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SPHERICAL RBF INTERPOLATION EMPLOYING PARTICULAR GEODESIC METRICS AND TREND FUNCTIONS

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Abstract: The paper is concerned with spherical radial basis function (SRBF) interpolation. We introduce particular SRBF interpolants employing several different geodesic metrics and a single trend function. Interpolation on a sphere is an important tool serving to processing data measured on the Earth's surface by satellites. Nevertheless, our model physical quantity is the magnetic susceptibility of rock measured in different directions. We construct a general SRBF formula and prove conditions sufficient for its existence. Particular formulae with specified geodesic metrics, trend and SRBFs are then constructed and tested on a series of magnetic susceptibility examples. The results show that this interpolation is sufficiently robust in general.

Keywords: radial basis function, spherical interpolation, spherical radial basis function, geodesic metric, trend, multiquadric, magnetic susceptibility

MSC: 65D05, 65D12, 65Z05

1. Introduction

In many geophysical applications there is a demand to compute an approximate representation of data measured on the sphere. We introduce a radial basis function (RBF) or spherical radial basis function (SRBF) interpolant in a real Euclidean space $\mathbb{R}^d$ for data measured at nodes on the $(d-1)$-dimensional surface of the unit sphere in $\mathbb{R}^d$ ([2], [10], [14]) in Section 2. Further we present sufficient conditions for the existence of such an interpolation formula.

Physical quantities measured on a sphere have brought an increasing interest with very advanced satellite technology of acquiring such data on the Earth surface. In the paper, the model physical quantity, having extensive applications, is different. It is concerned with the laboratory determined scalar physical data, the values of magnetic susceptibility of rock measured in different directions.

We introduce the spherical data interpolation formula and give sufficient conditions for its existence in Section 2. We describe the ways of approximating raw data

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starting from the primary statistical treatment, important for the choice of the trend of the interpolation formula, in Section 3, see, e.g., [15]. We use a single trend in the formula, the second order polynomial in three Cartesian variables, that follows from these considerations and fits the data measured as well as possible.

Several geodesic metrics, functions necessary for the construction of spherical radial basis function interpolation, are considered in the paper, cf. [9], [10], [11]. We employ only one SRBF in the experiments presented, the multiquadric $\psi(r) = \sqrt{r^2 + c^2}$, see Sections 4 and 5. Further RBFs often used can be found in [2], [10], [12] etc.

The choice of a grid for the measurements performed is an important part of interpolation [1], [5], [6], [7]. An apparent drawback of the simplest grid equidistant in the spherical coordinates φ and ϑ is considered in Section 7. In this section, numerical experiments employ as input the exact data given by the formula for trend, but perturbed randomly. The results given in Sections 6 and 7 show that the interpolation considered is sufficiently reliable.

2. Spherical data interpolation

We start with the notation necessary for introducing spherical data interpolation. Let d be the dimension of a real Euclidean space $\mathbb{R}^d$. Then $S^{d-1} = \{x \in \mathbb{R}^d \mid \|x\| = 1\}$, where the norm $\|\cdot\|$ is Euclidean, is the $(d-1)$-*dimensional surface of unit sphere* in the d-dimensional space $\mathbb{R}^d$.

Further, a function $\sigma(x, y)$ of two variables $x, y \in \mathbb{R}^d$ is called *radial* if there exists a function $\tau(r)$, $r \geq 0$, such that $\sigma(x, y) = \tau(r)$, where $r \in \mathbb{R}$ is usually the Euclidean distance between x and y in case of non-spherical data.

Let N and M be integers, $N > 0$, $M \geq 0$, $N \geq M$, and $X = \{x_j\}_{j=1}^N$ be a set of mutually distinct *interpolation nodes* $x_j = (x_{j1}, x_{j2}, \ldots, x_{jd})$ on S^{d-1}. The real *spherical interpolant* v for $x \in S^{d-1}$ is constructed as

$$v(x) = \sum_{j=1}^{N} a_j \psi(g(x, x_j)) + \sum_{k=1}^{M} b_k p_k(x), \tag{1}$$

where a_j, $j = 1, \ldots, N$, and b_k, $k = 1, \ldots, M$, are real coefficients to be found. If $M = 0$, the second sum is empty.

In the interpolant, g is a nonnegative function called the *geodesic metric*, usually $g\colon S^{d-1} \times S^{d-1} \to [0, 1]$ is based on the angle between the radius vectors corresponding to the two arguments of g, see Section 3. Examples are given in Section 4. Further, $\psi\colon [0, 1] \to \mathbb{R}$ is a continuous real function, called the *radial basis function* (RBF) or *spherical radial basis function* (SRBF), and p_k is a polynomial from $\Pi_t(\mathbb{R}^d)$, where $\Pi_t(\mathbb{R}^d)$ is the set of all polynomials (*trends*) $p\colon \mathbb{R}^d \to \mathbb{R}$ with real coefficients and of total degree less than or equal to some nonnegative integer t.

Let us formulate the interpolation problem to be solved. Given a continuous real *target function* $f\colon S^{d-1} \to \mathbb{R}$, find the *spherical interpolant*, i.e., a continuous function $v\colon S^{d-1} \to \mathbb{R}$ that satisfies the *interpolation conditions*

$$v(x_i) = f(x_i), \quad i = 1, \ldots, N, \tag{2}$$

where $f(x_i)$ is the value measured at the node x_i. Multiple measurements in a single direction x_i with different results lead to a singular linear algebraic system for coefficients of the interpolation formula.

We confine ourselves only to real-valued functions and real data to make the exposition clearer. Substitute $x_i,\ i = 1, \ldots, N$, for x in the formula (1) for v to get

$$v(x_i) = \sum_{j=1}^{N} a_j \psi(g(x_i, x_j)) + \sum_{k=1}^{M} b_k p_k(x_i), \quad i = 1, \ldots, N,$$

and replace the left hand parts $v(x_i)$ of the interpolation conditions (2) with these expressions.

In the matrix notation, introduce an $N \times N$ symmetric square matrix Ψ with the entries $\psi_{ij} = \psi(g(x_i, x_j)),\ i, j = 1, \ldots, N$, and an $N \times M$ matrix P with the entries $p_{jk} = p_k(x_j),\ j = 1, \ldots, N,\ k = 1, \ldots, M$. Let $a \in \mathbb{R}^N$, $b \in \mathbb{R}^M$, and $f \in \mathbb{R}^N$ be the vectors of the unknowns and the vector of the right hand parts $f(x_i)$ of the interpolation conditions (2).

Note that if $M > 0$ then we have only N interpolation conditions for $N+M$ interpolation coefficients a_j and b_k of the interpolant. Thus we can impose M additional linear constraints for the individual trends p_k,

$$\sum_{j=1}^{N} a_j p_k(x_j) = \sum_{j=1}^{N} a_j p_{jk} = 0, \quad k = 1, \ldots, M.$$

Now the system of linear algebraic equations to be solved for the unknown vectors a and b reads

$$Q \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}, \quad \text{where } Q = \begin{bmatrix} \Psi & P \\ P^{\mathrm{T}} & 0 \end{bmatrix} \tag{3}$$

is a symmetric $(N + M) \times (N + M)$ matrix of the system.

Theorem 1. *Let the $N \times N$ principal submatrix Ψ of the $(N+M) \times (N+M)$ matrix Q be positive definite and let* rank $P = M$. *Then the matrix Q is nonsingular.*

Proof. The proof follows from Theorem 1 of [13]. □

In Theorem 1, we use the hypothesis that the matrix Ψ is positive definite and rank $P = M$. However, in Micchelli [12] and many other sources, the condition that the spherical basis function ψ is *conditionally (strictly) positive definite* is employed to prove that the matrix Q is nonsingular.

A problem similar to data interpolation is *data smoothing (fitting)* but we are not concerned with that problem in this contribution.

3. Model problem

For a model problem, we have chosen the laboratory determination of raw susceptibility data, see, e.g., [8], [15]. The 3D rock sample rotates in magnetic field and the scalar data items s_i measured in a set of selected directions u_i are of the form

$$s_i = u_i^{\mathrm{T}} K u_i + e_i, \tag{4}$$

where u_i is a unit vector in Cartesian coordinates in $\mathbb{R}^3$, whose initial point is at the origin and whose end point is on the unit sphere at x_i. K is a tensor, and e_i are deviations from the theoretical tensor model. Assuming the equation (4), we carry out linear regression and find an estimation of the tensor K. Then an appropriate rotation of the coordinate system can make the tensor K diagonal with the *principal susceptibilities* K_1, K_2, K_3 on the diagonal.

We call the graphical representation of the directional susceptibilities the *lemniscate surface*, see Figure 1. Two-dimensional surfaces in $\mathbb{R}^3$ are depicted as endpoints of the corresponding vectors $s_i u_i$, as usual.

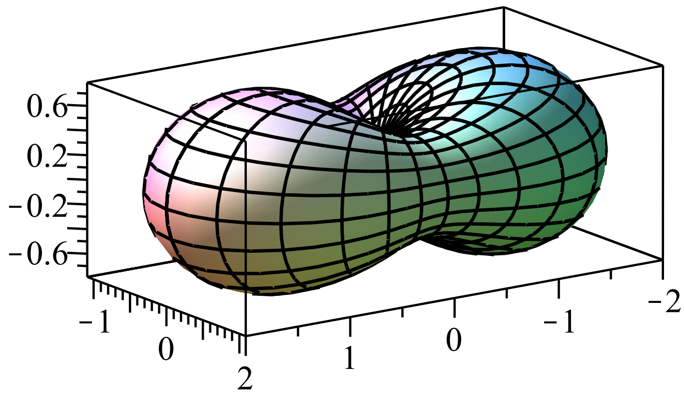

Figure 1: Model problem. Lemniscate surface v obtained by the interpolation formula (10) (with interpolation conditions (2)) employing the theoretical values $f(x_i) = s_i$ given by (4) with principal susceptibilities $K_1 = 2.00$, $K_2 = 1.00$, $K_3 = 0.10$, and $e_i = 0$.

The magnitude of directional susceptibility in the ith direction z_i is given by the distance between the origin and the surface measured along the vector z_i. The polynomial $s(z) = z^{\mathrm{T}} K z$ determined by the tensor K is taken for the only trend in our further considerations, see Section 5.

4. Geodesic metric

Employing a RBF in the interpolation formula, we are supposed to define the distance between two nodes (i.e., between two unit vectors) x and y on the unit sphere S^{d-1}. The angle α of these two vectors is given as

$$\cos\alpha = x \cdot y,$$

where $x \cdot y$ is the inner product of two vectors from $\mathbb{R}^d$. Since $\cos\alpha = \cos(2\pi - \alpha)$ we can choose for computation either the angle α or its complement to 2π, i.e. $2\pi - \alpha$. Only few geodesic metrics g are used in practice. They usually satisfy $g\colon S^{d-1} \times S^{d-1} \to [0, 1]$.

The simplest geodesic metric is the angle α itself,

$$g_0(x, y) = \alpha/(2\pi) = \cos^{-1}(x \cdot y)/(2\pi). \tag{5}$$

Further two geodesic metrics, g_1 and g_2, are based on $\cos\alpha$, $\alpha \in [0, 2\pi]$. We put

$$g_1(x, y) = \sqrt{1 - \cos^2\alpha} = |\sin\alpha|. \tag{6}$$

Central symmetry of the data measured is expected when we apply the geodesic metric g_1. Every unit vector x is considered as a part of an axis coming through the center of the sphere and from its two possible directions no direction is prescribed. Our quantity measured (magnetic susceptibility of rock) is just of this kind. If the angle α of two vectors x and y equals π (i.e., $y = -x$) then the values measured on the sphere at x and y should be identical since the nodes x and $-x$ of interpolation are not distinguished.

Therefore, in what follows, when using g_1, we assume that the elements x_j of the set X are mutually distinct and, moreover, that it is $x_i \neq -x_j$ for every $i, j = 1, \dots, N$. The geodesic metric g_1 is periodic in α with the period π, and it holds $g_1(x, y) = 0$ for $\alpha = 0, \pi, 2\pi$.

The next geodesic metric considered is

$$g_2(x, y) = \sqrt{\tfrac{1}{2}(1 - \cos\alpha)}. \tag{7}$$

No symmetry of data measured is supposed when we employ the geodesic metric g_2. Apparently, g_2 is periodic in α with the period 2π, and it holds $g_2(x, y) = 0$ for $\alpha = 0, 2\pi$.

5. A particular trend function

Let us turn back to our 3D problem introduced in Sec. 3. We take the second degree polynomial corresponding to (4), i.e.

$$s(z) = K_1 z_1^2 + K_2 z_2^2 + K_3 z_3^2, \quad z = (z_1, z_2, z_3) \in S^2, \tag{8}$$

where K_1, K_2, K_3 are known positive constants, for the only trend, i.e. $M = 1$.

Notice that the single argument of the SRBF function ψ is from the interval $[0, 1]$ due to the geodesic metric, while the argument z of the trend s is from S^2.

The advantage of the formula proposed is apparent in cases when we know that the physical field measured does not principally differ from the ideal field whose values can be computed from some explicit formula, in our case from (4). Description of the ideal field is then fitted by the trend part of the formula and the corrections resulting from the first, spherical part of the formula are only small.

6. The SRBF formula employed

We put $d = 3$ in our model problem, then S^2 is the usual two-dimensional unit sphere surface in the three-dimensional Euclidean space $\mathbb{R}^3$. Choose a fixed positive integer N and put $M = 1$.

We take the *multiquadric*

$$\psi(r) = \sqrt{r^2 + c^2} \tag{9}$$

for the spherical radial basis function, where $r \in [0, 1]$ (the range of the geodesic function) and c is a positive shape parameter.

Apparently, the trend s given by (8) belongs to $\Pi_2(\mathbb{R}^3)$, which is the set of all polynomials $p\colon \mathbb{R}^3 \to \mathbb{R}$ of three variables with real coefficients and of total degree less then or equal to 2.

Consider the interpolation formula (1) in the form

$$v(x) = \sum_{j=1}^{N} a_j \psi(g(x, x_j)) + bs(x), \tag{10}$$

where $x, x_j \in S^2$, i.e., in the interpolation system (3), P is a single column N-vector and b and 0 are scalars.

We add a single constraint

$$\sum_{j=1}^{N} a_j s(x_j) = 0$$

to the interpolation conditions.

The following theorem is a particular case of Theorem 1 that covers our model problem.

Theorem 2. *Let the linear algebraic system* (3) *correspond to the interpolation formula* (10)*. Let the block P in the block matrix Q have rank* 1*. Then the interpolation problem has the unique solution a_j, $j = 1, \dots, N$, and b.*

Proof. It is known that the principal submatrix Ψ of the block matrix Q of the linear algebraic system (3) is positive definite when ψ is an inverse multiquadric (Micchelli [12]). On the assumption that rank $P = 1$, the matrix Q is nonsingular by Theorem 1 and the system has the unique solution a_j, $j = 1, \dots, N$, and b. □

Remark 1. P is a single column N-vector, $P^{\mathrm{T}} = (s(x_1), \dots, s(x_N))$. The assumption of Theorem 2 that rank $P = 1$ is apparently fulfilled if at least one of the entries $p_k = s(x_k)$ is nonzero.

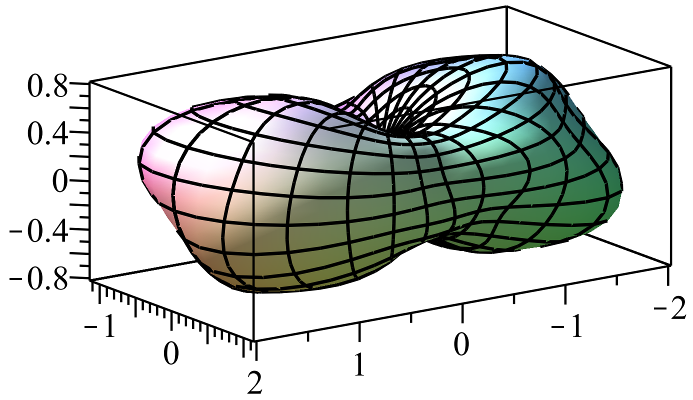

Figure 2: Lemniscate surface with $K_1 = 2.00, K_2 = 1.00, K_3 = 0.10$. The exact values given by (4) were multiplied by a random factor σ from the range $[0.9, 1.1]$ resulting in the corresponding e_i. The geodesic metric g_0, $N = 74$, $c^2 = 0.25$.

7. Computational experiments

We have accomplished several series of computational experiments with the SRBF interpolation of the theoretical as well as perturbed theoretical lemniscate surfaces in the model problem with $d = 3$, where S^2 is the usual two-dimensional unit sphere surface in the three-dimensional Euclidean space. We have employed the SRBF interpolation formula (10) and different grids, geodesic metrics g_0, g_1, g_2, and several SRBF functions. See Figures 2, 3, 4.

The simplest grid used on a unit sphere is the grid equidistant in both the spherical coordinates φ and ϑ. The drawback of this grid is the fact that its nodes are dense in the vicinity of poles and sparse around the equator. For g_1, the interpolation nodes should satisfy the condition $x_j \neq \pm x_i$ mentioned above. The results presented in this paper have been computed in such grids.

Grids on a unit sphere are often used also for numerical integration. For interpolation, we have tried three such systems of grids: Bažant grids [1], Fibonacci grids [4], [6], and triangular grids stemming from an icosahedron [7], but we have found that they bring no significant advantage. A general treatment of data sampling on a unit sphere is provided in [5].

In literature (see, e.g., [2], [10], [12]), one can find several SRBFs ψ known to provide a positive definite matrix Ψ of (3). For example, the (direct) multiquadric (9), inverse multiquadric $1/\sqrt{r^2 + c^2}$, Gaussian function $\exp(-cr^2)$ or thin plate spline [3]. The results presented in this contribution have been computed with the direct multiquadric ψ with a positive parameter c. The results may strongly depend on the constant c.

The resulting linear algebraic system (3) for the coefficients of the formula can be easily solved by the LU decomposition method for N of order tens. For higher N, the system may be very ill-conditioned and special solution methods should be used. We apply, e.g., the Gauss-Jordan method.

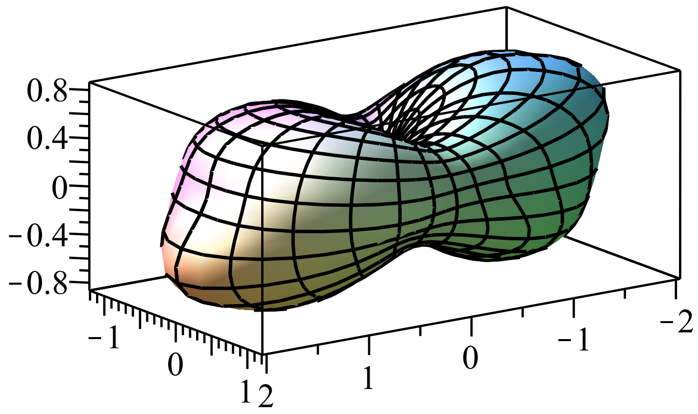

Figure 3: Lemniscate surface with $K_1 = 2.00, K_2 = 1.00, K_3 = 0.10$. The exact values given by (4) were multiplied by a random factor σ from the range $[0.9, 1.1]$ resulting in the corresponding e_i. The geodesic metric g_1, symmetric grid and data, $N = 40$, $c^2 = 0.25$.

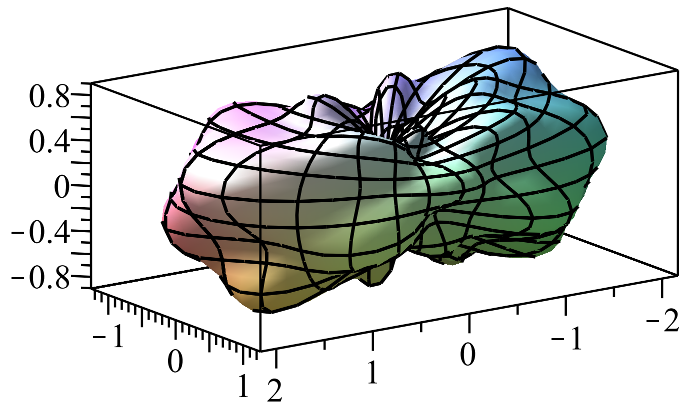

Figure 4: Lemniscate surface with $K_1 = 2.00, K_2 = 1.00, K_3 = 0.10$. The exact values given by (4) were multiplied by a random factor σ from the range $[0.999, 1.001]$ resulting in the corresponding e_i. The geodesic metric g_2, $N = 74$, $c^2 = 0.25$.

8. Conclusions

We have carried out numerical tests with the interpolation formula (10), three geodesic metrics (5), (6) and (7), and SRBF (9). The formula performs efficiently and the results exhibit dependence on the parameter c. Further research shall provide a comparison of results obtained using various other SRBFs, e.g. thin plate splines, inverse multiquadrics, or the Gaussian function.

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