ell$ such that for every $i \leq k$, either

(1) $a_{ij_1} = a_{ij_2} = \ldots = a_{ij_p}$ or

(2) $a_{ij_x} \neq a_{ij_y}$ for $x \neq y$, $x, y\leq p$.

Proof. For $k = 1$, the proof is trivial. Let us suppose that Lemma is proved for $k-1$. Let $A = \{a_{ij}\}$ be a $k \times \ell$ matrix. According to the induction assumption (applied to $p' = (p-1)^2 + 1$ and $A' = \{a_{ij}\}$ where $i \leq k-1$, $j \leq \ell$) there exist $\bar{j}_1, \bar{j}_2, \ldots, \bar{j}_{p'} \leq \leq \ell$ such that, for every $i \leq k-1$, either

$$a_{i\bar{j}_1} = a_{i\bar{j}_2} = \ldots = a_{i\bar{j}_{p'}}, \text{ or}$$

$$a_{i\bar{j}_x} \neq a_{i\bar{j}_y} \quad \text{for } x \neq y, \ x, y \leq p'.$$

On the other hand, there exists $\{j_1, j_2, \ldots, j_p\} \subset \{\bar{j}_1, \bar{j}_2, \ldots, \bar{j}_{p'}\}$ such that either

$$a_{kj_1} = a_{kj_2} = \ldots = a_{kj_p}, \text{ or}$$

$$a_{kj_x} \neq a_{kj_y} \quad \text{for } x \neq y, \ x, y \leq p.$$

The proof is finished.

2.3. Lemma. Let $A = \{a_{ij}\}$ be a $k \times p$ matrix with the following properties:

(i) $p \geq ((s-1)s + 1)m$ where $s = \left[\frac{k+1}{2}\right]$,

(ii) $a_{ij} \neq a_{i'j}$ iff $i \neq i'$ for every $j \leq p$,

(iii) for every $i \leq k$, either

$$a_{i1} = a_{i2} = \ldots = a_{ip}, \text{ or}$$

$$a_{ix} \neq a_{iy} \quad \text{for } x \neq y, \ x, y \leq p.$$

Then there exist $j_1, j_2, \ldots, j_m, i_1, i_2, \ldots, i_s$ such that either

(3) $$a_{i_x j_1} = a_{i_x j_2} = \ldots = a_{i_x j_m} \quad \text{for every } x \leq s, \text{ or}$$

(4) $$a_{i_x j_u} \neq a_{i_y j_v} \quad \text{iff } \langle x,u \rangle \neq \langle y,v \rangle.$$

Proof. Obviously, there exist $i_1, \ldots, i_s$ such that either

(5) $$a_{i_x 1} = a_{i_x 2} = \ldots = a_{i_x p} \quad \text{for every } x \leq s, \text{ or}$$

(6) $$a_{i_x j} \neq a_{i_x j'} \quad \text{iff } j \neq j', \ j, j' \leq p \text{ for every } x \leq s.$$

If (5) is true then the proof is finished. Let us suppose that (6) holds. Let $B = \{a_{ij}\}$ where $i \quad \{i_1, i_2, \ldots, i_s\}$ and $j \leq p$. Let C be a graph the vertices of which are columms of B and and two columms j and j' are joined by an edge if the sets $\{a_{i_1 j}, \ldots, a_{i_s j}\}$ and $\{a_{i_1 j'}, \ldots a_{i_s j'}\}$ have non-empty intersection. It is not difficult to see from (6) that the degree of every vertex of this graph is at most $s(s-1)$. As the number of vertices is at least $(s(s-1)+1)\,m$, there is an independent set in this graph /no two vertices are joined by an edge/ of size $\dfrac{p}{s(s-1)+1} = m$, see[1], p. 284. In other words, there are $j_1, \ldots, j_m$ such that (4) is true.

2.4. Lemma. Let $A = \{a_{ij}\}$ be a $k \times \ell$ matrix with the following properties:

(i) $\ell > ((s(s-1)+1)m-1)^{2^k} + 1$, where $s = \left[\dfrac{k+1}{2}\right]$,

(ii) $a_{ij} \neq a_{i'j}$ iff $i \neq i'$ for every $i \leq k$.

Then there exist $j_1, \ldots, j_m$ and $i_1, \ldots, i_s$ such that (3) or (4) holds.

Proof. See 2.2. and 2.3.

The following theorem and its corollary provide main results of this section.

2.5. Theorem. Let n, m be positive integers. Then there exists a positive integer $N = N^n(m)$ such that for every mapping $\varphi : \hat{N}^n \longrightarrow R$ /where R is an infinite set/ there is a partiti[on] $\hat{n} = A \cup B$ and an embedding $\psi : \hat{m}^n \longrightarrow \hat{N}^n$ such that

$$\varphi\psi(f) = \varphi\psi(g) \quad \text{iff} \quad f/A = g/A .$$

Proof. We shall prove the theorem by induction on n. It is easy to see that $N^1(m) = (m-1)^2 + 1$. For $n > 1$, denote

$$\varkappa = \max\left(\varkappa_1(m), \varkappa_2(m), \dots, \varkappa_{n-1}(m)\right) \text{ /see 2.1./,}$$

$$\ell = \left(\left(\left[\tfrac{\varkappa}{2}\right]^{n-1}\left(\left[\tfrac{\varkappa}{2}\right]^{n-1} - 1\right) + 1\right)m - 1\right)^{2^{\varkappa^{n-1}}} + 1,$$

$$r = 2^{n-1}\ell$$

and define $N_0, N_1, \dots, N_r$ by

$$N_r = \varkappa, \quad N_{q-1} = N^{n-1}(N_q), \quad q = 1, 2, \dots, r.$$

Finally, put $N^n(m) = N_0$.

Let us consider a mapping $\varphi : \hat{N}^n \longrightarrow R$. Identifying the set $F_1 = \{f \in \hat{N}^n ;\ f(n) = 1\}$ with $\hat{N}^{n-1}$ and using the induction assumption we get an embedding $\psi_1 : \hat{N}_1^{\,n-1} \longrightarrow F_1$ and a corresponding partition $A_1 \cup B_1$. Redefine all functions $f \in \psi_1(\hat{N}_1^{\,n-1})$ by $f(n) = 2$ /instead of $f(n) = 1$ / to obtain a set F_2 which can be identified with $\hat{N}_1^{\,n-1}$. Let us repeate the procedure to obtain an embedding $\psi_2 : \hat{N}_2^{\,n-1} \longrightarrow F_2$ and a corresponding partition $A_2 \cup B_2$. After r-fold repeating we get an embedding $\psi_r : \hat{N}_r^{\,n-1} \longrightarrow F_r$ and a partition $A_r \cup B_r$.

Consider the embeddings $\psi_1, \dots, \psi_r$ restricted to the set $\hat{N}_r^{\,n-1}$. Then for every $i \leq r$ and for every $f, g \in \psi_i(\hat{N}_r^{\,n-1})$,

$$\varphi\psi_i(f) = \varphi\psi_i(g) \quad \text{iff} \quad f/A_i = g/A_i.$$

As $r = 2^{n-1}\ell$, there exist numbers $\alpha_1, \dots, \alpha_\ell$ and a partition $\widehat{n-1} = A \cup B$ such that $A_{\alpha_i} = A$ and $B_{\alpha_i} = B$ for $i \leq \ell$.

Let us consider the equivalence $\sim$ on the set $\hat{N}_r^{\,n-1}$ defined by

$$f \sim g \quad \text{iff} \quad f/A = g/A.$$

Let us denote the equivalence classes of $\sim$ by $C_1, \dots, C_k$, where $k = N_r^{|A|}$. Then for every $j \leq \ell$,

(8) $\qquad \varphi\psi_{\alpha_j}(f) = \varphi\psi_{\alpha_j}(g)$ iff $f, g \in C_i$ for some $i \leq k$.

Let us consider a $k \times \ell$ matrix $A = \{a_{ij}\}$ where $a_{ij} = \varphi\psi_{\alpha_j}$ for $f \in C_i$, $i \leq k$, $j \leq \ell$. By (8), the matrix satisfies the condition (ii) of 2.4. Thus, it follows from the definition of ℓ that there exist $j_1, \dots, j_m$ and $i_1, \dots, i_s$ where $s = \left[\frac{\varkappa^{|A|}}{2} + 1\right]$ such that (3) or (4) is true. From 2.1. applied to the set $F = \bigcup_{x \leq s} C_{i_x}$, we obtain an embedding $\psi : \hat{m}^n \to \hat{N}^n$ with required properties. The proof is finished.

2.6. If $\mathcal{P} = \{P_i\}$ is a partition of a set X and $A \subset X$, then A is said to be <u>$\mathcal{P}$-selective</u> (<u>$\mathcal{P}$-fine</u>) if $|A \cap P_i| \leq 1$ for every i (if $A \subset P_i$ for some i, respectively).

Theorem 2.5. has the following

<u>Corollary.</u> Let n be a positive integer. Then there exists a positive integer N such that for each partition $\mathcal{P}$ of $\hat{N}^N$ there is an embedding $\psi : \hat{n}^n \to \hat{N}^N$ whose image is either $\mathcal{P}$-selective or $\mathcal{P}$-fine.

<u>Proof.</u> Put $N = \hat{N}^{2n-1}(n)$, see the preceding theorem. If $\mathcal{P}$ is a partition of $\hat{N}^{2n-1}$, define a mapping $\varphi : \hat{N}^{2n-1} \to R$ such that $\mathcal{P} = \{\varphi^{-1}r \,;\, r \in R\}$ and apply the preceding theorem; we obtain an embedding $\psi : \hat{n}^{2n-1} \to \hat{N}^{2n-1}$ and a partition $\widehat{2n-1} = A \cup B$. Choose an arbitrary embedding $\psi' : \hat{n}^n \to \hat{n}^{2n-1}$ such that the elements $a_1, \dots, a_n$ from the definition of an embedding are in A or in B according as $|A| \geq n$ or $|B| \geq n$. In the former case, the embedding $\psi\psi' : \hat{n}^n \to \hat{N}^{2n-1}$ is $\mathcal{P}$-selective and it is $\mathcal{P}$-fine in the latter one. As $\hat{N}^{2n-1}$ can be embedded into $\hat{N}^N$, the corollary follows.

3. The construction.

3.1. Denote Y the disjoint union $\bigcup_{n=1} \hat{n}^n$. Let us extend the metric on cubes by putting $\rho(f, g) = \infty$ if f, g belongs to distinct cubes (for the sake of convenience, we admit the value ∞ in the definition of a metric). This makes Y a metric space.

3.2. Convention. Writing Y', Y_n or any other symbol containing the capital Y we shall always mean a subset of Y which (equipped with the induced metric) is an isometric copy of Y.

3.3. A partition $\mathcal{P} = \{P_i\}$ of a metric space is said to be bounded if there is K such that $\operatorname{diam} P_i < K$ for every i.

3.4. Lemma. Let $\mathcal{P}$ be a partition of Y which is bounded (finite). Then there is $Y' \subset Y$ which is $\mathcal{P}$-selective ($\mathcal{P}$-fine, respectively).

Proof. Let $\mathcal{P} = \{P_i\}$ be a bounded partition of Y, let $\operatorname{diam} P_i < K$ for every i. Let $n > K$. Let $N = N(n)$ be from 2.6. Consider the trace of $\mathcal{P}$ on $\hat{N}^N$. As $\operatorname{diam} \hat{n}^n = n > K$, no member of $\mathcal{P}$ can contain a copy of $\hat{n}^n$. Thus, 2.6. gives an embedding $\psi_n : \hat{n}^n \to \hat{N}^N$ such that $\psi_n(\hat{n}^n)$ is $\mathcal{P}$-selective. We may assume that $N(n) \neq N(m)$ for $n \neq m$. Then $A = \bigcup_{n>K} \psi_n(\hat{n}^n)$ is $\mathcal{P}$-selective. As the latter space is an isometric copy of $\bigcup_{n>K} \hat{n}^n$ and Y can be isometrically embedded into $\bigcup_{n>K} \hat{n}^n$, there is $Y' \subset A$ which is a $\mathcal{P}$-selective isometric copy of Y.

If $\mathcal{P}$ is a finite partition of Y, we proceed quite analogously: define only K to be the cardinality of $\mathcal{P}$ to obtain that the images of the above embeddings ψ_n are $\mathcal{P}$-fine.

3.5. Lemma. Let $Y = Y_1 \supset Y_2 \supset Y_3 \supset \ldots$. Then there exists $Y_\infty \subset Y$ such that for each $Y' \subset Y_\infty$ there are $Y_1', Y_2', Y_3', \ldots$ such that $Y' = Y_1' \supset Y_2' \supset Y_3' \supset \ldots$ and $Y_i' \subset Y_i$ for every i.

Proof. Choose an isometric copy K_n of $\hat{n}^n$ in each Y_n such that the K_n's are pairwise disjoint. Put $Y_\infty = \bigcup_{n=1} K_n$. Now, if $Y' \subset Y_\infty$ and $Y' = \bigcup_{n=1} K_n'$, each K_n' being an isometric copy of $\hat{n}^n$, then for each n there exists n' with $K_n' \subset K_{n'}$. Then $\bigcup_{i \geq n} K_i' \subset \bigcup_{i \geq n} K_i \subset Y_n$. Put $Y_1' = Y'$. Let Y_n' be defined such that $Y_n' \subset Y_n \cap \bigcup_{i \geq n} K_i'$. Then choose $Y_{n+1}' \subset Y_n' \cap \bigcup_{i \geq n+1} K_i'$. It follows that $Y' = Y_1' \supset Y_2' \supset Y_3' \supset \dots$ and that $Y_n' \subset \bigcup_{i \geq n} K_i' \subset Y_n$.

3.6. Lemma. Let $\{\mathcal{P}_\alpha ; \alpha < \omega_1\}$ be a collection of partitions of Y, each $\mathcal{P}_\alpha$ being either bounded or finite. Then there exists a family $\{Y_K\}$ where K runs over all finite subsets of ω_1 such that

(i) $L \subset K \Rightarrow Y_L \supset Y_K$,

(ii) For every α $Y_{\{\alpha\}}$ is $\mathcal{P}_\alpha$-selective if $\mathcal{P}_\alpha$ is bounded and $\mathcal{P}_\alpha$-fine if $\mathcal{P}_\alpha$ is finite.

Proof. We shall proceed by induction on max K. If max K = 0 then $K = \{0\}$ and we choose Y_K to be $\mathcal{P}_0$-selective or $\mathcal{P}_0$-fine, see 3.4.

Let the Y_K's be defined for $\max K < \alpha$. First, we shall defi[ne] $Y_{\{\alpha\}}$. Let $\{n_i\}_{i=1}^\infty$ be a sequence such that $\{n_i;\ i < \omega_0\}$ is the set of all ordinals $< \alpha$. Put $K_i = \{n_1, \dots, n_i\}$ and $Y_i = Y_{K_i}$. By (i) we have $Y_1 \supset Y_2 \supset Y_3 \supset \dots$ and we can apply 3.5. to obtain Y_∞. Put $Y_{\{\alpha\}} = Y'$ where Y' is from 3.4. applied to the trace of $\mathcal{P}_\alpha$ on Y_∞. By 3.5., we have also obtained a sequence $Y' = Y_1' \supset Y_2' \supset Y_3' \supset \dots$ such that $Y_i' \subset Y_i$ for every [i]. Finally, if $K \subset \omega_1$ is a finite set with $\max K = \alpha$, $K \neq \{\alpha\}$ consider the smallest i with $K - \{\alpha\} \subset K_i$ and put $Y_K = Y_i'$. Then (i) and (ii) holds for $\max K, \max L \leq \alpha$, too.

.7. Let Z be a disjoint union $\bigcup_{n=1}^{\infty} Z_n$ where $Z_n = Y$ for very n. Define a metric σ on Z by

$$\sigma(x,y) = \frac{\rho(x,y)}{n} \quad \text{if } x, y \in Z_n \text{ for some } n,$$
$$\sigma(x,y) = \infty \quad \text{otherwise.}$$

.8. Assume the CH. Then we can assume that the collection n 3.6. contains all bounded partitions and all finite partitions f Y. Then the family $\{Y_K\}$ is a basis f an ultrafilter on Y which will be denoted by $\mathcal{F}$. Further, efine a filter $\mathcal{G}$ on Z by $G \in \mathcal{G} \Leftrightarrow G \cap Z_n \in \mathcal{F}$ for every n. inally, let $\mathcal{X}$ be an arbitrary ultrafilter on the set of positive ntegers; put $\mathcal{Y} = \mathcal{X}\mathcal{F}$, i.e. $\mathcal{Y}$ is an ultrafilter on Z a basis f which consists of sets of the form $\bigcup_{n \in H} C_n$ where $H \in \mathcal{X}$ nd $G_n \in \mathcal{F}$ for every $n \in H$.

In contrary to Y, the metric space Z is not uniformly disrete. Observe also that a partition of Z is bounded iff its race on Z_n is bounded for every n. Thus, by 3.6. we have the llowing

Proposition. (CH) (i) The metric uniformity of Z is not uniformly iscrete.

(ii) The filter $\mathcal{G}$ on Z posses a basis consisting of isometric pies of Z.

(iii) The ultrafilter $\mathcal{Y}$ on Z posses a basis consisting of iformly homeomorphic copies of Z.

(iv) The filter $\mathcal{G}$ is selective with respect to bounded partions of Z, i.e. for every bounded partition $\mathcal{P}$ of Z there is $\in \mathcal{G}$ which is $\mathcal{P}$-selective. Analogously for $\mathcal{Y}$.

9. Theorem. (CH) There is an atom in the lattice of uniformities a countable set which is non-zerodimensional. In fact, all oms refining $\mathcal{U}_Z \wedge \mathcal{U}_{\mathcal{G}}$ (where $\mathcal{U}_Z$ is the metric uniformity of Z) e non-zerodimensional; analogously for $\mathcal{U}_Z \wedge \mathcal{U}_{\mathcal{Y}}$.

Proof. Recall that $\mathcal{U}_{\mathcal{G}}$ is the uniformity consisting of all covers $\mathcal{C}$ with $\mathcal{C} \cap \mathcal{G} \neq \emptyset$. Further observe that $\mathcal{U}_Z \wedge \mathcal{U}_{\mathcal{G}}$ is generated by the family ~~xx~~ $\{\sigma_G ; G \in \mathcal{G}\}$ where

$$\sigma_G(x,y) = \sigma(x,y) \text{ if } x,y \in G,$$
$$\sigma_G(x,y) = \infty \quad \text{for } x \neq y \text{ otherwise.}$$

By 3.8. (i), (ii), no $G \in \mathcal{G}$ is $\mathcal{U}_Z$-uniformly discrete and so $\mathcal{V} = \mathcal{U}_Z \wedge \mathcal{U}_{\mathcal{G}}$ is not uniformly discrete as well. Let $\mathcal{a}$ be an atom refining $\mathcal{V}$ (recall from [3] that each uniformly non-discrete uniformity can be refined by an atom). Then the cover $\mathcal{C}$ consisting of b with r = 1 with respect to σ belongs to $\mathcal{a}$. However, any partition $\mathcal{P}$ refining $\mathcal{C}$ is bounded and so $\mathcal{P} \notin \mathcal{a}$ according to 3.8.(iv). Thus, $\mathcal{a}$ is non-zerodimensional. The proof for $\mathcal{U}_{\mathcal{Y}}$ is quite analo

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